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ON HOLOMORPHY OF FENYVES BCI-ALGEBRAS

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ABSTRACT. Fenyves BCI-algebras are BCI-algebras that satisfy the Bol-Moufang identities. In this paper, the holomorphy of BCI-algebras are studied. It is shown that whenever a loop and its holomorph are BCI-algebras, the former is p-semisimple if and only if the latter is p-semisimple. Whenever a loop and its holomorph are BCI-algebras, it is established that the former is a BCK-algebra if and only if the latter has a BCK-subalgebra. Moreover, the holomorphy of the associative and some non-associative Fenyves BCI-algebras are also studied.

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1. INTRODUCTION

BCK-algebras and BCI-algebras are abreviated to two B-algebras. They were introduced by Imai and Iseki [16]. The two algebras originated from two different sources. One of the motivations is based on set theory. In set theory, there are three most elementary and fundamental operations. They are the union, intersection, and set difference. If we consider those three operations and their properties, then as a generalization of them, we have the notion of Boolean algebras. If we take both the union and intersection, then as a general algebra, the notion of distributive lattices is obtained. Moreover, if we consider the union or the intersection alone, we have the notion of upper semilattices or lower semilattices. However, the set difference together with its properties had not been considered systematically before Imai and Iseki. Jaiyéọlá et al. [15] introduced new kinds of BCI-algebras known as Fenyves BCI-algebras. In this paper, we will study the holomorphy of the classical BCI-algebras as well as the holomorphy of Fenyves BCI-algebras.

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2. PRELIMINARY

Definition 1([28]**):** A triple $(X;*,0)$ is called a BCI-algebra if the following conditions are satisfied for any $x,y,z\in X$

(1) $((x*y)*(x*z))*(z*y)=0$;
(2) $x*0=x$;
(3) $x*y=0$ and $y*x=0 \Longrightarrow x=y$.

We call the binary operation $*$ on X the multiplication on X, and the constant 0 of X the zero element of X. We often write X instead of $(X;*,0)$ for a BCI-algebra in brevity.

Example 1([28]**):** Let S be a set. Denote 2^S for the power set of S in the sense that 2^S is the collection of all subsets of S, $-$ the set difference and $\emptyset$ for the empty set. Then, $(2^S;-,\emptyset)$ is a BCI-algebra.

Example 2([28]**):** Suppose $(G;\cdot,e)$ is an abelian group with e as the identity element. Define a binary operation $*$ on G by $x*y=xy^{-1}$. Then $(G,*,e)$ is a BCI-algebra.

Example 3([28]**):** Let $\mathbb{Z}$ be the set of integers. Then, $(\mathbb{Z},-,0)$ is a BCI-algebra.

The following theorem gives necessary and sufficient conditions for the existence of a BCI-algebra.

Theorem 1([28]**):** Let X be a non-empty set,$*$ a binary operation on X and 0 a constant element of X. Then, $(X,*,0)$ is a BCI-algebra if and only if the following conditions hold:

(1) $((x*y)*(x*z))*(z*y)=0$;
(2) $(x*(x*y)*y=0$;
(3) $x*x=0$;
(4) $x*y=0$ and $y*x=0$ imply $x=y$.

3. THE HEART OF THE MATTER

Definition 2: A BCI- algebra $(X,*,0)$ is called a Fenyves BCI-algebra if it satisfies any of the identities of Bol-Moufang type.
The identities of Bol-Moufang type are given below:

F_1**::** $xy\cdot zx=(xy\cdot z)x$
F_2**::** $xy\cdot zx=(x\cdot yz)x$ (Moufang identity)
F_3**::** $xy\cdot zx=x(y\cdot zx)$
F_4**::** $xy\cdot zx=x(yz\cdot x)$ (Moufang identity)
F_5**::** $(xy\cdot z)x=(x\cdot yz)x$
F_6**::** $(xy\cdot z)x=x(y\cdot zx)$ (extra identity)
F_7**::** $(xy\cdot z)x=x(yz\cdot x)$
F_8**::** $(x\cdot yz)x=x(y\cdot zx)$
F_9**::** $(x\cdot yz)x=x(yz\cdot x)$
F_{10}**::** $x(y\cdot zx)=x(yz\cdot x)$
F_{11}**::** $xy\cdot xz=(xy\cdot x)z$

F_{12}:: $xy \cdot xz = (x \cdot yx)z$
F_{13}:: $xy \cdot xz = x(yx \cdot z)$ (extra identity)
F_{14}:: $xy \cdot xz = x(y \cdot xz)$
F_{15}:: $(xy \cdot x)z = (x \cdot yx)z$
F_{16}:: $(xy \cdot x)z = x(yx \cdot z)$
F_{17}:: $(xy \cdot x)z = x(y \cdot xz)$ (Moufang identity)
F_{18}:: $(x \cdot yx)z = x(yx \cdot z)$
F_{19}:: $(x \cdot yx)z = x(y \cdot xz)$ (left Bol identity)
F_{20}:: $x(yx \cdot z) = x(y \cdot xz)$
F_{21}:: $yx \cdot zx = (yx \cdot z)x$
F_{22}:: $yx \cdot zx = (y \cdot xz)x$ (extra identity)
F_{23}:: $yx \cdot zx = y(xz \cdot x)$
F_{24}:: $yx \cdot zx = y(x \cdot zx)$
F_{25}:: $(yx \cdot z)x = (y \cdot xz)x$
F_{26}:: $(yx \cdot z)x = y(xz \cdot x)$ (right Bol identity)
F_{27}:: $(yx \cdot z)x = y(x \cdot zx)$ (Moufang identity)
F_{28}:: $(y \cdot xz)x = y(xz \cdot x)$
F_{29}:: $(y \cdot xz)x = y(x \cdot zx)$
F_{30}:: $y(xz \cdot x) = y(x \cdot zx)$
F_{31}:: $yx \cdot xz = (yx \cdot x)z$
F_{32}:: $yx \cdot xz = (y \cdot xx)z$
F_{33}:: $yx \cdot xz = y(xx \cdot z)$
F_{34}:: $yx \cdot xz = y(x \cdot xz)$
F_{35}:: $(yx \cdot x)z = (y \cdot xx)z$
F_{36}:: $(yx \cdot x)z = y(xx \cdot z)$ (RC identity)
F_{37}:: $(yx \cdot x)z = y(x \cdot xz)$ (C identity)
F_{38}:: $(y \cdot xx)z = y(xx \cdot z)$
F_{39}:: $(y \cdot xx)z = y(x \cdot xz)$ (LC identity)
F_{40}:: $y(xx \cdot z) = y(x \cdot xz)$
F_{41}:: $xx \cdot yz = (x \cdot xy)z$ (LC identity)
F_{42}:: $xx \cdot yz = (xx \cdot y)z$
F_{43}:: $xx \cdot yz = x(x \cdot yz)$
F_{44}:: $xx \cdot yz = x(xy \cdot z)$
F_{45}:: $(x \cdot xy)z = (xx \cdot y)z$
F_{46}:: $(x \cdot xy)z = x(x \cdot yz)$ (LC identity)
F_{47}:: $(x \cdot xy)z = x(xy \cdot z)$
F_{48}:: $(xx \cdot y)z = x(x \cdot yz)$ (LC identity)
F_{49}:: $(xx \cdot y)z = x(xy \cdot z)$
F_{50}:: $x(x \cdot yz) = x(xy \cdot z)$
F_{51}:: $yz \cdot xx = (yz \cdot x)x$
F_{52}:: $yz \cdot xx = (y \cdot zx)x$
F_{53}:: $yz \cdot xx = y(zx \cdot x)$ (RC identity)
F_{54}:: $yz \cdot xx = y(z \cdot xx)$
F_{55}:: $(yz \cdot x)x = (y \cdot zx)x$
F_{56}:: $(yz \cdot x)x = y(zx \cdot x)$ (RC identity)
F_{57}:: $(yz \cdot x)x = y(z \cdot xx)$ (RC identity)
F_{58}:: $(y \cdot zx)x = y(zx \cdot x)$
F_{59}:: $(y \cdot zx)x = y(z \cdot xx)$
F_{60}:: $y(zx \cdot x) = y(z \cdot xx)$

Consequent upon this definition, there are sixty varieties of Fenyves BCI -algebras. We give some varieties of Fenyves BCI-algebras as follows:
Definition 3: A BCI-algebra $(X, *, 0)$ is called an F_1-algebra if it satisfies the following condition: $xy * zx = (xy * z) * x \; \forall \; x, y, z \in X$.
Definition 4: A BCI-algebra $(X, *, 0)$ is called an F_5- algebra if it satisfies the following condition: $(xy * z)x = (x * yz)x \; \forall \; x, y, z \in X$.
Definition 5: A BCI-algebra $(X, *, 0)$ is called an F_{47}-algebra if it satisfies the following condition: $(x * xy)z = x(xy * z) \; \forall \; x, y, z \in X$.

Let us now give some examples of a Fenyves' BCI-algebra:

Example 4: Let $(G; \cdot, e)$ be an abelian group with e as the identity element. Define a binary operation $*$ on G by $x * y = xy^{-1}$. Then $(G, *, e)$ is an F_8-algebra, F_{19}-algebra, F_{29}-algebra, F_{39}-algebra, F_{46}-algebra, F_{52}-algebra, F_{54}-algebra, F_{59}-algebra.

Example 5: Let S be a set. Denote 2^S for the power set of S in the sense that 2^S is the collection of all subsets of S, $-$ the set difference and $\emptyset$ for the empty set. Then $(2^S; -, \emptyset)$ is an F_5-algebra, F_{42}-algebra, F_{54}-algebra.

Definition 6([28]): A BCI-algebra $(X, *, 0)$ is called associative if $(x * y) * z = x * (y * z)\ \forall\ x, y, z \in X$.

Definition 7([28]): A BCI-algebra $(X, *, 0)$ is called p-semisimple if $0 * (0 * x) = x\ \forall\ x \in X$.

The following theorems give equivalent conditions for associativity and p-semisimplicity in a BCI-algebra:

Theorem 2 ([28]): Given a BCI-algebra X, the following are equivalent:

(1) X is associative.
(2) $0 * x = x\ \forall\ x \in X$.
(3) $x * y = y * x\ \forall\ x, y \in X$.

Theorem 3 ([28]): Let X be a BCI-algebra. Then the following conditions are equivalent for any $x, y, z, u \in X$:

(1) X is p-semisimple.
(2) $(x * y) * (z * u) = (x * z) * (y * u)$.
(3) $0 * (y * x) = x * y$.
(4) $(x * y) * (x * z) = z * y$.
(5) $z * x = z * y$ implies $x = y$. (the left cancellation law)
(6) $x * y = 0$ implies $x = y$.

Theorem 4 ([28]): Let X be a BCI-algebra. X is p-semisimple if and only if one of the following conditions holds for any $x, y, z \in X$:

(1) $x * z = y * z$ implies $x = y$. (the right cancellation law)
(2) $(y * x) * (z * x) = y * z$.
(3) $(x * y) * (x * z) = 0 * (y * z)$.

