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ON AMENABLE RESTRICTED SEMIGROUP

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ABSTRACT. We study amenability of restricted semigroup using some classes of inverse semigroups such as Clifford and Brandt semigroups. We particularly show that for a Clifford semigroup $S = \cup_{1=1}^{n} G_i$ with a finite set of idempotent elements, an inverse semigroup S is equivalent to a restricted semigroup S_r. It was equally shown that for a restricted semigroup $S_r = \cup_{i\in I} S_i$ where S_i is a Brandt semigroup, the amenability of a restricted semigroup is equivalent to the amenability of a Brandt semigroup.

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1. INTRODUCTION

After Day's first use of the word 'amenable' in his work in 1949, many authors have studied the notion of amenability in Banach algebras on different algebraic structures.
In 1972, this notion was initiated by B.E Johnson for Banach algebra when he showed that a locally compact group G is amenable if and only if a group algebra $L^1(G)$ is amenable as a Banach algebra. Many equivalent definitions which characterize the notion of amenability in Banach algebra have been given by many authors. It has thus been shown that the amenability of a locally compact group G is equivalent to many fundamental properties in harmonic analysis of the group. Some of these properties are the Folner condition, the fixed point property and the existence of a left (or right) invariant mean on $L^\infty(G)$ and so on.
Ever since this notion was initiated and developed by Johnson, this area of study in functional analysis has become a very fruitful area of research. Various other notions have also been introduced through modification of the original definition given by Johnson in

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1972. See [5] for some of these notions.
Massoud and Alireza in [4], recently introduced a restricted semigroup S_r, and its corresponding Banach algebra called the semigroup algebra, $l^1(S_r)$ on a restricted semigroup. The authors in [6], [7] and [8] respectively studied the character amenability, amenability and module amenability of restricted semigroup algebra, $l_r^1(S)$ and this class of Banach algebra.
In this work, we give a characterization of an amenable restricted semigroup using an invariant mean and show some of its topological properties.

2. PRELIMINARY

Throughout this work S is a discrete semigroup.
Let S be a semigroup.
An element $s^* \in S$ is called an inverse of $s \in S$ if $ss^*s = s$ and $s^*ss^* = s^*$.
An element $s \in S$ is called regular if there exists $t \in S$ with $sts = s$.
An element $s \in S$ is called completely regular if there exists $t \in S$ with $sts = s$ and $ts = st$.
If each $s \in S$ is a regular element, S is called regular.
S is called completely regular if each $s \in S$ is a completely regular element. Completely regular semigroups are those which can be regarded as the disjoint unions of their maximal subgroups.
S is called an inverse semigroup if S is regular and every element in S has a unique inverse.
An element $p \in S$ is called an idempotent if $p^2 = p$, the set of idempotents of S is denoted by $E(S)$.
S is called a semilattice if it commutes and $E(S) = S$.
S is commutative if $ab = ba$ for each $a, b \in S$.
An inverse semigroup S is called a Clifford semigroup if $ss^{-1} = s^{-1}s$ for each $s \in S$.
Let S be a Clifford semigroup and let $s \in S$. Then $s \in G_{ss^{-1}}$ and hence S is a disjoint union of the groups $G_p \quad (p \in E(S))$. Thus, $S = \cup_{p\in E(S)} G_p$ where $G_p s$ are the maximal subgroups of S.
An element $\mu \in l^\infty(S)^*$ is called left(right) invariant if $\mu(l_s x) = \mu x(\mu(r_s x) = \mu(x))$ for all $x \in l^\infty(S)$ and $s \in S$.
A semigroup S is left (right) amenable if it has left (right) invariant mean.
A semigroup S is called amenable if there is a mean μ on $l^\infty(S)$ which is both left and right invariant.
For any inverse semigroup S, the restricted product of elements s

and t of S is st if $s^*s = tt^*$ and undefined, otherwise. The set S with this product forms a discrete groupoid and if we adjoin a zero element 0 to this groupoid with $0^* = 0$, we get an inverse semigroup denoted by S_r, with the multiplication:

$$s \bullet t = \begin{cases} st & s^*s = tt^* \\ 0 & \text{otherwise} \end{cases}$$

$(s, t \in S \cup \{0\})$ which is called the **restricted semigroup** of an inverse semigroup S.
It is clear that $E(S_r) = E(S) \cup \{0\}$.
Suppose S is a $*$-semigroup, given a Banach space $l^1(S)$ with the usual l^1-norm, we set $\tilde{f}(x) = \overline{f(x)}$ and define the following multiplication on $l^1(S)$.

