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HIGHER ORDER BOUNDARY VALUE PROBLEMS WITH INTEGRAL BOUNDARY CONDITIONS AT RESONANCE ON THE HALF-LINE

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ABSTRACT. In this paper, we study the following higher order boundary value problems at resonance on the half-line:

$$(q(t)u^{(n-1)}(t))' = f(t, u(t), u'(t), \cdots, u^{(n-1)}(t)), \quad \text{a.e. } t \in (0, \infty),$$

subject to the boundary conditions

$$u^{(n-2)}(0) = \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} u(t)dt,\ u^{(i)}(0) = 0, i = 1, 2, \cdots, (n-3),$$

$$\lim_{t\to\infty} q(t)u^{(n-1)}(t) = 0.$$

By using coincidence degree arguments we establish some existence criteria under the resonant condition $\sum_{i=1}^{m-2} \alpha_i \xi_i^{n-1} = (n-1)!$.

Keywords and phrases: Boundary value problems, coincidence degree, half-line, higher order, integral boundary conditions, resonance.
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1. INTRODUCTION

In this article, we are concerned with the existence of solutions for the following higher-order boundary value problems on the half-line of the form:

$$(q(t)u^{(n-1)}(t))' = f(t, u(t), u'(t), \cdots, u^{(n-1)}(t)), \quad \text{a.e. } t \in (0, \infty), \quad (1)$$

$$u^{(n-2)}(0) = \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} u(t)dt, u^{(i)}(0) = 0, i = 1, 2, \cdots, (n-3),$$

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$$\lim_{t\to\infty} q(t)u^{(n-1)}(t) = 0, \tag{2}$$

where $f : [0,\infty] \times \mathbb{R}^n \to \mathbb{R}$ is a Caratheodory's function, $\alpha_i \in \mathbb{R}$ $(1 \leq i \leq m)$, $0 < \xi_1 < \xi_2 < \xi_3 < \cdots \xi_m < 1$, $q > 0$, $q \in C[0,\infty) \cap C^{n-2}(0,\infty)$, $\frac{1}{q} \in L^1[0,\infty]$ and
$\sum\limits_{i=1}^{m} \alpha_i \xi_i^{n-1} = (n-1)!$. The boundary value problem (1)-(2) is called a problem at resonance if $Lu = (q(t)u^{(n-1)(t))} = 0$ has nontrivial solutions under the boundary conditions (2), that is when dim $\ker L \geq 1$. In this case, the condition $\sum\limits_{i=1}^{m} \alpha_i \xi_i^{n-1} = (n-1)!$ is critical since we require a nontrivial kernel for our differential operator, if (1)-(2) is to be at resonance. In recent years there has been an increasing interest in the study of boundary value problems on finite intervals especially for second, third and fourth order boundary value problems. See [7, 9, 10, 13, 15, 16] and references therein. To the best of our knowledge there are few papers on higher order boundary value problems on the half-line or unbounded domains with integral boundary conditions. For some recent literature on higher order boundary value problems see [1, 3, 4, 5, 6, 8, 11, 12] and references therein. Boundary value problems with integral boundary conditions are encountered in various applications such as in population dynamics, blood flow models and cellular systems.

In this work, we will utilise the coincidence degree theory of Mawhin [14] to derive our existence results.

In section 2 of this paper, we shall provide some background definitions and some preliminary lemmas while section 3 is devoted to stating and proving the main existence theorems. in section 4, we will validate our main existence results with an example.

2. PRELIMINARY

We provide here some background definitions and coincidence degree theory.

Definition 1: Let X and Z be two real Banach spaces. A linear mapping
$L : \text{Dom } L \subset X \to Z$ is called a Fredholm mapping if

(i) $\ker L$ has a finite dimension,
(ii) Im L is closed and has a finite codimension.

If L is a Fredholm mapping then the Fredholm index is given by Ind$L$ =dim $\ker L-$ codim Im L. This definition implies that there exists continuous projections $P : X \to X$, $Q : Z \to Z$ such that

Im P = $\ker L$, $\ker Q$ = Im L, $X = \ker L \oplus \ker P$, Z = Im $L \oplus$ Im Q and that the mapping $L|_{\text{Dom } L \cap \ker P}$: Dom $L \cap \ker P \to \text{Im}L$ is invertible. The inverse is denoted by K_p while the generalised inverse is denoted by $K_{P,Q} : Z \to$ Dom $L \cap \ker P$ and defined by $K_{P,Q} = K_p(I - Q)$.