Definition 8 (Jaiyéọlá [29]): Let L be a non-empty set. Define a binary operation $(\cdot)$ on L . If $x \cdot y \in L$ for all $x, y \in L$, $(L, \cdot)$ is called a groupoid. If the equations:

$$a \cdot x = b \qquad \text{and} \qquad y \cdot a = b$$

have unique solutions for x and y respectively, then $(L, \cdot)$ is called a quasigroup. If there exists a unique element $e \in L$ called the identity element such that for all $x \in L$, $x \cdot e = e \cdot x = x$, $(L, \cdot)$ is called a loop.

Theorem 5 ([28]**):** Suppose that $(X; *, 0)$ is a BCI-algebra. X is associative if and only if X is p-semisimple and X is quasi-associative.

Theorem 6 ([28]**):** Suppose that $(X; *, 0)$ is a BCI-algebra. Then $(x * y) * z = (x * z) * y$ for all $x, y, z \in X$.

Remark 1: In Theorem 5, quasi-associativity in BCI-algebra plays a similar role which weak associativity (i.e. the F_i identities) plays in quasigroup and loop theory.

Definition 9([28]**):** A BCI-algebra $(X; *, 0)$ is called a BCK-algebra if $0 * x = 0$ for all $x \in X$.

Since late 1970s, much attention has been paid to the study of BCI and BCK algebras. In particular, the participation in the research of polish mathematicians Tadeusz Traczyk and Andrzej Wronski as well as Australian mathematician William H. Cornish, etc. has made the study of BCI-algebras to attract interest of many mathematicians. Many interesting and important results are discovered continuously. Now, the theory of BCI-algebras has widely spread to areas such as general theory which include congruences, quotient algebras, BCI-Homomorphisms, direct sums and direct products, commutative BCK-algebras, positive implicative and implicative BCK-algebras, derivations of BCI-algebras, and ideal theory of BCI-algebras ([16, 17, 14, 22, 27]).

Jaiyéọlá *et al* [15] looked at Fenyves identities on the platform of BCI-algebras. They classified the Fenyves BCI-algebras into 46 associative and 14 non-associative types and showed that some Fenyves identities played the role of quasi-associativity, vis-a-vis Theorem in BCI-algebras. They clarified the relationship between a BCI-algebra, a quasigroup and a loop. Some of their results are stated below.

Theorem 7 ([15]**):**

1. A BCI algebra X is a quasigroup if and only if it is p-semisimple.
2. A BCI algebra X is a loop if and only if it is associative.
3. An associative BCI algebra X is a Boolean group.

Theorem 8 [15]**:** Let $(X; *, 0)$ be a BCI-algebra. If X is any of the following Fenyves BCI-algebras, then X is associative.

(1) F_1-algebra	(3) F_4-algebra	(5) F_7-algebra
(2) F_2-algebra	(4) F_6-algebra	(6) F_9-algebra

(7) F_{10}-algebra
(8) F_{11}-algebra
(9) F_{12}-algebra
(10) F_{13}-algebra
(11) F_{14}-algebra
(12) F_{15}-algebra
(13) F_{16}-algebra
(14) F_{17}-algebra
(15) F_{18}-algebra
(16) F_{20}-algebra
(17) F_{22}-algebra
(18) F_{23}-algebra
(19) F_{24}-algebra
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(40) F_{49}-algebra
(41) F_{50}-algebra
(42) F_{51}-algebra
(43) F_{53}-algebra
(44) F_{57}-algebra
(45) F_{58}-algebra
(46) F_{60}-algebra

Remark 2: All other F_i's which are not mentioned in Theorem 8 were found to be non-associative. Every BCI-algebra is naturally an F_{54} BCI-algebra.

Definition 10 (Jaiyéọlá [29]**):**(Holomorph). Let $(Q,\cdot)$ be a groupoid (quasigroup, loop) and $A(Q) \leq AUM(Q,\cdot)$ be a boolean group of automorphisms of the groupoid (quasigroup, loop) $(Q,\cdot)$. Let $H = A(Q) \times Q$. Define $\circ$ on H as

$$(\alpha, x) \circ (\beta, y) = (\alpha\beta, x\beta \cdot y) \text{ for all } (\alpha, x), (\beta, y) \in H.$$

$(H, \circ)$ is a groupoid (quasigroup, loop) and is called the A-holomorph of $(Q,\cdot)$.

Definition 11 (Jaiyéọlá [29]**):**(λ-regular and ρ-regular bijections). A bijection U of a groupoid $(G,\cdot)$ is called λ-regular if there exists an autotopism (U, I, U) of $(G,\cdot)$ such that $xU \cdot y = (x \cdot y)U\ \forall\ x, y \in G$.

A bijection U of a groupoid $(G,\cdot)$ is called ρ-regular if there exists an autotopism (I, U, U) of $(G,\cdot)$ such that $x \cdot yU = (x \cdot y)U\ \forall\ x, y \in G$.

The holomorph of a loop is a loop according to Bruck [4]. Since then, the concept of holomorphy of loops has caught the attention of some researchers. Interestingly, Adéníran [1] and Robinson [25], Chein and Robinson[9], Adéníran *et al* [2], Chiboka and Solarin [10], [12], Bruck [4], Bruck and Paige [5], Robinson [24], Huthnance [26], Adéníran et tal [3] and, Jaiyéọlá and Popoola [20] have respectively studied the holomorphic structures of Bol/Bruck loops, Moufang loops, central loops, conjugacy closed loops, inverse property loops, A-loops, extra loops, weak inverse property loops and generalized

Bol loops. Isere *et al* [18] looked at the holomorphic structure of Osborn loops.

In this present work, the holomorphy of BCI-algebras are studied. The holomorphy of both the associative and non-associative Fenyves BCI-algebras are also studied.

4. MAIN RESULTS

4.1. REGULAR BIJECTIONS AND AUTOMORPHISMS OF BCI-ALGEBRAS

We first present a result on regular bijections and automorphisms of BCI-algebras.

Lemma 1: Let $(G,*)$ be a BCI-algebra with $\delta \in \mathrm{SYM}(G,*)$. Then the following hold:

(i) δ is λ-regular $\Leftrightarrow \delta R_x = R_x\delta \Leftrightarrow L_{x\delta} = L_x\delta$ for all $x \in G$.
(ii) δ is ρ-regular $\Leftrightarrow \delta L_x = L_x\delta \Leftrightarrow R_{x\delta} = R_x\delta$ for all $x \in G$.
(iii) δ is μ-regular $\Leftrightarrow \delta R_x = R_{x\delta} \Leftrightarrow L_{x\delta} = \delta L_x$ for all $x \in G$.

Proof:

(i) δ is λ-regular $\Leftrightarrow (\delta, I, \delta) \in$ AUT $(G,*) \Leftrightarrow y\delta * xI = (y * x)\delta \Leftrightarrow y\delta R_x = yR_x\delta \Leftrightarrow \delta R_x = R_x\delta \Leftrightarrow y\delta R_x = yR_x\delta \Leftrightarrow y\delta * x = (y * x)\delta \Leftrightarrow xL_{y\delta} = xL_y\delta \Leftrightarrow L_{y\delta} = L_y\delta$.
(ii) δ is ρ-regular $\Leftrightarrow (I, \delta, \delta) \in$ AUT $(G,*) \Leftrightarrow xI * y\delta = (x * y)\delta \Leftrightarrow y\delta L_x = yL_x\delta \Leftrightarrow \delta L_x = L_x\delta \Leftrightarrow y\delta L_x = yL_x\delta \Leftrightarrow x * y\delta = (x * y)\delta \Leftrightarrow xR_{y\delta} = xR_y\delta \Leftrightarrow R_{y\delta} = R_y\delta$.
(iii) δ is μ-regular with adjoint $\delta' = \delta \Leftrightarrow (\delta, \delta'^{-1}, I) \in$ AUT $(G,*) \Leftrightarrow x\delta * y\delta'^{-1} = (x * y)I \Leftrightarrow x\delta * y\delta\delta'^{-1} = x * y\delta$ (by replacing y by $y\delta$) $\Leftrightarrow x\delta * y = x * y\delta \Leftrightarrow x\delta R_y = xR_{y\delta} \Leftrightarrow \delta R_y = R_{y\delta} \Leftrightarrow x\delta R_y = xR_{y\delta} \Leftrightarrow x\delta * y = x * y\delta \Leftrightarrow yL_{x\delta} = y\delta L_x \Leftrightarrow L_{x\delta} = \delta L_x$.

Lemma 2: Let A be an automorphism on a BCI-algebra $(G,\cdot)$. Then the following hold for all $x, y \in G$:

(i) $R_yA = AR_{yA}$
(ii) $L_xA = AL_{xA}$

Proof: Suppose A is an automorphism on $(G,\cdot)$. Then $(x \cdot y)A = xA \cdot yA$ for all $x, y \in G$.

(i) Fixing x, we have $xR_yA = xAR_{yA} \Rightarrow R_yA = AR_{yA}$.
(ii) Fixing y, we have $yL_xA = yAL_{xA} \Rightarrow L_xA = AL_{xA}$.

4.2. HOLOMORPHY OF BCI-ALGEBRAS

We now present results on holomorphy of BCI-algebras.

Theorem 9: Let $(G,\cdot,0)$ be a BCI-algebra, and let $(H,\circ)$ be the A-holomorph of $(G,\cdot)$. $(H,\circ)$ is a BCI-algebra if and only if $[(x\delta\cdot y\delta)\cdot(x\cdot z\gamma)]\cdot(z\cdot y)=0$ for all $x,y,z\in G,\delta,\gamma\in A(G)$.

Proof: We shall be using Definition 1. Now, $(I,0)\in H$ Consider $(\alpha,x)\circ(I,0)=(\alpha I,xI\cdot 0)=(\alpha,x)$. So, $(I,0)$ is the zero element of H.

Now, let $(\alpha,x),(\beta,y)\in H$. Suppose $(\alpha,x)\circ(\beta,y)=(I,0)$ and $(\beta,y)\circ(\alpha,x)=(I,0)$. Then $(\alpha\beta,x\beta\cdot y)=(I,0)$ and $(\beta\alpha,y\alpha\cdot x)=(I,0)\Leftrightarrow \alpha\beta=I,x\beta\cdot y=0$ and $\beta\alpha=I,y\alpha\cdot x=0\Leftrightarrow\alpha=\beta^{-1},x\beta\cdot y=0$ and $\beta=\alpha^{-1},y\alpha\cdot x=0\Leftrightarrow\alpha=\beta,x\beta\cdot y=0$ and $y\alpha\cdot x=0\Leftrightarrow x\alpha\cdot y=0$ and $y\alpha\cdot x=0$. Replace x with $x\alpha^{-1}$ in $x\alpha\cdot y=0$ and y with $y\alpha^{-1}$ in $y\alpha\cdot x=0$ to get $x\cdot y=0$ and $y\cdot x=0\Rightarrow x=y\Rightarrow(\alpha,x)=(\beta,y)$.