$$(f \bullet g)(s) = \sum_{s^*s=tt^*} f(st)g(t^*) \qquad (s \in S).$$

Then $(l^1(S), \bullet,)$ with the l^1-norm is a Banach $*$-algebra denoted by $l^1_r(S)$, called the restricted semigroup algebra of S.
Since S is discrete, $l^1_r(S)$ is a discrete semigroup algebra.

$$l^1_r(S) = \{f : S \to \mathbb{C} : \sum_{s \in S} |f(s)| < \infty\},$$

$||f||_1 = \sum_{s \in S} |f(s)|$.
For a restricted semigroup S_r of an inverse semigroup S, $l^1(S_r)$ is called the semigroup algebra on restricted semigroup S_r.

3. MAIN RESULTS

Proposition 3.1 : Let $\Psi : S \to S_r$ be a canonical embedding of an inverse semigroup S into a restricted semigroup S_r. Then $ker\Psi$ is a closed ideal of S_r.

Proof: We define $ker\Psi = \{x \in S : \Psi(x) = 0\}$. Let $I = ker\Psi$. If I is a zero ideal of S_r, then the result is trivial. Suppose I is non-zero. Let $z \in I$, then $x.z = xz$ if $x^*x = zz^*$ and 0 otherwise, $(x \in S_r)$.
Suppose $z = 0$, then $x.0 = 0.x = 0 \quad (x^*x \neq zz^*)$.
If $x = 0$, we have $0.z = z.0 = 0 \quad (x^*x \neq zz^*)$ (by the second condition of the restricted product). Clearly, I is an ideal of S_r as $0 \in I \subset S_r$.
Now suppose $\{z_n\}_{n=1}^{\infty}$ is a sequence in I, we shall show that $I =$

$ker\Psi$ is a closed ideal of S_r by showing that $z_n \to x \in I$.
We have

$$\begin{aligned} lim_{n\to\infty} z_n = x \Rightarrow lim_{n\to\infty}||x - z_n|| = 0 \\ = |x - y + y - z_n| \quad (y \in S_r) \\ 0 = lim_{n\to\infty}|x - y| + lim_{n\to\infty}|y - z_n| \\ 0 = lim_{n\to\infty}|x - y| + 0 \Rightarrow |x - y| \to 0 \end{aligned}$$

This is a contradiction. So $ker\Psi = I$. Since $x \neq y \neq 0 \in S_r$, then $x \in I = ker\Psi$.
Hence the proof is complete.
Proposition 3.2 : Let S be an inverse semigroup and let S_r be a restricted semigroup. Then S_r is amenable if and only if S is amenable.
Proof : Let $\Phi : S_r \to S$ be a norm decreasing map. We recall that $S_r = S \cup \{0\}$. Now, let the map $\Phi : S_r \to S$ be defined by $\Phi(s + I) = s \quad \forall s \in S, \quad I = ker\Phi \subset S_r$. Then

$$\begin{aligned} \Phi((s + I)(r + I)) = \Phi((s + I) \bullet (r + I)) = \Phi(sr + I) \\ = \Phi(s + I)\Phi(r + I) = sr. \end{aligned}$$

This shows that Φ is a homomorphism. Now suppose S_r is amenable, then by [1, section 4, No C], S is also amenable.
Conversely, suppose S is amenable, then there exists an invariant mean μ on $l^\infty(S)$. Let $\Phi : S_r \to S$ be defined by $\Phi(s_r) = s$ where $s_r = s+I \quad \forall s \in S$. For $f \in l^\infty(S)$, we have $\mu(fs) = \mu f(\Phi_{s_r}) \quad \forall s \in S$.
Let M be a continuous linear functional on $l^\infty(S_r)$ and $\hat{f} \in l^\infty(S_r)$, then

$$M(\hat{f}s_r) = M\hat{f}(s_r) = \mu f \Phi(s_r).$$

Now we have $\mu f \Phi = M\hat{f} = \mu(f) \qquad (||\Phi|| = 1)$.
Thus $M(\hat{f}) = \mu(f)$ for $\hat{f} \in l^\infty(S_r)$. Clearly, M is an invariant mean on $l^\infty(S_r)$.
We then conclude that S_r is an amenable restricted semigroup. Hence the result is complete.