Definition 2: Let L : Dom $L \subset X \to Z$ be a Fredholm mapping. E a metric space and $N : E \to Z$ be a mapping. N is said to be L-compact on E if $QN : E \to Z$ and $Kp(I - Q) : E \to X$ are compact on E. N is L-completely continuous if it is L-compact on every bounded $E \subset X$.

The boundary value problem (1)-(2) will be formulated in the abstract form

$Lu = Nu$ and the following coincidence degree theorem of Mawhin [14] will be employed to derive existence of solutions.

Theorem 1:[14] Let $E \subset X$ be open and bounded and let L be a Fredholm operator of index zero and N be L-compact on $\bar{E}$. Assume that the following conditions are satisfied:

(1) $Lu \neq \lambda Nu$ for every $(u, \lambda) \in [(\text{Dom } L \setminus \ker L) \cap \partial E] \times (0, 1)$,
(2) $Nu \notin$ Im L for $u \in \ker L \cap \partial E$,
(3) $\deg(QN|_{\ker L}, E \cap \ker L, 0) \neq 0$ with $Q : Z \to Z$ a continuous projection such that $\ker Q$ = Im L.

The equation $Lu = Nu$ has at least one solution in Dom $L \cap \bar{E}$.

Definition 3: The map $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ is an $L^1[0, \infty)$ Caratheodory if the following conditions are satisfied:

(1) for each $u \in \mathbb{R}^n$, $f(t, u)$ is Lebesgue measurable,
(2) for a.e $t \in [0, \infty)$, $f(t, u)$ is continuous on $\mathbb{R}^n$,
(3) for each $r > 0$ there exists $\varphi_r \in L^1[0, \infty)$ such that for a.e. $t \in [0, \infty)$ and every u such that $|u| \leq r$ we have $|f(t, u)| \leq \varphi_r(t)$.

Let $AC[0, \infty)$ denote the space of absolutely continuous functions on $[0, \infty)$. Let $X = \{u \in C^{n-1}[0, \infty), u, u', \cdots, u^{(n-2)}, qu^{(n-2)} \in AC[0, \infty), \lim_{t \to \infty} e^{-t}|u^{(i)}(t)|$ exists , $0 \leq i \leq n-1$, $(qu^{(n-1)})' \in L^1[0, \infty)\}$ endowed with the norm

$\| u \| = \max_{0 \leq i \leq n-1} \left[\sup_{t \in [0,\infty)} e^{-t}|u^{(i)}(t)| \right]$. Then X is a Banach space. Let $Z = L^1[0, \infty)$ with the norm $\| y \|_1 = \int_0^\infty |y(t)| dt$, $y \in Z$. **Theorem**

2: [1] Let F be a subset of $C_\infty = \{y \in C([0,\infty)) : \lim_{t\to\infty} y(t)\text{exists}\}$ that is equipped with the norm $\| y \|_\infty = \sup_{t\in[0,\infty)} |y(t)|$. Then F is relatively compact if the following conditions hold:

(1) F is bounded in X,
(2) the functions belonging to F are equicontinuous on any compact subinterval of $[0,\infty)$,
(3) the functions from F are equicontinous at ∞.

We will use the following adaptation of theorem (2) to show that $K_p(I-Q) : \bar{E} \to X$ is relatively compact in X.

Lemma 1: [8]. Let $D \subset X$, then D is relatively compact in X if the following conditions are satisfied:

(1) D is bounded in X,
(2) the family $W = \{w_i : w_i(t) = e^{-t}u^{(i)}(t), t \geq 0, u \in D\}$ is equicontinuous on any compact subinterval of $[0,\infty)$ for $i = 0,1,\cdots,n-1$,
(3) the family $W = \{w_i : w_i(t) = e^{(-t)}u^i(t),\ t \geq 0,\ u \in D\}$ is equiconvergent at ∞ for $i = 0,1,\cdots,n-1$.

