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# A NOTE ON REPLENISHMENT OF INFRASTRUCTURE

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ABSTRACT. This paper focuses on the subject of replenishment of infrastructure in higher education institutions from the perspective of optimal control theory. The situation where an institution has autonomy over structuring its replenishment schedule and the fees that it charges constitute a major source of revenue is studied. Propositions on the pattern of replenishment are stated along with their proofs. The pattern of replenishment can help guide educational planners towards steering up infrastructure in higher institutions.

**Keywords and phrases:** autonomy, Hamilton-Pontryagin equation, higher education institution, optimal control theory, replenishment.

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## 1. INTRODUCTION

Infrastructure is one of the key components of any educational system. Higher education is a part of the educational system which is saddle with the responsibility of producing high level manpower. In the Nigerian case, the government has been concerned with opening many higher institutions without considering the state of infrastructure. Consequent upon this the level of productivity is low [1]. In recent times, trade unions in the education sector agitated through strikes to register their discontentment over the poor state of infrastructure vis-à-vis the proposed implementation of autonomy for higher education institutions. Even so, both the trade unions and the government have not been able to reach a consensus on an objective replenishment plan to steer up infrastructure. There is no doubt that the replenishment of infrastructure constitutes cost to the system and drains finances. More so, the infrastructure of an institution may not have a second-hand value as it is usually engraved to indicate ownership and oftentimes the phrase 'Not for Sale' is inscribed to prevent it from being sold elsewhere. However,

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these demerits are not enough to limit the provision of adequate infrastructure. Infrastructure (hereafter referred to as capital stock) may change due to: depreciation and the rate of replenishment. The rate of replenishment has a positive effect, while depreciation has a negative effect.

A wealth of literature on optimal capital accumulation abounds. The works of [2, 3, 4, 5] are just a few examples. Solution to optimal control problems may be derived from: the Green's theorem [6], the Pontryagin's principle [2], the turnpike theorem [7, 8, 9] and the most rapid approach path [10]. This study utilises the Pontryagin's principle.

The aim of this study is to develop simple formulas to describe the replenishment schedule for capital stock for higher education institutions where the policy of autonomy is operational. Edwards [11] had earlier posited that the ease of use of a model is more important than the theoretical sophistication. In the setting we consider the fees charged by an institution are the major source of revenue. Robst [12] had earlier examined the reduced importance of state appropriations and increased importance of tuition revenue in public universities. We use the stream of profit accruing to an institution in a fixed planning period as a measure of performance. We formulate an optimal control model consisting of the rate of replenishment as a control and the capital stock as the state variable. Nonetheless, we do not consider the efficiency of the system as this has already been discussed elsewhere [12, 13].

## 2. MODEL DEVELOPMENT

This section presents the model build-up for the dynamics of capital stock in a higher institution for a facility. We formulate the state equation for the institution based on the assumption that the rate of replenishment is gradual and that the

change in capital stock =

$$\text{replenishment less depreciation of capital stock.} \tag{1}$$

The capital stock is measured in monetary terms. We assume a fixed planning period $[0, T], T > 0$, so that an instant of time is $t \in [0, T]$. Let $x(t)$ denote the capital stock for a particular facility at moment $t$ and $u(t)$ the fraction of desired capital stock to be supplied for the facility at moment $t$ (or simply the rate of replenishment at moment $t$). We assume that the initial state of the capital stock is known and that a desired state is being sought for. Let $x_0$ denote the initial capital stock and $B$ the desired stock. Using these notations, the discontentment arising from the available capital stock at the initial stage is expressed by the relation $x_0 < B$. We assume that the depreciation of capital stock is proportional to the available capital stock. Based on these assumptions, Eq. (1) is technically written as

$$\dot{x}(t) = u(t)B - \gamma x(t), \gamma > 0, 0 < u(t) < 1, t \in (0, T), \tag{2}$$