Now, let $(\alpha,x),(\beta,y),(\gamma,z)\in H$. Consider $\Big\{\big[(\alpha,x)\circ(\beta,y)\big]\circ\big[(\alpha,x)\circ(\gamma,z)\big]\Big\}\circ\big[(\gamma,z)\circ(\beta,y)\big]=\Big[(\alpha\beta,x\beta\cdot y)\circ(\alpha\gamma,x\gamma\cdot z)\Big]\circ\big(\gamma\beta,z\beta\cdot y\big)=\Big[(\alpha\beta\alpha\gamma,(x\beta\cdot y)\alpha\gamma\cdot(x\gamma\cdot z))\Big]\cdot\big(\gamma\beta,z\beta\cdot y\big)=\Big[\alpha\beta\alpha\gamma\gamma\beta,((x\beta\cdot y)\alpha\gamma\cdot(x\gamma\cdot z))\gamma\beta\cdot(z\beta\cdot y)\Big]$ (as α,β,γ are homomorphisms)$=[I,((x\beta\alpha\gamma\cdot y\alpha\gamma)\cdot(x\gamma\cdot z)\gamma\beta\cdot(z\beta\cdot y)]=\Big(I,((x\alpha\cdot y\alpha\beta)\cdot(x\beta\cdot z\gamma\beta))\cdot(z\beta\cdot y)\Big)=(I,0)\Leftrightarrow\Big[(x\alpha\cdot y\alpha\beta)\cdot(x\beta\cdot z\beta\gamma)\Big]\cdot(z\beta\cdot y)=0$. Replace z with $z\beta^{-1}$, x with $x\beta^{-1}$ to get $\Big[(x\alpha\beta\cdot y\alpha\beta)\cdot(x\cdot z\gamma)\Big]\cdot(z\cdot y)=0$ (as $\alpha\beta=\beta\alpha$ for Boolean group) $\Leftrightarrow\big[(x\delta\cdot y\delta)\cdot(x\cdot z\gamma)\big]\cdot(z\cdot y)=0$ (where $\delta=\alpha\beta$).

Conversely, Suppose $\big[(x\delta\cdot y\delta)\cdot(x\cdot z\gamma)\big]\cdot(z\cdot y)=0$ for all $x,y,z\in G,\delta,\gamma\in A(G)$. Doing the reverse on $\big[(x\alpha\beta\cdot y\alpha\beta)\cdot(x\cdot z\gamma)\big]\cdot(z\cdot y)$ in the equation $\Big(I,((x\alpha\cdot y\alpha\beta)\cdot(x\beta\cdot z\gamma\beta))\cdot(z\beta\cdot y)\Big)=(I,0)$ we have $\Big(\alpha\beta\alpha\gamma\gamma\beta,\Big[(x\alpha\cdot y\alpha\beta)\cdot(x\beta\cdot z\beta\gamma)\Big]\cdot(z\beta\cdot y)\Big)=(I,0)$ which subsequently leads to $\Big\{[(\alpha,x)\circ(\beta,y)]\circ[(\alpha,x)\circ(\gamma,z)]\Big\}\circ\big[(\gamma,z)\circ(\beta,y)\big]=(I,0)$. Hence, $(H,\circ)$ is a BCI-algebra.

Theorem 10: Let $(G,\cdot)$ be a groupoid with A-holomorph $(H,\circ)$. $(H,\circ)$ is a BCI-algebra if and only if $(G,\cdot)$ is a BCI-algebra and $[(x\delta\cdot y\delta)\cdot(x\cdot z\gamma)]\cdot(z\cdot y)=0$ for all $x,y,z\in G,\delta,\gamma\in A(G)$.

Proof: Suppose $(H,\circ)$ is a BCI-algebra. The set $S=\{(I,x):x\in G\}$ is a subalgebra of $(H,\circ)$. Consider the function $f:(G,\cdot)\to S$ defined by $f(x)=(I,x)\forall x\in G$. We claim that f is an isomorphism.

Now, let $(I,x),(I,y)\in S$. Then $(I,x)\circ(I,y)=(II,xI\cdot y)=(I,x\cdot y)$.

Let $x,y\in G$. Consider $f(x\cdot y)=(I,x\cdot y)=(II,xI\cdot y)=(I,x)\circ(I,y)=f(x)\circ f(y)$. So, f is a homomorphism.

Now, let $x,y\in G\ni f(x)=f(y)\Rightarrow(I,x)=(I,y)\Rightarrow x=y$. So, f is one to one. Clearly, by definition, f is onto. So, f is an isomorphism.

Now, since $(H,\circ)$ is a BCI-algebra, its subalgebra S is also a BCI-algebra, and by the isomorphism, $(G,\cdot)$ is a BCI-algebra. By Theorem 9, we have $[(x\delta\cdot y\delta)\cdot(x\cdot z\gamma)]\cdot(z\cdot y)=0$.

Conversely, suppose $(G,\cdot)$ is a BCI-algebra and $[(x\delta\cdot y\delta)\cdot(x\cdot z\gamma)]\cdot(z\cdot y)=0$, then by Theorem 9, $(H,\circ)$ is a BCI-algebra.

Theorem 11: Let $(G,\cdot)$ be a BCI-algebra with an A-holomorph $(H,\circ)$ which is also a BCI-algebra. $(H,\circ)$ is p-semisimple if and only if $(G,\cdot)$ is p-semisimple.

Proof: $(H,\circ)$ is p-semisimple $\Leftrightarrow(I,0)\circ[(I,0)\circ(\alpha,x)]=(\alpha,x)\Leftrightarrow(I,0)\circ[(I,0)\circ(\alpha,x)]=(\alpha,x)\Leftrightarrow(I,0)\circ[(I\alpha,0\alpha\cdot x)]=(\alpha,x)\Leftrightarrow(I,0)\circ(\alpha,0\cdot x)=(\alpha,x)\Leftrightarrow(I\alpha,0\alpha\cdot 0x)=(\alpha,x)\Leftrightarrow(\alpha,0\cdot(0\cdot x))=(\alpha,x)\Leftrightarrow 0\cdot(0\cdot x)=x\Leftrightarrow(G,\cdot)$ is p-semisimple.

Corollary 1: Let $(G,\cdot)$ be a groupoid with A-holomorph $(H,\circ)$. $(H,\circ)$ is a p-semisimple BCI-algebra if and only if $(G,\cdot)$ is a p-semisimple BCI-algebra and $[(x\delta\cdot y\delta)\cdot(x\cdot z\gamma)]\cdot(z\cdot y)=0$.

Proof: Follows from Theorem 10 and Theorem 11.

Theorem 12: Let $(G,\cdot)$ be a BCI-algebra with A-holomorph $(H,\circ)$ which is also a BCI-algebra. $(G,\cdot)$ is a BCK-algebra if and only if $\{(I,x):x\in G\}$ is a BCK-algebra.

Proof: Now, $\{(I,x):x\in G\}$ is BCK $\Leftrightarrow(I,0)\circ(I,x)=(I,0)\Leftrightarrow(II,0I\cdot x)=(I,0)\Leftrightarrow(I,0\cdot x)=(I,0)\Leftrightarrow 0\cdot x=0\Leftrightarrow(G,\cdot)$ is BCK.

Corollary 2: Let $(G,\cdot)$ be a groupoid with A-holomorph $(H,\circ)$. $(H,\circ)$ is a BCK-algebra if and only if $(G,\cdot)$ is a BCK-algebra and $[(x\delta\cdot y\delta)\cdot(x\cdot z\gamma)]\cdot(z\cdot y)=0$ for all $x,y,z\in G,\delta,\gamma\in A(G)$.

Proof: Follows from Theorem 11 and Theorem 12.

4.3. HOLOMORPHY OF ASSOCIATIVE FENYVES BCI-ALGEBRAS

We now present results on holomorphy of associative Fenyves BCI-algebras.

Theorem 13: Let $(G,\cdot,0)$ be a BCI-algebra with an A-holomorph $(H,\circ)$ which is also a BCI-algebra. $(H,\circ)$ is associative if and only if $(G,\cdot)$ is associative.

Proof: Now, $(H,\circ)$ is associative if and only if $(I,0)\circ(\alpha,x)=(\alpha,x)\Leftrightarrow(I\alpha,0\alpha\cdot x)=(\alpha,x)\Leftrightarrow 0\cdot x=x\Leftrightarrow(G,\cdot)$ is associative.

Corollary 3: Let $(G,\cdot,0)$ be a BCI-algebra with an A-holomorph $(H,\circ)$ which is also a BCI-algebra. $(H,\circ)$ is an F_i-algebra if and only if $(G,\cdot)$ is an F_i-algebra where $i=1,2,4,6,7,9,10,11,12,13,14,$ $15,16,17,18,20,22,23,24,25,26,27,28,30,31,32,33,34,35,36,37,$ $38,40,41,43,44,45,47,48,49,50,51,53,5758,60.$

Proof: Follows from Theorem 8 and Theorem 13.

Corollary 4: Let $(G,\cdot)$ be a groupoid with A-holomorph $(H,\circ)$. $(H,\circ)$ is an associative BCI-algebra if and only if $(G,\cdot)$ is an associative BCI-algebra and $[(x\delta\cdot y\delta)\cdot(x\cdot z\gamma)]\cdot(z\cdot y)=0$ for all $x,y,z\in G,\delta,\gamma\in A(G)$.

Proof: Follows from Theorem 8, Theorem 10 and Theorem 13.

4.4. HOLOMORPHY OF BCI-ALGEBRAS AND NON-ASSOCIATIVE FENYVES BCI-ALGEBRAS

We now present results on holomorphy of BCI-algebras and non-associative Fenyves BCI-algebras.

Theorem 14: Let $(G,\cdot)$ be a BCI-algebra with A-holomorph $(H,\circ)$ which is also a BCI-algebra such that every automorphism δ on $(G,\cdot)$ is ρ-regular and $|\delta|=2$. Then $(H,\circ)$ is an F_i-algebra if and only if $(G,\cdot)$ is an F_i-algebra; where $i=27,38$.

Proof: Let $(H,\circ)$ be an F_{27}-algebra, and let $(\alpha,x),(\beta,y),(\gamma,z)\in(H,\circ)$. Then $\{[(\beta,y)\circ(\alpha,x)]\circ(\gamma,z)\}\circ(\alpha,x)=(\beta,y)\circ\{(\alpha,x)\circ[(\gamma,z)\circ(\alpha,x)]\}\Leftrightarrow\{[\beta\alpha,y\alpha\cdot x]\circ(\gamma,z)\}\circ(\alpha,x)=(\beta,y)\circ\{(\alpha,x)\circ[\gamma\alpha,z\alpha\cdot x]\}\Leftrightarrow\{\beta\alpha\gamma,(y\alpha\cdot x)\gamma\cdot z\}\circ(\alpha,x)=(\beta,y)\circ\{\alpha\gamma\alpha,x\gamma\alpha\cdot(z\alpha\cdot x)\}\Leftrightarrow\{\beta\alpha\gamma,(y\alpha\gamma\cdot x\gamma)\cdot z\}\cdot(\alpha,x)=(\beta,y)\circ\{\alpha\gamma\alpha,x\gamma\alpha\cdot(z\alpha\cdot x)\}\Leftrightarrow\{\beta\alpha\gamma\alpha,[(y\alpha\gamma\cdot x\gamma)\cdot z]\alpha\cdot x\}=\{\beta\alpha\gamma\alpha,y\alpha\gamma\alpha\cdot[x\gamma\alpha\cdot(z\alpha\cdot x)]\}\Leftrightarrow\{\beta\alpha\gamma\alpha,[(y\alpha\gamma\cdot x\gamma)\alpha\cdot z\alpha]\cdot x\}=\{\beta\alpha\gamma\alpha,y\alpha\gamma\alpha\cdot[x\gamma\alpha\cdot(z\alpha\cdot x)]\}\Leftrightarrow\{\beta\alpha\gamma\alpha,[(y\alpha\gamma\alpha\cdot x\gamma\alpha)\cdot z\alpha]\cdot x\}=\{\beta\alpha\gamma\alpha,y\alpha\gamma\alpha\cdot[x\gamma\alpha\cdot(z\alpha\cdot x)]\}\Leftrightarrow[(y\alpha\gamma\alpha\cdot x\gamma\alpha)\cdot z\alpha]\cdot x=y\alpha\gamma\alpha\cdot[x\gamma\alpha\cdot(z\alpha\cdot x)]$.