Let $L : \text{Dom } L \subset X \to Z$ be defined by $Lu = (q(t)u^{(n-1)})'$, $t \in [0,\infty)$ where $\text{Dom } L = \{u \in X : u^{(n-2)}(0) = \sum_{i=1}^m \alpha_i \int_0^{\xi_i} u(t)dt,\ u^{(i)}(0) = 0,\ i = 0,1,2,\cdots,n-3, \lim_{t\to\infty} q(t)u^{(n-1)}(t) = 0\}$. We define the operator $N : X \to Z$ by $Nu(t) = f(t,u(t),\cdots,u^{(n-1}(t)),\ t \in [0,\infty)$. Then (1)-(2) takes the form

$$Lu = Nu. \tag{3}$$

Lemma 2: If $\sum_{i=1}^m \alpha_i \xi_i^{n-1} = (n-1)!$ and
$\sum_{i=1}^m \alpha_i \int_0^{\xi_i}\int_0^t\int_0^{\tau_n}\cdots\int_0^{\tau_3} \frac{e^{-s}}{q(s)} ds d\tau_3\cdots d\tau_n dt \neq 0$
then

(i) $\ker L = \{u \in \text{Dom } L : u(t) = dt^{n-2}, d \in \mathbb{R},\ t \in [0,\infty)\}$,
(ii) $\text{Im } L = \Bigg\{ y \in Z :$
$\sum_{i=1}^m \alpha_i \int_0^{\xi_i}\int_0^t\int_0^{\tau_n}\cdots\int_0^{\tau_3} \frac{1}{q(\tau_2)} \int_{\tau_2}^\infty y(\tau_1)d\tau_1 d\tau_2\cdots d\tau_n dt = 0 \Bigg\}.$

Proof: (i) For $u \in \ker L$, we have $(q(t)u^{(n-1)}(t))' = 0$. Then we obtain

$$u^{(n-1)}(t) = 0$$

or

$$u(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_{n-2} t^{n-2},$$

where $a_i (i = 0, 1, \cdots, n-2) \in \mathbb{R}$. From $u^{(i)}(0) = 0,\ i = 0, 1, \cdots,$ $n-3$ we derive $a_0 = a_1 = a_2 = \cdots = a_{n-3} = 0$. Hence

$$\ker L = \{u \in \text{Dom } L : u = dt^{n-2},\ d \in \mathbb{R},\ t \in [0, \infty)\}.$$

(ii) We show that $\text{Im } L = \Bigg\{ y \in Z :$

$$\sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} \int_0^{t} \int_0^{\tau_n} \cdots \int_0^{\tau_3} \frac{1}{q(\tau_2)} \int_{\tau_2}^{\infty} y(\tau_1) d\tau_1 \cdots d\tau_n dt = 0 \Bigg\}.$$

To do this, we consider the problem

$$(q(t)u^{(n-1)}(t))' = y(t), \tag{4}$$

subject to the boundary conditions (2). We show that (4) subject to (2) has a solution $u(t)$ if and only if

$$\sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} \int_0^{t} \int_0^{\tau_n} \cdots \int_0^{\tau_3} \frac{1}{q(\tau_2)} \int_{\tau_2}^{\infty} y(\tau_1) d\tau_1 \cdots d\tau_n dt = 0. \tag{5}$$

If $u(t)$ has a solution $u(t)$ satisfying (2) then from (4) we obtain

$$u(t) = -\int_0^{t} \int_0^{\tau_n} \cdots \int_0^{\tau_3} \frac{1}{q(\tau_2)} \int_{\tau_2}^{\infty} y(\tau_1) d\tau_1 \cdots d\tau_n + \frac{u^{(n-2)}(0)}{(n-2)!} t^{n-2}. \tag{6}$$

$$\begin{aligned} u^{(n-2)}(0) &= \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} u(t) dt \\ &= -\sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} \int_0^{t} \int_0^{\tau_n} \cdots \int_0^{\tau_3} \frac{1}{q(\tau_2)} \int_{\tau_2}^{\infty} y(\tau_1) d\tau_1 \cdots d\tau_n dt \\ &+ \frac{u^{(n-2)}(0)}{(n-2)!} \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} t^{(n-2)} dt, \end{aligned}$$

since $\sum\limits_{i=1}^{m} \alpha_i \xi_i^{n-1} = (n-1)!$. We derive that

$$\sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} \int_0^{t} \int_0^{\tau_n} \cdots \int_0^{\tau_3} \frac{1}{q(\tau_2)} \int_{\tau_2}^{\infty} y(\tau_1) d\tau_1 \cdots d\tau_n dt = 0.$$

If (5) holds then setting

$$u(t) = dt^{n-2} - \int_0^t \int_0^{\tau_n} \cdots \int_0^{\tau_3} \frac{1}{q(\tau_2)} \int_{\tau_2}^{\infty} y(\tau_1) d\tau_1 \cdots d\tau_n, \qquad (7)$$

where d is an arbitrary constant, then $u(t)$ is a solution of (4) and is such that (2) is satisfied.