where $\gamma$ is the rate of depreciation and the over dot on $x(t)$ denotes differentiation with respect to time $t$. We assume the absence of free education and that the fees charged by an institution are partly constant and partly varies as the capital stock as school fees is related to capital stock [14]. We take the population of students in the institution as an exogenous variable. This is because the population of students is affected by uncontrollable factors such as academic performance and wastage (e.g. voluntary withdrawal, financial insolvency, medical challenges, death, etc.). We express the revenue, $R(x(t))$, accruing to the institution as

$$R(x(t)) = \bar{N}(\phi_1 + \phi_2 x(t)), \phi_1, \phi_2 > 0, \tag{3}$$

where $\bar{N}$ is the population of students and the bar indicates that it is an exogenous variable. The population $\bar{N}$ may be obtained using either the fractional flow model [15, 16] or the imbedded Markov chain model [17]. In either case, $\bar{N}$ is the sum of the expected enrolment structure of the system. We express the value of the replenishment as

$$V(u(t)) = cu^{\alpha}(t), c > 0, \alpha > 1, \tag{4}$$

where $V(u(t))$ is the money's worth of replenishment $u(t)$, $c$ is the cost of replenishment and $\alpha$ models the diseconomies when scaling up the rate of replenishment [4]. As mentioned earlier, we adopt the stream of profit accruing to the institution during the planning period as a measure of performance. Inflation is not considered within the profit function so that no discounting factor is required. The institution is assumed to maximize the stream of profit over the fixed planning period based on the control. In addition, we assume that the salvage value of the capital stock is negligible at the terminal year, $T$. This assumption is in line with the inscription on facilities, which prevents them from being sold. In the light of the foregoing, we formulate an optimal control problem (OCP) for the institution as follows.

Objective function:

$$\max_{u(t)} \int_0^T \left(\bar{N}(\phi_1 + \phi_2 x(t)) - cu^{\alpha}(t)\right) dt, \tag{5}$$

subject to

- the state equation: $\dot{x}(t) = u(t)B - \gamma x(t)$, $t \in (0, T)$,
- the initial condition: $x(0) = x_0$,
- the control constraint: $0 < u(t) < 1$, $t \in (0, T)$,
- the endpoint constraint: $x(T) = B$.

The task is to obtain optimal values for the rate of replenishment and the capital stock. We use the notation, *opt*, to denote an optimal value.

## 3. MODEL SOLUTION

This section contains propositions derived from solutions to the OCP for two cases: the variable replenishment plan and the fixed replenishment plan. The propositions provide useful insights on the replenishment plans.

**Proposition 1:** The optimal trajectory under a variable gradual replenishment plan is a pair $(x^{opt}(t), u^{opt}(t))$, with

$$u^{opt}(t) = \frac{\alpha\gamma}{\alpha - 1}\left(1 - \frac{x_0}{B}\exp(-\gamma T)\right)\left(\frac{1 - \exp(\gamma(t-T))}{(1 - \exp(-\gamma T))^{\alpha}}\right)^{\frac{1}{\alpha-1}}, \tag{6}$$

and

$$x^{opt}(t) = \exp(-\gamma t)\left(x_0 + (B\exp(\gamma T) - x_0)(1 - \Gamma)\right), \tag{7}$$

where $\Gamma = \left(\frac{1-\exp(\gamma(t-T))}{(1-\exp(-\gamma T))}\right)^{\frac{\alpha}{\alpha-1}}$.