Replace z with $z\alpha^{-1}$, y with $y(\alpha\gamma\alpha)^{-1}$ to get $[(y\cdot x\gamma\alpha)\cdot z]\cdot x=y\cdot[x\gamma\alpha\cdot(z\cdot x)]$.

Put $\delta=\gamma\alpha$ to get $[(y\cdot x\delta)\cdot z]\cdot x=y\cdot[x\delta\cdot(z\cdot x)]$.

Replace x with $x\delta$ to get $[(y\cdot x\delta^2)\cdot z]\cdot x\delta=y\cdot[x\delta^2\cdot(z\cdot x\delta)]\Leftrightarrow[(y\cdot x)\cdot z]\cdot x\delta=y\cdot[x\cdot(z\cdot x\delta)]\Leftrightarrow zL_{yx}R_{x\delta}=zR_{x\delta}L_xL_y\Leftrightarrow L_{yx}R_{x\delta}=R_{x\delta}L_xL_y\Leftrightarrow\delta L_{yx}R_{x\delta}=\delta R_{x\delta}L_xL_y\Leftrightarrow L_{yx}\delta R_{x\delta}=R_x\delta L_xL_y\Leftrightarrow$

$L_{yx}R_x\delta = R_xL_xL_y\delta \Leftrightarrow L_{yx}R_x = R_xL_xL_y \Leftrightarrow zL_{yx}R_x = zR_xL_xL_y$ as required.

Let $(H,\circ)$ be an F_{38}-algebra, and let $(\alpha,x),(\beta,y),(\gamma,z)\in(H,\circ)$. Then $[(\beta,y)\circ(I,0)]\circ(\gamma,z) = (\beta,y)\circ[(I,0)\circ(\gamma,z)] \Leftrightarrow [\beta I, yI\cdot 0]\circ(\gamma,z) = (\beta,y)\circ[I\gamma, 0\gamma\cdot z] \Leftrightarrow [\beta\gamma, y\gamma\cdot z] = [\beta\gamma, y\gamma\cdot(0\cdot z)] \Leftrightarrow y\gamma\cdot z = y\gamma\cdot(0\cdot z) \Leftrightarrow zL_{y\gamma} = zL_0L_{y\gamma} \Leftrightarrow L_{y\gamma} = L_0L_{y\gamma} \Leftrightarrow \gamma L_{y\gamma} = \gamma L_0L_{y\gamma} \Leftrightarrow L_y\gamma = L_0\gamma L_{y\gamma} \Leftrightarrow L_y\gamma = L_0L_y\gamma \Leftrightarrow L_y = L_0L_y \Leftrightarrow zL_y = zL_0L_y$ as required.

Theorem 15: Let $(G,\cdot)$ be a BCI-algebra with A-holomorph $(H,\circ)$ which is also a BCI-algebra such that every automorphism δ on $(G,\cdot)$ is λ-regular, ρ-regular and $\mid \delta \mid = 2$. Then $(H,\circ)$ is an F_i-algebra if and only if $(G,\cdot)$ is an F_i-algebra; where $i = 30, 40, 50, 53, 55, 56, 58$.

Proof: Let $(H,\circ)$ be an F_{30}-algebra, and let $(\alpha,x),(\beta,y),(\gamma,z)\in H$. Then $(\beta,y)\circ\{[(\alpha,x)\circ(\gamma,z)]\circ(\alpha,x)\} = (\beta,y)\circ\{(\alpha,x)\circ[(\gamma,z)\circ(\alpha,x)]\} \Leftrightarrow (\beta,y)\circ\{[\alpha\gamma, x\gamma\cdot z]\cdot(\alpha,x)\} = (\beta,y)\circ\{(\alpha,x)\circ[\gamma\alpha, z\alpha\cdot x]\} \Leftrightarrow (\beta,y)\circ\{\alpha\gamma\alpha, (x\gamma\cdot z)\alpha\cdot x\} = (\beta,y)\circ\{\alpha\gamma\alpha, x\gamma\alpha\cdot(z\alpha\cdot x)\} \Leftrightarrow (\beta,y)\circ\{\alpha\gamma\alpha, (x\gamma\alpha\cdot z\alpha)\cdot x\} = (\beta,y)\circ\{\alpha\gamma\alpha, x\gamma\alpha\cdot(z\alpha\cdot x)\} \Leftrightarrow \{\beta\alpha\gamma\alpha, y\alpha\gamma\alpha\cdot[(x\alpha\gamma\cdot z\alpha)\cdot x]\} = \{\beta\alpha\gamma\alpha, y\alpha\gamma\alpha\cdot[x\gamma\alpha\cdot(z\alpha\cdot x)]\} \Leftrightarrow \{y\alpha\gamma\alpha\cdot[(x\alpha\gamma\cdot z\alpha)\cdot x]\} = \{y\alpha\gamma\alpha\cdot[x\gamma\alpha\cdot(z\alpha\cdot x)]\}$.

Replace z with $z\alpha^{-1}$, y with $y(\alpha\gamma\alpha)^{-1}$ to get

$y\cdot[(x\alpha\gamma\cdot z)\cdot x] = y\cdot[x\alpha\gamma\cdot(z\cdot x)]$.

Put $\alpha\gamma = \delta$ to get

$y\cdot[(x\delta\cdot z)\cdot x] = y\cdot[x\delta\cdot(z\cdot x)] \Leftrightarrow zL_{x\delta}R_xL_y = zR_xL_{x\delta}L_y \Leftrightarrow L_{x\delta}R_xL_y = R_xL_{x\delta}L_y \Leftrightarrow \delta L_{x\delta}R_xL_y = \delta R_xL_{x\delta}L_y \Leftrightarrow L_x\delta R_xL_y = R_x\delta L_{x\delta}L_y \Leftrightarrow L_xR_x\delta L_y = R_xL_x\delta L_y \Leftrightarrow L_xR_xL_y\delta = R_xL_xL_y\delta \Leftrightarrow L_xR_xL_y = R_xL_xL_y \Leftrightarrow zL_xR_xL_y = zR_xL_xL_y$ as required.

Let $(H,\circ)$ be an F_{40}-algebra, and let $(\alpha,x),(\beta,y),(\gamma,z)\in H$. Then $(\beta,y)\circ\{[(\alpha,x)\circ(\alpha,x)]\circ(\gamma,z)\} = (\beta,y)\circ\{(\alpha,x)\circ[(\alpha,x)\circ(\gamma,z)]\} \Leftrightarrow (\beta,y)\circ\{[\alpha\alpha, x\alpha\cdot x]\circ(\gamma,z)\} = (\beta,y)\circ\{(\alpha,x)\circ[\alpha\gamma, x\gamma\cdot z]\} \Leftrightarrow (\beta,y)\circ[\alpha\alpha\gamma, (x\alpha\cdot x)\gamma\cdot z] = (\beta)\circ[\alpha\alpha\gamma, x\alpha\gamma\cdot(x\gamma\cdot z)] \Leftrightarrow (\beta,y)\circ[\alpha\alpha\gamma, (x\alpha\gamma\cdot x\gamma)\cdot z] = (\beta,y)\circ[\alpha\alpha\gamma, x\alpha\gamma\cdot(x\gamma\cdot z)] \Leftrightarrow \{\beta\alpha\alpha\gamma, y\alpha\alpha\gamma\cdot[(x\alpha\gamma\cdot x\gamma)\cdot z]\} = \{\beta\alpha\alpha\gamma, y\alpha\alpha\gamma\cdot[x\alpha\gamma\cdot(x\gamma\cdot z)]\} \Leftrightarrow y\alpha\alpha\gamma\cdot[(x\alpha\gamma\cdot x\alpha)\cdot z] = y\alpha\alpha\gamma\cdot[x\alpha\gamma\cdot(x\gamma\cdot z)]$. Replace y with $y(\alpha\alpha\gamma)^{-1}$ to get $y\cdot[(x\alpha\gamma\cdot x\gamma)\cdot z] = y\cdot[x\alpha\gamma\cdot(x\gamma\cdot z)]$. Put $\alpha\gamma = \delta$ to get $y\cdot[(x\delta\cdot x\gamma)\cdot z] = y\cdot[x\delta\cdot(x\gamma\cdot z)] \Leftrightarrow zL_{(x\delta\cdot x\gamma)}L_y = zL_{x\gamma}L_{x\delta}L_y \Leftrightarrow L_{(x\delta\cdot x\gamma)}L_y = L_{x\gamma}L_{x\delta}L_y$. Replace x with $x\gamma$ to get $L_{(x\gamma\delta\cdot x\gamma^2)}L_y = L_{x\gamma^2}L_{x\gamma\delta}L_y \Leftrightarrow L_{(x\gamma\delta\cdot x)}L_y = L_xL_{x\gamma\delta}L_y$. Put $\gamma\delta = \theta$ to get $L_{(x\theta\cdot x)}L_y = L_xL_{x\theta}L_y \Leftrightarrow L_{(x\cdot x)\theta}L_y = L_xL_{x\theta}L_y \Leftrightarrow \theta L_{(x\cot x)\theta}L_y = \theta L_xL_{x\theta}L_y \Leftrightarrow \theta L_{(x\cot x)\theta}L_y = L_x\theta L_{x\theta}L_y \Leftrightarrow L_{x\cdot}\theta L_y = L_xL_x\theta L_y \Leftrightarrow$

$L_{x\cdot x}L_y\theta = L_xL_xL_y\theta \Leftrightarrow L_{x\cdot x}L_y = L_xL_xL_y \Leftrightarrow zL_{x\cdot x}L_y = zL_xL_xL_y$ as required.