3. THE HEART OF THE MATTER

In this section, we shall establish existence results for (1)-(2). We assume the following conditions:

(A0) $\sum_{i=1}^{m} \alpha_i \xi_i^{n-1} = (n-1)!$ and
$$\sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} \int_0^{\tau_n} \cdots \int_0^{\tau_3} \frac{e^{-\tau_2}}{q(\tau_2)} d\tau_2 d\tau_3 \cdots d\tau_n \neq 0,$$

(A1) There exists constant $M_1 > 0$ such that for each $u \in \text{Dom } L \backslash \ker L$ satisfying $|u^{(n-2)}(t)| > M_1$ for all $t \in [0, \infty)$ we have $QNu \neq 0$,

(A2) There exist positive functions $a_i (i = 1, 2, \cdots, n),\ r \in L^1[0, \infty)$ such that for all $(u_1, u_2, \cdots, u_n) \in \mathbb{R}^n,\ t \in [0, \infty)$ and $\theta \in [0, 1)$ the following inequality holds

$$|f(t, u_1, \cdots, u_n| \leq e^{-t} \sum_{i=1}^{n-1} \alpha_i(t) |u_i(t)| + e^{-t} a_n(t) |u_n(t)|^{\theta} + r(t), \quad (8)$$

(A3) There exists a constant $M_2 > 0$ such that for every $d \in \mathbb{R}$ with $|d| > M_2$ we have either

$$\begin{aligned} &d \cdot \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} \int_0^{\tau_n} \cdots \int_0^{\tau_3} \\ &\frac{1}{q(t_2)} \int_{t_2}^{\infty} f(t_1, dt_1^{n-1}, \cdots, d(n-2)!, 0) d\tau_1 d\tau_2 \cdots d\tau_n > 0 \end{aligned} \qquad (9)$$

or

$$d \cdot \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} \int_0^{\tau_n} \cdots \int_0^{\tau_3} \frac{1}{q(t_2)} \int_{t_2}^{\infty} f(t_1, dt_1^{n-1}, \cdots, d(n-2)!, 0) d\tau_1 d\tau_2 \cdots d\tau_n < 0. \tag{10}$$

Theorem 3: If (A0)-(A3) holds then the boundary value problem (1.1)-(1.2) has at least one solution provided

$$\sum_{i=1}^{m} \| a_i \|_1 < \frac{1}{2D_n \| \frac{1}{q} \|_1}. \tag{11}$$

To prove theorem 3, we require the following lemmas

Lemma 3: If the conditions of Lemma 2 holds then:

(i) $L : \text{Dom } L \subset X \to Z$ is a Fredholm operator of index zero and the linear continuous projection $Q : Z \to Z$ can be written as

$$(Qy)(t) = h(t) \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} \int_0^{t} \int_0^{\tau_n} \cdots \int_0^{\tau_3} \frac{1}{q(\tau_2)} \int_{\tau_2}^{\infty} y(\tau_1) d\tau_1 \cdots d\tau_n dt$$

where

$$h(t) = \frac{e^{-t}}{\sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} \int_0^{t} \int_0^{\tau_n} \cdots \int_0^{\tau_3} \frac{e^{-s}}{q(s)} ds d\tau_3 \cdots d\tau_n dt}. \tag{12}$$

(ii) The inverse $K_p : \text{Im } L \to \text{Dom } L \cap \ker P$ is of the form

$$K_p y(t) = -\int_0^{t} \int_0^{\tau_n} \cdots \int_0^{\tau_3} \frac{1}{q(\tau_2)} \int_{\tau_2}^{\infty} y(\tau_1) d\tau_1 \cdots d\tau_n \tag{13}$$

with

$$\begin{aligned} \| K_p y \| &\leq \max \left[\max_{0 \leq i \leq n-2} \left(\sup_{t \in [0,\infty)} e^{-t} t^{n-2-i} \right), \frac{\| \frac{1}{q} \|_\infty}{\| \frac{1}{q} \|_1} \right] \| \frac{1}{q} \|_1 \| y \|_1 \\ &= D_n \| \frac{1}{q} \|_1 \| y \|_1 . \end{aligned} \tag{14}$$