Let G be a group, a Brandt semigroup S over a group G with index set J is the semigroup consisting of elementary $J \times J$ matrices over $G \cup \{0\}$ and a zero matrix $\{0\}$. We write $S = \{(g)_{ij} : g \in G, i, j \in J\} \cup \{\{0\}\}$, with multiplication given by:

$$(g)_{ij}(h)_{kl} = \begin{cases} (gh)_{il} & \text{if } j = k \\ 0 & \text{if } j \neq k. \end{cases}$$

The Brandt semigroup is an inverse semigroup.
For an inverse semigroup S, it was shown in [7] that the restricted semigroup $S_r = \bigcup_{i \in I} S_i$ for Brandt semigroup S_i with $S_i \cap S_j = S_i S_j = \{0\}$, if $i \neq j$.
Corollary 3.3 : Let S_r be a restricted semigroup and let $S_r = \cup_{i \in I} S_i$ where S_i is a Brandt semigroup and each $S_i \cap S_j = S_i S_j = \{0\}$ if $i \neq j$. Then S_r is amenable if and only the Brandt semigroup is amenable.
Proof: Suppose S_i is amenable for each i, then its union is amenable and so the restricted semigroup S_r is amenable [1, section 4, No F]. The converse follows easily from Proposition 3.2.
Proposition 3.4: Let $S = M^0(G, I, n)$ be a Brandt semigroup for some group G. Then the restricted semigroup S_r is amenable if and only if S is amenable.
Proof: By Example 1.2 [7], $S = S_r$. It then follows that S_r is amenable if and only if S is amenable.
Theorem 3.5: Let S be a Clifford semigroup with $E(S)$ finite such that $S = \cup_{i=1}^{n} G_i$. Then the following conditions are equivalent.
(i) Each group G_i is amenable,
(ii) S is an amenable semigroup,
(iii) S_r is amenable.
Proof: $(i) \Rightarrow (ii)$ Suppose each G_i is amenable. Since $E(S)$ is finite, S is a union of finite groups. Then by [1, section 4, No I], S is an amenable semigroup.
$(ii) \Rightarrow (iii)$ Suppose S is amenable. If $S_r = \cup_{i \in I} S_i$ where S_i is a Brandt semigroup and $S_i \cap S_j = S_i S_j = \{0\}$ if $i \neq j$. Then by Corollary 3.3, S_r is amenable.
$(iii) \Rightarrow (i)$ Suppose S_r is amenable, if S is a Brandt semigroup, then the result follows from Proposition 3.4. The following result in [7] gives another example of amenable restricted semigroup.
Example 3.6 : For every finite meet semilattice S, we have $S_r = \cup_{i=1}^{n} S_i$, where $S_i = \{0, a_i\}$ with $a_i \in S$.
Here S_i is a Brandt semigroup with corresponding $G_i = \{a_i\}$.
S_r is an amenable restricted semigroup.
Corollary 3.7: Let $S_r = \cup_{i \in I} S_i$ be an amenable restricted semigroup where S_i is a Brandt semigroup. Let $S = \cup_{i=1}^{n} G_i$ be a Clifford semigroup. Then the Clifford semigroup S is amenable if and only if the Brandt semigroup is amenable.
Proof: By Corollary 3.3 and Theorem 3.5, the result follows.

Theorem 3.8: Let $S = M^0(G, I, n)$ be a Brandt semigroup for some group G. Let $\xi : S \to G_p$ be a canonical homomorphism. Then S_r is amenable if and only if G_p is amenable.

Proof: Suppose S_r is amenable, then by Proposition 3.4, the Brandt semigroup S is amenable. It then follows from [3, Theorem 1] that G_p is amenable. Conversely, suppose G_p is amenable and it is a maximal homomorphic image of S. By [3, Theorem 1] and Proposition 3.3, the result follows.

Example 3.9: Let S be an inverse semigroup and S_r be a restricted semigroup. Suppose there is a short exact sequence :$0 \to I \xrightarrow{\psi} S_r \xrightarrow{\phi} S_r/I \to 0$. Then S_r is amenable if and only if I and S_r/I are amenable.

Proof: Clearly, the sequence is exact at S_r if $ker\phi = Im\psi$ and $ker\psi \subset I \subset S_r$ [2]. It is well-known that if there is an exact sequence of discrete groups: $H \hookrightarrow G \to K, G$ is amenable if and only if H and K are amenable.

Using this analogous result, then S_r is amenable if and only if I and S_r/I are amenable.

4. CONCLUDING REMARKS

The results obtained in this work show that a restricted semigroup S_r has an invariant mean structure as well as some topological properties.

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