Let $(H,\circ)$ be an F_{50}-algebra, and let $(\alpha,x),(\beta,y),(\gamma,z) \in H$. Then $(\alpha,x)\circ\{(\alpha,x)\circ[(\beta,y)\circ(\gamma,z)]\} = (\alpha,x)\circ\{[(\alpha,x)\circ(\beta,y)]\circ(\gamma,z)\} \Leftrightarrow (\alpha,x)\circ\{(\alpha,x)\circ[\beta\gamma, y\gamma\cdot z]\} = (\alpha,x)\circ\{[\alpha\beta, x\beta\cdot y]\circ(\gamma,z)\} \Leftrightarrow (\alpha,x)\circ\{\alpha\beta\gamma, x\beta\gamma\cdot(y\gamma\cdot z)\} = (\alpha,x)\circ\{\alpha\beta\gamma,(x\beta\cdot y)\gamma\cdot z\} \Leftrightarrow \{\alpha\alpha\beta\gamma, x\alpha\beta\gamma\cdot(x\beta\gamma\cdot y\gamma\cdot z)\} = \{\alpha\alpha\beta\gamma, x\alpha\beta\gamma\cdot(x\beta\cdot y)\gamma\cdot z\} \Leftrightarrow \{x\alpha\beta\gamma\cdot(x\beta\gamma\cot y\gamma\cdot z)\} = \{x\alpha\beta\gamma\cdot(x\beta\gamma\cdot y\gamma)\cdot z\}$. Replace y with $y\gamma^{-1}$ to get $x\alpha\beta\gamma\cdot[x\beta\gamma\cdot(y\cdot z)] = x\alpha\beta\gamma\cdot[(x\beta\gamma\cdot y)\cdot z]$. Put $\delta=\alpha\beta\gamma$ to get $x\delta\cdot[x\beta\gamma\cdot(y\cdot z)] = x\delta\cdot[(x\beta\gamma\cdot y)\cdot z]$. Put $\theta=\beta\gamma$ to get $x\delta\cdot[x\theta\cdot(y\cdot z)] = x\delta\cdot[(x\theta\cdot y)\cdot z] \Leftrightarrow yR_zL_{x\theta}L_{x\delta} = yL_{x\theta}R_zL_{x\delta} \Leftrightarrow R_zL_{x\theta}L_{x\delta} = L_{x\theta}R_zL_{x\delta} \Leftrightarrow \theta R_zL_{x\theta}L_{x\delta} = \theta L_{x\theta}R_zL_{x\delta} \Leftrightarrow R_zL_x\theta L_{x\delta} = L_x\theta R_zL_{x\delta} \Leftrightarrow R_zL_xL_{x\delta}\theta = L_xR_zL_{x\delta}\theta \Leftrightarrow \delta R_zL_xL_{x\delta}\theta = \delta L_xR_zL_{x\delta}\theta \Leftrightarrow R_zL_x\delta L_{x\delta}\theta = L_xR_z\delta L_{x\delta}\theta \Leftrightarrow R_zL_xL_x\delta\theta = L_xR_zL_x\delta\theta \Leftrightarrow R_zL_xL_x = L_xR_zL_x \Leftrightarrow yR_zL_xL_x = yL_xR_zL_x$ as required.

Let $(H,\circ)$ be an F_{53}-algebra, and let $(\alpha,x),(\beta,y),(\gamma,z) \in H$. Then $[(\beta,y)\circ(\gamma,z)]\circ[(\alpha,x)\circ(\alpha,x)] = (\beta,y)\circ\{[(\gamma,z)\circ(\alpha,x)]\circ(\alpha,x)\} \Leftrightarrow [\beta\gamma, y\gamma\cdot z]\circ[\alpha\alpha, x\alpha\cdot x] = (\beta,y)\circ\{[\gamma\alpha, z\alpha\cdot x]\circ(\alpha,x)\} \Leftrightarrow [\beta\gamma\alpha\alpha,(y\gamma\cdot z)\alpha\alpha\cdot(x\alpha\cdot x)] = (\beta,y)\circ[\gamma\alpha\alpha,(z\alpha\cdot x)\alpha\cdot x] \Leftrightarrow [\beta\gamma\alpha\alpha,(y\gamma\alpha\alpha\cdot z\alpha\alpha)\cdot(x\alpha\cdot x)] = \{\beta\gamma\alpha\alpha, y\gamma\alpha\alpha,\cdot[(z\alpha\alpha\cdot x\alpha)\cdot x]\} \Leftrightarrow (y\gamma\alpha\alpha\cdot z\alpha\alpha)\cdot(x\alpha\cdot x) = y\gamma\alpha\alpha\cdot[(z\alpha\alpha\cdot x\alpha)\cdot x]$. Replace y with $y(\gamma\alpha\alpha)^{-1}$, z with $z(\alpha\alpha)^{-1}$ to get $(y\cdot z)\cdot(x\alpha\cdot x) = y\cdot[(z\cdot x\alpha)\cdot x] \Leftrightarrow zL_yR_{x\alpha\cdot x} = zR_{x\alpha}R_xL_y \Leftrightarrow L_yR_{x\alpha\cdot x} = R_{x\alpha}R_xL_y \Leftrightarrow L_yR_{(x\cdot x)\alpha} = R_{x\alpha}R_xL_y \Leftrightarrow \alpha L_yR_{(x\cdot x)\alpha} = \alpha R_{x\alpha}R_xL_y \Leftrightarrow L_y\alpha R_{(x\cdot x)\alpha} = \alpha R_{x\alpha}R_xL_y \Leftrightarrow L_yR_{(x\cdot x)}\alpha = R_x\alpha R_xL_y \Leftrightarrow L_yR_{(x\cdot x)}\alpha = R_xR_xL_y\alpha \Leftrightarrow L_yR_{(x\cdot x)} = R_xR_xL_y \Leftrightarrow zL_yR_{(x\cdot x)} = zR_xR_xL_y$ as required.

Let $(H,\circ)$ be an F_{55}-algebra, and let $(\alpha,x),(\beta,y),(\gamma,z) \in H$. Then $\{[(\beta,y)\circ(\gamma,z)]\circ(\alpha,x)\}\circ(\alpha,x) = \{(\beta,y)\circ[(\gamma,z)\circ(\alpha,x)]\}\circ(\alpha,x) \Leftrightarrow \{[\beta\gamma, y\gamma\cdot z]\circ(\alpha,x)\}\circ(\alpha,x) = \{(\beta,y)\circ[\gamma\alpha, z\alpha\cdot x]\}\circ(\alpha,x) \Leftrightarrow \{\beta\gamma\alpha,(y\gamma\cdot z)\alpha\cdot x\}\circ(\alpha,x) = \{\beta\gamma\alpha, y\gamma\alpha\cdot(z\alpha\cdot x)\}\circ(\alpha,x) \Leftrightarrow \{\beta\gamma\alpha\alpha,[(y\gamma\alpha\cdot z\alpha)\cdot x]\alpha\cdot x\} = \{\beta\gamma\alpha\alpha,[y\gamma\alpha\cdot(z\alpha\cdot x)]\alpha\cdot x\} \Leftrightarrow \{\beta\gamma\alpha\alpha,[(y\gamma\alpha\cdot z\alpha)\cdot x]\alpha\cdot x\} = \{\beta\gamma\alpha\alpha,[y\gamma\alpha\alpha\cdot(z\alpha\alpha\cdot x\alpha)]\cdot x\} \Leftrightarrow \{\beta\gamma\alpha\alpha,[(y\gamma\alpha\alpha\cdot z\alpha\alpha)\cdot x\alpha]\cdot x\} = \{\beta\gamma\alpha\alpha,[y\gamma\alpha\alpha\cdot(z\alpha\alpha\cdot x\alpha)]\cdot x\} \Leftrightarrow [(y\gamma\alpha\alpha\cdot z\alpha\alpha)\cdot x\alpha]\cdot x = [y\gamma\alpha\alpha\cdot(z\alpha\alpha\cdot x\alpha)]\cdot x$. Replace y with $y(\gamma\alpha\alpha)^{-1}$, z with $z(\alpha\alpha)^{-1}$ to get $[(y\cdot z)\cdot x\alpha]\cdot x = [y\cdot(z\cdot x\alpha)]\cdot x \Leftrightarrow zL_yR_{x\alpha}R_x = zR_{x\alpha}L_yR_x \Leftrightarrow L_yR_{x\alpha}R_x = R_{x\alpha}L_yR_x \Leftrightarrow \alpha L_yR_{x\alpha}R_x = \alpha R_{x\alpha}L_yR_x \Leftrightarrow L_yR_x\alpha R_x = R_x\alpha L_yR_x \Leftrightarrow L_yR_xR_x\alpha = R_xL_yR_x\alpha \Leftrightarrow L_yR_xR_x = R_xL_yR_x \Leftrightarrow zL_yR_xR_x = zR_xL_yR_x$ as required.

Let $(H,\circ)$ be an F_{56}-algebra, and let $(\alpha,x),(\beta,y),(\gamma,z) \in H$. Then $\{[(\beta,y)\circ(\gamma,z)]\circ(\alpha,x)\}\circ(\alpha,x) = (\beta,y)\circ\{[(\gamma,z)\circ(\alpha,x)]\circ(\alpha,x)\} \Leftrightarrow \{[\beta\gamma, y\gamma\cdot z]\circ(\alpha,x)\} = (\beta,y)\circ\{[\gamma\alpha, z\alpha\cdot x]\circ(\alpha,x)\} \Leftrightarrow$

$\{\beta\gamma\alpha,(y\gamma\cdot z)\alpha\cdot x\}\circ(\alpha,x)=(\beta,y)\circ\{\gamma\alpha\alpha,(z\alpha\cdot x)\alpha\cdot x\}\Leftrightarrow\{\beta\gamma\alpha\alpha,[(y\gamma\cdot z)\alpha\cdot x]\alpha\cdot x\}=\{\beta\gamma\alpha\alpha,y\gamma\alpha\alpha\cdot[(z\alpha\cdot x)\alpha\cdot x]\}\Leftrightarrow\{\beta\gamma\alpha\alpha,[(y\gamma\alpha\cdot z\alpha)\cdot x]\alpha\cdot x\}=\{\beta\gamma\alpha\alpha,y\gamma\alpha\alpha\cdot(z\alpha\alpha\cdot x\alpha)\cdot x\}\Leftrightarrow\{\beta\gamma\alpha\alpha,[(y\gamma\alpha\cdot z\alpha)\alpha\cdot x\alpha]\cdot x\}=\{\beta\gamma\alpha\alpha,y\gamma\alpha\alpha\cdot(z\alpha\alpha\cdot x\alpha)\cdot x\}\Leftrightarrow\{\beta\gamma\alpha\alpha,[(y\gamma\alpha\alpha\cdot z\alpha\alpha)\cdot x\alpha]\cdot x\}=\{\beta\gamma\alpha\alpha,y\gamma\alpha\alpha\cdot(z\alpha\alpha\cdot x\alpha)\cdot x\}\Leftrightarrow\{[(y\gamma\alpha\alpha\cdot z\alpha\alpha)\cdot x\alpha]\cdot x\}=y\gamma\alpha\alpha\cdot\{(z\alpha\alpha\cdot x\alpha)\cdot x\}$. Replace y with $y(\gamma\alpha\alpha)^{-1}$, z with $z(\alpha\alpha)^{-1}$ to get $[(y\cdot z)\cdot x\alpha]\cdot x=y\cdot[(z\cdot x\alpha)\cdot x]\Leftrightarrow zL_yR_{x\alpha}R_x=zR_{x\alpha}R_xL_y\Leftrightarrow L_yR_{x\alpha}R_x=R_{x\alpha}R_xL_y\Leftrightarrow\alpha L_yR_{x\alpha}R_x=\alpha R_{x\alpha}R_xL_y\Leftrightarrow L_y\alpha R_{x\alpha}R_x=R_x\alpha R_xL_y\Leftrightarrow L_yR_x\alpha R_x=R_x\alpha R_xL_y\Leftrightarrow L_yR_xR_x\alpha=R_xR_xL_y\alpha\Leftrightarrow L_yR_xR_x=R_xR_xL_y\Leftrightarrow zL_yR_xR_x=zR_xR_xL_y$ as required.