Proof: (i) For any $y \in Z$, we define the projection Q as

$$(Qy)(t) = h(t)\sum_{i=1}^{m}\alpha_i \int_0^{\xi_i}\int_0^{t}\int_0^{\tau_n}\cdots\int_0^{\tau_3}\frac{1}{q(\tau_2)}\int_{\tau_2}^{\infty} y(\tau_1)d\tau_1\cdots d\tau_n dt,$$

then

$$\begin{aligned}(Q^2y)(t) &= Q(Qy)(t)\\ &= h(t)\sum_{i=1}^{m}\alpha_i \int_0^{\xi_i}\int_0^{t}\int_0^{\tau_n}\cdots\int_0^{\tau_3}\frac{1}{q(\tau_2)}\int_{\tau_2}^{\infty} y(\tau_1)d\tau_1\cdots d\tau_n dt\\ &\times \frac{\sum_{i=1}^{m}\alpha_i \int_0^{\xi_i}\int_0^{t}\int_0^{\tau_n}\cdots\int_0^{\tau_3}\frac{e^{-s}}{q(s)}ds d\tau_3\cdots d\tau_2 dt}{\sum_{i=1}^{m}\alpha_i \int_0^{\xi_i}\int_0^{t}\int_0^{\tau_n}\cdots\frac{e^{-s}}{q(s)}ds d\tau_3\cdots d\tau_2 dt}\\ &= Qy.\end{aligned}$$

For $y \in Z$, $y - Qy \in \ker Q = \text{Im } L$ and $Qy \in \text{Im } Q$. Hence, $Z = \text{Im } L + \text{Im Q}$. Since $\text{Im L} \cap \text{Im } Q = \{0\}$, we have $Z = \text{Im } L \oplus \text{Im } Q$. Therefore $\dim \ker L = \dim \text{Im } Q = 1$. Thus, L is a Fredholm operator of index zero.
(ii) Taking $P : X \to X$ as

$$Pu(t) = \frac{u^{(n-2)}(0)}{(n-2)!}t^{n-2},\ t \in [0,\infty), \tag{15}$$

then it is obvious that $\text{Im } P = \ker L$ and $P^2u(t) = Pu(t)$. We can write $u \in X$ as $u(t) = Pu(t) + (u(t) - Pu(t))$. Thus $X = \ker P + \ker L$. Since $\ker L \cap \ker P = 0$, we obtain that $X = \ker P \oplus \ker L$.

For $y \in \text{Im } L$, we have

$$(LK_p)y(t) = [q(t)(K_p y(t))^{(n-1)}]' = y(t),\ t \in [0,\infty)$$

and for $u \in \text{Dom } L \cap \ker P$ one has

$$\begin{aligned}(K_pL)u(t) &= K_p[q(t)u^{(n-1)}(t)]' \\ &= -\int_0^t\int_0^{\tau_n}\cdots\int_0^{\tau_3}\frac{1}{q(\tau_2)}\int_{\tau_2}^{\infty}[q(\tau_1)u^{(n-1)}(\tau_1)]'d\tau_1 d\tau_2\cdots d\tau_n \\ &= \int_0^t\int_0^{\tau_n}\cdots\int_0^{\tau_3}u^{(n-1)}(\tau_2)d\tau_2\cdots d\tau_n \\ &= u(t) - t^{n-2}\frac{u^{(n-2)}(0)}{(n-2)!}\end{aligned}$$

since $u \in \text{Dom } L \cap \ker P$, $(K_pL)u(t) = u(t)$.
Using the definition of K_p we have

$$e^{-t}|(K_py)^{(i)}(t)| \le \max_{0\le i\le n-2}\left(\sup_{t\in[0,\infty)} e^{-t}t^{n-2-i}\right) \parallel \frac{1}{q} \parallel_1 \parallel y \parallel_1 .$$

For $i = n-1$

$$e^{-t}|(K_py)^{(n-1)}(t)| \le \sup_{t\in(0,\infty)} e^{-t} \parallel \frac{1}{q} \parallel_\infty \parallel y \parallel_1 . \tag{16}$$

Therefore,

$$\begin{aligned}\parallel K_py \parallel &\le \max\left[\max_{0\le i\le n-2}\left(\sup_{t\in[0,\infty)} e^{-t}t^{n-2-i}\right), \frac{\parallel \frac{1}{q} \parallel_\infty}{\parallel \frac{1}{q} \parallel_1}\right] \parallel \frac{1}{q} \parallel_1 \parallel y \parallel_1 \\ &= D_n \parallel \frac{1}{q} \parallel_1 \parallel y \parallel_1 .\end{aligned} \tag{17}$$