**Proof:** Consider the Hamilton-Pontryagin equation (or simply the Hamiltonian) for the OCP

$$H(x(t), u(t), \lambda(t)) = \left(\bar{N}(\phi_1 + \phi_2 x(t)) - cu^{\alpha}(t)\right) + \lambda(t)\left(u(t)B - \gamma x(t)\right), \tag{8}$$

where $\lambda(t)$ is the marginal valuation of the capital stock at moment $t$. The optimality conditions for the OCP are

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -\phi_2 \bar{N} + \gamma\lambda, \tag{9}$$

$$\lambda(T) = 0, \tag{10}$$

$$\frac{\partial H}{\partial u} = -c\alpha u^{\alpha-1} + \lambda B = 0, \tag{11}$$

$$\dot{x} = \frac{\partial H}{\partial \lambda} = uB - \gamma x. \tag{12}$$

We have suppressed the argument(s) in the functions for convenience. It is worth noting that $\lambda(T) = 0$ because the salvage value of capital stock is assume to be negligible. Eq. (9) and Eq. (10) yield

$$\lambda = \frac{\phi_2 \bar{N}}{\gamma} \left(1 - \exp(\gamma(t - T))\right). \tag{13}$$

Substituting Eq. (13) into Eq. (11) and the result into Eq. (12) with reference to the initial condition and the endpoint constraint, we obtain Eq. (6) and Eq. (7). This completes the proof.

Eq. (6) is the formula for the variable replenishment plan. Note that $u^{opt}(t) = 0$ occurs only at the terminal moment $t = T$. $u^{opt}(T) = 0$ means that no supply is made at moment $t = T$. Thus $u^{opt}(T) = 0$ is enough motivation for the management of the institution to follow the optimal solution, as $x(T) = B$ at the terminal moment. Suppose $u^{opt}(t) \geq 1$ for at least one $t \in [0, T]$. Then the supply should be instantaneous. This is a contradiction as we have assumed a gradual replenishment plan. To circumvent this, the control should be chosen such that the control constraint is not violated. In this light, we propose a fixed rate of replenishment, say $u^*$, such that $0 < u^* < 1 \forall t \in (0, T)$.

**Proposition 2:** Under a fixed replenishment plan, the control is chosen such that it lies in the open interval $(\gamma, 1)$.

**Proof:** Under a fixed replenishment plan, the control is a constant, say $u^*$. Thus the state equation becomes

$$\dot{x}(t) = u^* B - \gamma x(t). \tag{14}$$

Using the initial condition $x(0) = x_0$ and the constraint $x(T) = B$, $u^*$ is found to be

$$u^* = \frac{\gamma \left(1 - \frac{x_0}{B} \exp(-\gamma T)\right)}{1 - \exp(-\gamma T)}. \tag{15}$$

Since $x_0 < B$, then

$$1 - \frac{x_0}{B} \exp(-\gamma T) > 1 - \exp(-\gamma T).$$

It follows that

$$\frac{\gamma\left(1-\frac{x_0}{B}\exp(-\gamma T)\right)}{1-\exp(-\gamma T)} > \gamma, \text{ as } \gamma > 0.$$

From the control constraint, $u^* < 1$. This completes the proof.

It should be noted that there is no guarantee that the fixed replenishment rate, $u^*$, is optimal. The pattern of the capital stock corresponding to $u^*$ is obtained from the state equation as

$$x^*(t) = \frac{u^* B}{\gamma}(1-\exp(-\gamma t)) + x_0\exp(-\gamma t). \tag{16}$$

## 4. ILLUSTRATIVE EXAMPLES

We take a 'snapshot' into the pattern of the optimal solution for a ten-year planning period using the examples below. We carry out all our computations and graphical displays in the MATLAB environment.

**Example 1:** Let $\gamma = 0.02$, $x(0) = 2$ billion (in monetary terms), $x(10) = 15$ billion (in monetary terms) for a facility. We compute $u^{opt}(t)$ and $x^{opt}(t)$ for $\alpha = 2, 3, 4, 5$. The results are depicted in Fig. 1 below with subplots **a** and **b**. The subplot **a** shows the dynam-