Let $(H,\circ)$ be an F_{58}-algebra, and let $(\alpha,x),(\beta,y),(\gamma,z)\in H$. Then $\{(\beta,y)\circ[(\gamma,z)\circ(\alpha,x)]\}\circ(\alpha,x)=(\beta,y)\circ\{[(\gamma,z)\circ(\alpha,x)]\circ(\alpha,x)\}\Leftrightarrow\{(\beta,y)\circ[\gamma\alpha,z\alpha\cdot x]\}\circ(\alpha,x)=(\beta,y)\circ\{[\gamma\alpha,z\alpha\cdot x]\circ(\alpha,x)\}\Leftrightarrow\{\beta\gamma\alpha,y\gamma\alpha\cdot(z\alpha\cdot x)\}\circ(\alpha,x)=(\beta,y)\circ\{\gamma\alpha\alpha,(z\alpha\cdot x)\alpha\cdot x\}\Leftrightarrow\{\beta\gamma\alpha\alpha,[y\gamma\alpha\cdot(z\alpha\cdot x)\alpha\cdot x]\}=(\beta,y)\circ\{\gamma\alpha\alpha,(z\alpha\alpha\cdot x\alpha)\cdot x\}\Leftrightarrow\{\beta\gamma\alpha\alpha,[y\gamma\alpha\alpha\cdot(z\alpha\cdot x)\alpha]\cdot x\}=\{\beta\gamma\alpha\alpha,y\gamma\alpha\alpha\cdot(z\alpha\alpha\cdot x\alpha)\cdot x\}\Leftrightarrow\{\beta\gamma\alpha\alpha,[y\gamma\alpha\alpha\cdot(z\alpha\alpha\cdot x\alpha)]\cdot x\}=\{\beta\gamma\alpha\alpha,y\gamma\alpha\alpha\cdot(z\alpha\alpha\cdot x\alpha)\cdot x\}\Leftrightarrow[y\gamma\alpha\alpha\cdot(z\alpha\alpha\cdot x\alpha)]\cdot x=y\gamma\alpha\alpha\cdot[(z\alpha\alpha\cdot x\alpha)\cdot x]$. Replace y with $y(\gamma\alpha\alpha)^{-1}$, z with $z(\alpha\alpha)^{-1}$ to get $[y\cdot(z\cdot x\alpha)]\cdot x=y\cdot[(z\cdot x\alpha)\cdot x]\Leftrightarrow zR_{x\alpha}L_yR_x=zR_{x\alpha}R_xL_y\Leftrightarrow R_{x\alpha}L_yR_x=R_{x\alpha}R_xL_y\Leftrightarrow\alpha R_{x\alpha}L_yR_x=\alpha R_{x\alpha}R_xL_y\Leftrightarrow R_x\alpha L_yR_x=R_x\alpha R_xL_y\Leftrightarrow R_xL_yR_x\alpha=R_xR_xL_y\alpha\Leftrightarrow R_xL_yR_x=R_xR_xL_y\Leftrightarrow zR_xL_yR_x=zR_xR_xL_y$ as required.

Theorem 16: Let $(G,\cdot)$ be a BCI-algebra with A-holomorph $(H,\circ)$ which is also a BCI-algebra such that every automorphism δ on $(G,\cdot)$ is λ-regular and $|\delta|=2$. Then $(H,\circ)$ is an F_i-algebra if and only if $(G,\cdot)$ is an F_i-algebra; where $i=4,5,6,10,20,21,25,31$.

Proof: Let $(H,\circ)$ be an F_4-algebra, and let $(\alpha,x),(\beta,y),(\gamma,z)\in H$. Then $[(\alpha,x)\circ(\beta,y)]\circ[(\gamma,z)\circ(\alpha,x)]=(\alpha,x)\circ\{[(\beta,y)\circ(\gamma,z)]\circ(\alpha,x)\}\Leftrightarrow[(\alpha\beta,x\beta\cdot y)\circ(\gamma\alpha,z\alpha\cdot x)]=(\alpha,x)\circ[(\beta\gamma,y\gamma\cdot z)\circ(\alpha,x)]\Leftrightarrow[\alpha\beta\gamma\alpha,(x\beta\cdot y)\gamma\alpha\cdot(z\alpha\cdot x)]=(\alpha,x)\circ[\beta\gamma\alpha,(y\gamma\cdot z)\alpha\cdot x]\Leftrightarrow[\alpha\beta\gamma\alpha,(x\beta\cdot y)\gamma\alpha\cdot(z\alpha\cdot x)]=\{\alpha\beta\gamma\alpha,x\beta\gamma\alpha\cdot[(y\gamma\cdot z)\alpha\cdot x]\}\Leftrightarrow[\alpha\beta\gamma\alpha,(x\beta\gamma\alpha\cdot y\gamma\alpha)\cdot(z\alpha\cdot x)]=\{\alpha\beta\gamma\alpha,x\beta\gamma\alpha\cdot[(y\gamma\alpha\cdot z\alpha)\cdot x]\}\Leftrightarrow(x\beta\gamma\alpha\cdot y\gamma\alpha)\cdot(z\alpha\cdot x)x\beta\gamma\alpha\cdot[(y\gamma\alpha\cdot z\alpha)\cdot x]$. Replace z with $z\alpha^{-1}$, y with $y(\gamma\alpha)^{-1}$ to get $(x\beta\gamma\alpha\cdot y)\cdot(z\cdot x)=x\beta\gamma\gamma\alpha\cdot[(y\cdot z)\cdot x]\Leftrightarrow(x\delta\cdot y)\cdot(z\cdot x)=x\delta\cdot[(y\cdot z)\cdot x]$(where $\delta=\beta\gamma\alpha$) $\Leftrightarrow yL_{x\delta}R_{z\cdot x}=yR_zR_xL_{x\delta}\Leftrightarrow L_{x\delta}R_{z\cdot x}=R_zR_xL_{x\delta}\Leftrightarrow L_x\delta R_{z\cdot x}=R_zR_xL_x\delta\Leftrightarrow L_xR_{z\cdot x}\delta=R_zR_xL_x\delta\Leftrightarrow L_xR_{z\cdot x}=R_zR_xL_x\Leftrightarrow yL_xR_{z\cdot x}=yR_zR_xL_x$ as required.

Let $(H,\circ)$ be an F_5-algebra, and let $(\alpha,x),(\beta,y),(\gamma,z)\in H$. Then $\{[(\alpha,x)\circ(\beta,y)]\circ(\gamma,z)\}\circ(\alpha,x)=\{(\alpha,x)\circ[(\beta,y)\circ(\gamma,z)]\}\circ$

$(\alpha, x) \Leftrightarrow \{[\alpha\beta, x\beta \cdot y] \circ (\gamma, z)\} \circ (\alpha, x) = \{(\alpha, x) \circ [\beta\gamma, y\gamma \cdot z]\} \circ (\alpha, x) \Leftrightarrow \{\alpha\beta\gamma, (x\beta \cdot y)\gamma \cdot z\} \circ (\alpha, x) = \{\alpha\beta\gamma, (x\beta\gamma) \cdot (y\gamma \cdot z)\} \circ (\alpha, x) \Leftrightarrow \{\alpha\beta\gamma\alpha, [(x\beta \cdot y)\gamma \cdot z]\alpha \cdot x\} = \{\alpha\beta\gamma\alpha, [(x\beta\gamma) \cdot (y\gamma \cdot z)]\alpha \cdot x\} \Leftrightarrow \{\alpha\beta\gamma\alpha, [(x\beta\gamma \cdot y\gamma) \cdot z]\alpha \cdot x\} = \{\alpha\beta\gamma\alpha, [(x\beta\gamma\alpha) \cdot (y\gamma \cdot z)\alpha] \cdot x\} \Leftrightarrow \{\alpha\beta\gamma\alpha, [(x\beta\gamma \cdot y\gamma) \cdot z]\alpha \cdot x\} = \{\alpha\beta\gamma\alpha, [(x\beta\gamma\alpha) \cdot (y\gamma\alpha \cdot z\alpha)] \cdot x\} \Leftrightarrow \{\alpha\beta\gamma\alpha, [(x\beta\gamma \cdot y\gamma)\alpha \cdot z\alpha] \cdot x\} = \{\alpha\beta\gamma\alpha, [(x\beta\gamma\alpha) \cdot (y\gamma\alpha \cdot z\alpha)] \cdot x\} \Leftrightarrow \{\alpha\beta\gamma\alpha, [(x\beta\gamma\alpha \cdot y\gamma\alpha) \cdot z\alpha] \cdot x\} = \{\alpha\beta\gamma\alpha, [(x\beta\gamma\alpha) \cdot (y\gamma\alpha \cdot z\alpha)] \cdot x\} \Leftrightarrow [(x\beta\gamma\alpha \cdot y\gamma\alpha) \cdot z\alpha] \cdot x = [(x\beta\gamma\alpha) \cdot (y\gamma\alpha \cdot z\alpha)] \cdot x.$

Replace z with $z\alpha^{-1}$, y with $y(\gamma\alpha)^{-1}$ to get

$[(x\beta\gamma\alpha \cdot y) \cdot z] \cdot x = [(x\beta\gamma\alpha) \cdot (y \cdot z)] \cdot x.$

Put $\delta = \beta\gamma\alpha$ to get

$[(x\delta \cdot y) \cdot z] \cdot x = [x\delta \cdot (y \cdot z)] \cdot x \Leftrightarrow yL_{x\delta}R_zR_x = yR_zL_{x\delta}R_x \Leftrightarrow L_{x\delta}R_zR_x = R_zL_{x\delta}R_x \Leftrightarrow \delta L_{x\delta}R_zR_x = \delta R_zL_{x\delta}R_x \Leftrightarrow L_xR_zR_x\delta = R_zL_xR_z\delta \Leftrightarrow L_xR_zR_x = R_zL_xR_z \Leftrightarrow yL_xR_zR_x = yR_zL_xR_z$ as required.