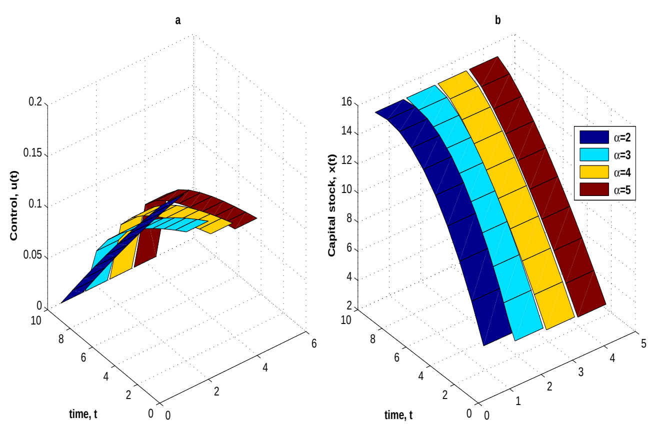


**Fig. 1.** The dynamics of replenishment and capital stock for a ten-year period.

ics of the optimal rate of replenishment for the facility at moment $t \in [0, 10]$, and the subplot **b** shows the corresponding pattern of the optimal capital stock given the rate of replenishment for the facility at moment $t \in [0, 10]$. From Fig. 1, the rate of replenishment decreases gradually with time, whereas the capital stock rises steadily with time. The rise in capital stock over time indicates an increase in money's worth of capital towards the desired state, and the decrease in the rate of replenishment indicates a reduction in investment as the time period elapses. Hence, the institution enjoys economies of scale towards the end of the planning interval.

**Example 2:** Let $\gamma = 0.7$, $x(0) = 2$ billion (in monetary terms), $x(10) = 15$ billion (in monetary terms). We compute $u^{opt}(t)$, $u^*$, and $x^{opt}(t)$, $x^*(t)$, respectively, for $\alpha = 2$. The results are depicted in the subplots **a** and **b** of Fig. 2. Fig. 2 shows that the optimum rate of replenishment, $u^{opt}(t)$, violates the control constraint under a variable replenishment plan for the first six years (see subplot **a**) and that $x^{opt}(t) > 15$ for $t \in [0, 10)$ (see subplot **b**). The practical implications of these are that some of the stock will be idle and that the institution incurs additional cost which is associated with the storage of the excess stock. Nonetheless, this problem is circumvented by choosing $u^*$ such that $\gamma < u^* < 1$, say $u^* = 0.7006$. The subplot **b** of Fig. 2 shows that with $u^* = 0.7006$, the capital stock rises gradually until the desired stock is attained at the terminal year.

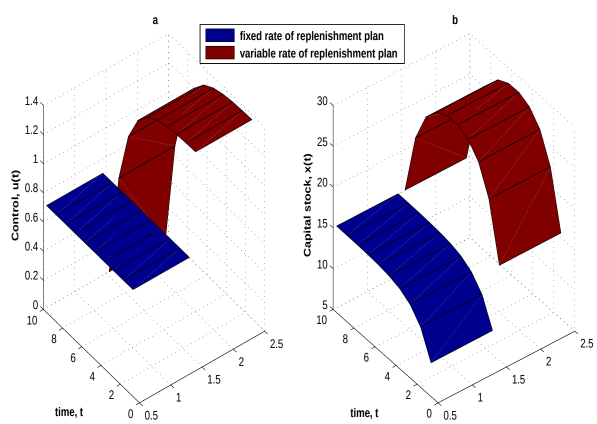


**Fig. 2.** Replenishment plan under fixed and variable rates.

## 5. CONCLUDING REMARKS

In this study, an attempt has been made to find a replenishment plan which is capable of achieving a desired capital stock over a fixed planning period. The task was to formulate an OCP. The educational system under the policy of autonomy was considered. A gradual replenishment plan in which the rate of replenishment is either fixed or variable is developed. The study proposes a fixed replenishment plan whenever the control constraint lies outside its bounds when the variable replenishment plan is implemented. In either the fixed or variable plan, the desired stock is attained at the terminal moment, $T$, of the planning period. The practical challenge of implementing the model proposed in this paper in an institution may include bottlenecks associated with the budget constraint of the institution. This challenge coupled with the relaxation of some of the assumptions in this study are areas for future research.

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