Let $(H, \circ)$ be an F_6-algebra, and let $(\alpha, x), (\beta, y), (\gamma, z) \in H$. Then $\{[(\alpha, x) \circ (\beta, y)] \circ (\gamma, z)\} \circ (\alpha, x) = (\alpha, x) \circ \{(\beta, y) \circ [(\gamma, z) \circ (\alpha, x)]\} \Leftrightarrow \{[\alpha\beta, x\beta \cdot y] \circ (\gamma, z)\} \circ (\alpha, x) = (\alpha, x) \circ \{(\beta, y) \circ [\gamma\alpha, z\alpha \cdot x]\} \Leftrightarrow [\alpha\beta\gamma, (x\beta \cdot y)\gamma \cdot z] \circ (\alpha, x) = (\alpha, x) \circ [\beta\gamma\alpha, y\gamma\alpha \cdot (z\alpha \cdot x)] \Leftrightarrow \{\alpha\beta\gamma\alpha, [(x\beta \cdot y)\gamma \cdot z]\alpha \cdot x\} = \{\alpha\beta\gamma\alpha, x\beta\gamma\alpha \cdot [y\gamma\alpha \cdot (z\alpha \cdot x)]\} \Leftrightarrow \{\alpha\beta\gamma\alpha, [(x\beta\gamma \cdot y\gamma) \cdot z]\alpha \cdot x\} = \{\alpha\beta\gamma\alpha, x\beta\gamma\alpha \cdot [y\gamma\alpha \cdot (z\alpha \cdot x)]\} \Leftrightarrow \{\alpha\beta\gamma\alpha, [(x\beta\gamma \cdot y\gamma)\alpha \cdot z\alpha] \cdot x\} = \{\alpha\beta\gamma\alpha, x\beta\gamma\alpha \cdot [y\gamma\alpha \cdot (z\alpha \cdot x)]\} \Leftrightarrow \{\alpha\beta\gamma\alpha, [(x\beta\gamma\alpha \cdot y\gamma\alpha) \cdot z\alpha] \cdot x\} = \{\alpha\beta\gamma\alpha, x\beta\gamma\alpha \cdot [y\gamma\alpha \cdot (z\alpha \cdot x)]\} \Leftrightarrow [(x\beta\gamma\alpha \cdot y\gamma\alpha) \cdot z\alpha] \cdot x = \{x\beta\gamma\alpha \cdot [y\gamma\alpha \cdot (z\alpha \cdot x)]\}$. Replace z with $z\alpha^{-1}$, y with $y(\gamma\alpha)^{-1}$ to get $[(x\beta\gamma\alpha \cdot y) \cdot z] \cdot x = \{x\beta\gamma\alpha \cdot [y \cdot (z \cdot x)]\}$. Put $\delta = \beta\gamma\alpha$ to get $[(x\delta \cdot y) \cdot z] \cdot x = x\delta \cdot [y \cdot (z \cdot x)] \Leftrightarrow yL_{x\delta}R_zR_x = yR_{z\cdot x}L_{x\delta} \Leftrightarrow L_{x\delta}R_zR_x = R_{z\cdot x}L_{x\delta} \Leftrightarrow \delta L_{x\delta}R_zR_x = \delta R_{z\cdot x}L_{x\delta} \Leftrightarrow L_x\delta R_zR_x = \delta R_{z\cdot x}L_{x\delta} \Leftrightarrow L_xR_zR_x\delta = R_{z\cdot x}\delta L_{x\delta} \Leftrightarrow L_xR_zR_x\delta = R_{z\cdot x}L_x\delta \Leftrightarrow L_xR_zR_x = R_{z\cdot x}L_x \Leftrightarrow yL_xR_zR_x = yR_{z\cdot x}L_x$ as required.

Let $(H, \circ)$ be an F_{10}-algebra, and let $(\alpha, x), (\beta, y), (\gamma, z) \in H$. Then $(\alpha, x) \circ \{(\beta, y) \circ [(\gamma, z) \circ (\alpha, x)]\} = (\alpha, x) \circ \{[(\beta, y) \circ (\gamma, z)] \circ (\alpha, x)\} \Leftrightarrow (\alpha, x) \circ \{(\beta, y) \circ [\gamma\alpha, z\alpha \cdot x]\} = (\alpha, x) \circ \{[\beta\gamma, y\gamma \cdot z] \circ (\alpha, x)\} \Leftrightarrow (\alpha, x) \circ \{\beta\gamma\alpha, y\gamma\alpha \cdot (z\alpha \cdot x)\} = (\alpha, x) \circ \{\beta\gamma\alpha, (y\gamma \cdot z)\alpha \cdot x\} \Leftrightarrow \{\alpha\beta\gamma\alpha, x\beta\gamma\alpha \cdot [y\gamma\alpha \cdot (z\alpha \cdot x)]\} = \{\alpha\beta\gamma\alpha, x\beta\gamma\alpha \cdot (y\gamma \cdot z)\alpha \cdot x\} \Leftrightarrow \{\alpha\beta\gamma\alpha, x\beta\gamma\alpha \cdot [y\gamma\alpha \cdot (z\alpha \cdot x)]\} = \{\alpha\beta\gamma\alpha, x\beta\gamma\alpha \cdot (y\gamma\alpha \cdot z\alpha) \cdot x\} \Leftrightarrow \{x\beta\gamma\alpha \cdot [y\gamma\alpha \cdot (z\alpha \cdot x)]\} = \{x\beta\gamma\alpha \cdot (y\gamma\alpha \cdot z\alpha) \cdot x\}$. Replace z with $z\alpha^{-1}$, y with $y(\gamma\alpha)^{-1}$ to get $\{x\beta\gamma\alpha \cdot [y \cdot (z \cdot x)]\} = \{x\beta\gamma\alpha \cdot (y \cdot z) \cdot x\}$. Put $\delta = \beta\gamma\alpha$ to get $x\delta \cdot [y \cdot (z \cdot x)] = x\delta \cdot [(y \cdot z) \cdot x] \Leftrightarrow yR_{z\cdot x}L_{x\delta} = yR_zR_xL_{x\delta} \Leftrightarrow R_{z\cdot x}L_{x\delta} = R_zR_xL_{x\delta} \Leftrightarrow \delta R_{z\cdot x}L_{x\delta} = \delta R_zR_xL_{x\delta} \Leftrightarrow R_{z\cdot x}\delta L_{x\delta} = R_zR_x\delta L_{x\delta} \Leftrightarrow R_{z\cdot x}L_x\delta = R_zR_xL_x\delta \Leftrightarrow R_{z\cdot x}L_x = R_zR_xL_x \Leftrightarrow yR_{z\cdot x}L_x = yR_zR_xL_x$ as required.

Let $(H, \circ)$ be an F_{20}-algebra, and let $(\alpha, x), (\beta, y), (\gamma, z) \in H$. Then $(\alpha, x) \circ \{[(\beta, y) \circ (\alpha, x)] \circ (\gamma, z)\} = (\alpha, x) \circ \{(\beta, y) \circ [(\alpha, x) \circ (\gamma, z)]\} \Leftrightarrow (\alpha, x) \circ \{[\beta\alpha, y\alpha \cdot x] \circ (\gamma, z)\} = (\alpha, x) \circ \{(\beta, y) \circ [\alpha\gamma, x\gamma \cdot z]\} \Leftrightarrow (\alpha, x) \circ \{\beta\alpha\gamma, (y\alpha \cdot x)\gamma \cdot z\} = (\alpha, x) \circ \{\beta\alpha\gamma, y\alpha\gamma \cdot (x\gamma \cdot z)\} \Leftrightarrow \{\alpha\beta\alpha\gamma, x\beta\alpha\gamma \cdot [(y\alpha \cdot x)\gamma \cdot z]\} = \{\alpha\beta\alpha\gamma, x\beta\alpha\gamma \cdot [y\alpha\gamma \cdot (x\gamma \cdot z)]\} \Leftrightarrow \{\alpha\beta\alpha\gamma, x\beta\alpha\gamma \cdot (y\alpha\gamma \cdot x\gamma) \cdot z\} = \{\alpha\beta\alpha\gamma, x\beta\alpha\gamma \cdot [y\alpha\gamma \cdot (x\gamma \cdot z)]\} \Leftrightarrow \{x\beta\alpha\gamma \cdot (y\alpha\gamma \cdot x\gamma) \cdot z\} = \{x\beta\alpha\gamma \cdot [y\alpha\gamma \cdot (x\gamma \cdot z)]\}$. Replace y with $y(\alpha\gamma)^{-1}$ to get $\{x\beta\alpha\gamma \cdot (y \cdot x) \cdot z\} = \{x\beta\alpha\gamma \cdot [y \cdot (x \cdot z)]\}$. Put $\delta = \beta\alpha\gamma$ to get $x\delta \cdot [(y \cdot x) \cdot z] = \{x\delta \cdot [y \cdot (x \cdot z)]\} \Leftrightarrow yR_xR_zL_{x\delta} = yR_{x\cdot z}L_{x\delta} \Leftrightarrow R_xR_zL_{x\delta} = R_{x\cdot z}L_{x\delta} \Leftrightarrow \delta R_xR_zL_{x\delta} = \delta R_{x\cdot z}L_{x\delta} \Leftrightarrow R_xR_z\delta L_{x\delta} = \delta R_{x\cdot z}\delta L_{x\delta} \Leftrightarrow R_xR_zL_x\delta = R_{x\cdot z}L_x\delta \Leftrightarrow R_xR_zL_x = R_{x\cdot z}L_x \Leftrightarrow yR_xR_zL_x = yR_{x\cdot z}L_x$ as required.

Let $(H, \circ)$ be an F_{21}-algebra, and let $(\alpha, x), (\beta, y), (\gamma, z) \in H$. Then $[(\beta, y) \circ (\alpha, x)] \circ [(\gamma, z) \circ (\alpha, x)] = \{[(\beta, y) \circ (\alpha, x)] \circ (\gamma, z)\} \circ (\alpha, x) \Leftrightarrow (\beta\alpha, y\alpha \cdot x) \circ (\gamma\alpha, z\alpha \cdot x) = \{[\beta\alpha, y\alpha \cdot x] \circ (\gamma, z)\} \circ (\alpha, x) \Leftrightarrow [\beta\alpha\gamma\alpha, (y\alpha \cdot x)\gamma\alpha \cdot (z\alpha \cdot x)] = [\beta\alpha\gamma, (y\alpha \cdot x)\gamma \cdot z] \circ (\alpha, x) \Leftrightarrow [\beta\alpha\gamma\alpha, (y\alpha \cdot x)\gamma\alpha \cdot (z\alpha \cdot x)] = \{\beta\alpha\gamma\alpha, [(y\alpha \cdot x)\gamma \cdot z]\alpha \cdot x\} \Leftrightarrow [\beta\alpha\gamma\alpha, (y\alpha\gamma\alpha \cdot x\gamma\alpha) \cdot (z\alpha \cdot x)] = \{\beta\alpha\gamma\alpha, [(y\alpha\gamma \cdot x\gamma) \cdot z]\alpha \cdot x\} \Leftrightarrow [\beta\alpha\gamma\alpha, (y\alpha\gamma\alpha \cdot x\gamma\alpha) \cdot (z\alpha \cdot x)] = \{\beta\alpha\gamma\alpha, [(y\alpha\gamma \cdot x\gamma)\alpha \cdot z\alpha] \cdot x\} \Leftrightarrow [\beta\alpha\gamma\alpha, (y\alpha\gamma\alpha \cdot x\gamma\alpha) \cdot (z\alpha \cdot x)] = \{\beta\alpha\gamma\alpha, [(y\alpha\gamma\alpha \cdot x\gamma\alpha) \cdot z\alpha] \cdot x\} \Leftrightarrow (y\alpha\gamma\alpha \cdot x\gamma\alpha) \cdot (z\alpha \cdot x) = [(y\alpha\gamma\alpha \cdot x\gamma\alpha) \cdot z\alpha] \cdot x$. Replace y with $y(\alpha\gamma\alpha)^{-1}$, z with $z\alpha^{-1}$ to get $(y \cdot x\gamma\alpha) \cdot (z \cdot x) = [(y \cdot x\gamma\alpha) \cdot z] \cdot x$. Put $\delta = \gamma\alpha$ to get $(y \cdot x\delta) \cdot (z \cdot x) = [(y \cdot x\delta) \cdot z] \cdot x \Leftrightarrow yR_{x\delta}R_{z\cdot x} = yR_{x\delta}R_zR_x \Leftrightarrow R_{x\delta}R_{z\cdot x} = R_{x\delta}R_zR_x \Leftrightarrow \delta R_{x\delta}R_{z\cdot x} = \delta R_{x\delta}R_zR_x \Leftrightarrow R_x\delta R_{z\cdot x} = R_x\delta R_zR_x \Leftrightarrow R_xR_{z\cdot x}\delta = R_xR_zR_x\delta \Leftrightarrow R_xR_{z\cdot x} = R_xR_zR_x \Leftrightarrow yR_xR_{z\cdot x} = yR_xR_zR_x$ as required.

Let $(H, \circ)$ be an F_{25}-algebra, and let $(\alpha, x), (\beta, y), (\gamma, z) \in H$. Then $\{[(\beta, y) \circ (\alpha, x)] \circ (\gamma, z)\} \circ (\alpha, x) = \{(\beta, y) \circ [(\alpha, x) \circ (\gamma, z)]\} \circ (\alpha, x) \Leftrightarrow \{[\beta\alpha, y\alpha \cdot x] \circ (\gamma, z)\} \circ (\alpha, x) = \{(\beta, y) \circ [\alpha\gamma, x\gamma \cdot z]\} \circ (\alpha, x) \Leftrightarrow \{[\beta\alpha\gamma, (y\alpha \cdot x)\gamma \cdot z]\} \circ (\alpha, x) = [\beta\alpha\gamma \cdot (x\gamma \cdot z)] \circ (\alpha, x) \Leftrightarrow \{[\beta\alpha\gamma, (y\alpha\gamma \cdot x\gamma) \cdot z]\} \circ (\alpha, x) = [\beta\alpha\gamma, y\alpha\gamma \cdot (x\gamma \cdot z)] \circ (\alpha, x) \Leftrightarrow \{\beta\alpha\gamma\alpha, [(y\alpha\gamma \cdot x\gamma) \cdot z]\alpha \cdot x\} = \{\beta\alpha\gamma\alpha, [y\alpha\gamma \cdot (x\gamma \cdot z)]\alpha \cdot x\} \Leftrightarrow \{\beta\alpha\gamma\alpha, [(y\alpha\gamma \cdot x\gamma)\alpha \cdot z\alpha] \cdot x\} = \{\beta\alpha\gamma\alpha, [y\alpha\gamma\alpha \cdot (x\gamma \cdot z)\alpha] \cdot x\} \Leftrightarrow \{\beta\alpha\gamma\alpha, [(y\alpha\gamma\alpha \cdot x\gamma\alpha) \cdot z\alpha] \cdot x\} = \{\beta\alpha\gamma\alpha, [y\alpha\gamma\alpha \cdot (x\gamma\alpha \cdot z\alpha)] \cdot x\} \Leftrightarrow [(y\alpha\gamma\alpha \cdot x\gamma\alpha) \cdot z\alpha] \cdot x = [y\alpha\gamma\alpha \cdot (x\gamma\alpha \cdot z\alpha)] \cdot x$. Replace y with $y(\alpha\gamma\alpha)^{-1}$, z with $z\alpha^{-1}$ to get $[(y \cdot x\gamma\alpha) \cdot z] \cdot x = [y \cdot (x\gamma\alpha \cdot z)] \cdot x$. Put $\delta = \gamma\alpha$ to get $[(y \cdot x\delta) \cdot z] \cdot x = [y \cdot (x\delta \cdot z)] \cdot x$. Replace x with $x\delta$ to get $[(y \cdot x\delta\delta) \cdot z] \cdot x\delta = [y \cdot (x\delta\delta \cdot z)] \cdot x\delta \Leftrightarrow [(y \cdot x) \cdot z] \cdot x\delta = [y \cdot (x \cdot z)] \cdot x\delta \Leftrightarrow yR_xR_zR_{x\delta} = yR_{x\cdot z}R_{x\delta} \Leftrightarrow R_xR_zR_{x\delta} = R_{x\cdot z}R_{x\delta} \Leftrightarrow \delta R_xR_zR_{x\delta} = \delta R_{x\cdot z}R_{x\delta} \Leftrightarrow R_xR_z\delta R_{x\delta} = R_{x\cdot z}\delta R_{x\delta} \Leftrightarrow R_xR_zR_x\delta = R_{x\cdot z}R_x\delta \Leftrightarrow R_xR_zR_x = R_{x\cdot z}R_x \Leftrightarrow yR_xR_zR_x = yR_{x\cdot z}R_x$ as required.

Let $(H, \circ)$ be an F_{31}-algebra, and let $(\alpha, x), (\beta, y), (\gamma, z) \in H$. Then $[(\beta, y) \circ (\alpha, x)] \circ [(\alpha, x) \circ (\gamma, z)] = \{[(\beta, y) \circ (\alpha, x)] \circ (\alpha, x)\} \circ (\gamma, z) \Leftrightarrow [\beta\alpha, y\alpha \cdot x] \circ [\alpha\gamma, x\gamma \cdot z] = \{[\beta\alpha, y\alpha \cdot x] \circ (\alpha, x)\} \circ (\gamma, z) \Leftrightarrow [\beta\alpha\alpha\gamma, (y\alpha \cdot x)\alpha\gamma \cdot (x\gamma \cdot z)] = \{[\beta\alpha\alpha, (y\alpha \cdot x)\alpha \cdot x]\} \circ (\gamma, z) \Leftrightarrow [\beta\alpha\alpha\gamma, (y\alpha\alpha\gamma \cdot x\alpha\gamma) \cdot (x\gamma \cdot z)] = \{[\beta\alpha\alpha, (y\alpha\alpha \cdot x\alpha) \cdot x]\} \circ (\gamma, z) \Leftrightarrow [\beta\alpha\alpha\gamma, (y\alpha\alpha\gamma \cdot x\alpha\gamma) \cdot (x\gamma \cdot z)] = \{\beta\alpha\alpha\gamma, [(y\alpha\alpha \cdot x\alpha) \cdot x]\gamma cdot z\} \Leftrightarrow [\beta\alpha\alpha\gamma, (y\alpha\alpha\gamma \cdot x\alpha\gamma) \cdot (x\gamma \cdot z)] = \{\beta\alpha\alpha\gamma, [(y\alpha\alpha\gamma \cdot x\alpha\gamma) \cdot x\gamma \cdot z]\} \Leftrightarrow (y\alpha\alpha\gamma \cdot x\alpha\gamma) \cdot (x\gamma \cdot z) = [(y\alpha\alpha\gamma \cdot x\alpha\gamma) \cdot x\gamma] \cdot z$. Replace y with $y(\alpha\alpha)^{-1}$ to get $(y \cdot x\alpha\gamma) \cdot (x\gamma \cdot z) = (y \cdot x\alpha\gamma) \cdot x\gamma \cdot z$. Put $\delta = \alpha\gamma$ to get $(y \cdot x\delta) \cdot (x\gamma \cdot z) = [(y \cdot x\delta) \cdot x\gamma] \cdot z \Leftrightarrow yR_{x\delta}R_{x\gamma \cdot z} = yR_{x\delta}R_{x\gamma}R_z \Leftrightarrow R_{x\delta}R_{x\gamma \cdot z} = R_{x\delta}R_{x\gamma}R_z$. Put $x = x\delta$ to get $R_{x\delta^2}R_{x\delta\gamma \cdot z} = R_{x\delta^2}R_{x\delta\gamma}R_z \Leftrightarrow R_xR_{x\delta\gamma \cdot z} = R_xR_{x\delta\gamma}R_z$. Put $\delta\gamma = \theta$ to get $R_xR_{x\theta \cdot z} = R_xR_{x\theta}R_z \Leftrightarrow R_xR_{(x \cdot z)\theta} = R_xR_{x\theta}R_z \Leftrightarrow \theta R_xR_{(x \cdot z)\theta} = \theta R_xR_{x\theta}R_z \Leftrightarrow R_x\theta R_{(x \cdot z)\theta} = R_x\theta R_{x\theta}R_z \Leftrightarrow R_xR_{x \cdot z}\theta = R_xR_x\theta R_z \Leftrightarrow R_xR_{x \cdot z}\theta = R_xR_xR_z\theta \Leftrightarrow R_xR_{x \cdot z} = R_xR_xR_z \Leftrightarrow yR_xR_{x \cdot z} = yR_xR_xR_z$ as required.

Theorem 17: Let $(G, \cdot)$ be a BCI-algebra with A-holomorph $(H, \circ)$ which is also a BCI-algebra. Then $(H, \circ)$ is an F_i-algebra if and only if $(G, \cdot)$ is an F_i-algebra; where $i = 42, 54$.

Proof: Let $(H, \circ)$ be an F_{42}-algebra, and let $(\alpha, x), (\beta, y), (\gamma, z) \in H$. Then $(I, 0) \circ [(\beta, y) \circ (\gamma, z)] = [(I, 0) \circ (\beta, y)] \circ (\gamma, z) \Leftrightarrow (I, 0) \circ [\beta\gamma, y\gamma \cdot z] = [I\beta, 0\beta \cdot y] \circ (\gamma, z) \Leftrightarrow [I\beta\gamma, 0\beta\gamma \cdot (y\gamma \cdot z)] = [I\beta\gamma, (0\beta \cdot y)\gamma \cdot z] \Leftrightarrow [\beta\gamma, 0\beta\gamma \cdot (y\gamma \cdot z)] = [\beta\gamma, (0\beta\gamma \cdot y\gamma) \cdot z] \Leftrightarrow 0\beta\gamma \cdot (y\gamma \cdot z) = (0\beta\gamma \cdot y\gamma) \cdot z \Leftrightarrow 0 \cdot (y\gamma \cdot z) = (0 \cdot y\gamma) \cdot z \Leftrightarrow y\gamma R_zL_0 = y\gamma L_0R_z \Leftrightarrow \gamma R_zL_0 = \gamma L_0R_z \Leftrightarrow R_zL_0 = L_0R_z \Leftrightarrow yR_zL_0 = yL_0R_z$ as required.

Let $(H, \circ)$ be an F_{54}-algebra, and let $(\beta, y), (\gamma, z) \in H$. Then $(\beta, y) \circ (\gamma, z) = (\beta, y) \circ (\gamma, z) \Leftrightarrow (\beta\gamma, y\gamma \cdot z) = (\beta\gamma, y\gamma \cdot z)$. Put $\beta\gamma = \delta$ to get $(\delta, y\gamma \cdot z) = (\delta, y\gamma \cdot z) \Leftrightarrow y\gamma \cdot z = y\gamma \cdot z \Leftrightarrow zL_{y\gamma} = zL_{y\gamma} \Leftrightarrow L_{y\gamma} = L_{y\gamma} \Leftrightarrow \gamma L_{y\gamma} = \gamma L_{y\gamma} \Leftrightarrow L_y\gamma = L_y\gamma \Leftrightarrow L_y = L_y \Leftrightarrow zL_y = zL_y$ as required.

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