Lemma 4: If f is a Caratheodory's function then N is L-compact.
Proof: Let $D \subset X$ be bounded with $r = \sup\{\parallel u \parallel : u \in D\}$. Since $f : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ satisfies the Caratheodory's condition with respect to $L^1[0,\infty)$ there exists a Lebesgue integrable function φ_r such that

$$|Nu(t)| = |f(t, u(t), u'(t), \cdots, u^{(n-1)}(t)| \le \varphi_r(t),\ t \in [0,\infty),\ u \in D,$$

$\| Nu \|_1 \leq \int\limits_0^\infty |Nu(s)|ds \leq \int_0^\infty |\varphi_r|dt = \| \varphi \|_1,$

$$\begin{aligned}
\| QNu \|_1 &\leq \int\limits_0^\infty |QNu(s)|ds \\
&\leq \int\limits_0^\infty |h(t)| \Big[\sum_{i=1}^m |\alpha_i| \int\limits_0^{\xi_i} \int\limits_0^{s} \int\limits_0^{\tau_n} \cdots \int\limits_0^{\tau_3} \\
&\quad \frac{1}{q(\tau_2)} \int\limits_{\tau_2}^\infty |Nu(\tau_1)| d\tau_1 d\tau_2 \cdots d\tau_n ds \Big] dt \\
&\leq \| h \|_1 \| \varphi_r \|_1 \ \Big[\sum_{i=1}^m |\alpha_i| \int\limits_0^{\xi_i} \int\limits_0^{s} \int\limits_0^{\tau_n} \cdots \int\limits_0^{\tau_3} \\
&\quad \frac{1}{q(\tau_2)} \int\limits_{\tau_2}^\infty |Nu(\tau_1)| d\tau_1 d\tau_2 \cdots d\tau_n ds dt \Big] \\
&\leq \| h \|_1 \| \varphi_r \|_1 \sum_{i=1}^m |\alpha_i| \frac{\xi_i^n}{n!} \| \frac{1}{q} \|_1 = B_n < \infty.
\end{aligned}$$

From (17) we have

$$\begin{aligned}
\| K_p(I-Q)Nu \| &\leq D_n \| \frac{1}{q} \|_1 \| (I-Q)Nu \|_1 \\
&\leq D_n \| \frac{1}{q} \|_1 \| Nu \|_1 + D_n \| \frac{1}{q} \|_1 \| QNu \|_1 \\
&\leq D_n \| \frac{1}{q} \|_1 [\| \varphi_r \| + B_n].
\end{aligned}$$

Thus $K_p(I-Q)N(D)$ is uniformly bounded in X.
Let $u \in D$ and $t_1,\, t_2 \in [0, M],\ M \in (0,\infty)$ with $t_1 < t_2$. Then

$$\begin{aligned}
&|e^{-t_2}(K_p(I-Q)Nu(t_2)) - e^{-t_1}(K_p(I-Q)Nu(t_1))| \\
&= \left| \int_{t_1}^{t_2} [e^{-s}(K_p(I-Q)Nu(s))]' ds \right| \\
&= \left| \int_{t_1}^{t_2} [-e^{-s}(K_p(I-Q)Nu(s))ds + e^{-s}(K_p(I-Q)Nu(s))']ds \right| \\
&\leq 2(t_2-t_1) \parallel K_p(I-Q)Nu \parallel \\
&\leq 2(t_2-t_1)D_n \parallel \frac{1}{q} \parallel_1 [\parallel \varphi_r + B_n] \to 0 \text{ as } t_1 \to t_2.
\end{aligned}$$

For $0 \leq i \leq n-2$

$$\begin{aligned}
&|e^{-t_2}(K_p(I-Q)Nu)^{(i)}(t_2) - e^{-t_1}(K_p(I-Q)Nu)^{(i)}(t_1)| \\
&= \left| \int_{t_1}^{t_2} [e^{-s}(K_p(I-Q)Nu)^{(i)}(s)]' ds \right| \\
&= \left| \int_{t_1}^{t_2} [-e^{-s}(K_p(I-Q)Nu)^{(i)}(s))ds + e^{-s}(K_p(I-Q)Nu)^{(i+1)}(s)]ds \right| \\
&\leq 2(t_2-t_1) \parallel K_p(I-Q)Nu \parallel \\
&\leq 2(t_2-t_1)D_n \parallel \frac{1}{q} \parallel_1 [\parallel \varphi_r\|_1 + B_n] \to 0 \text{ as } t_1 \to t_2.
\end{aligned}$$

For $i = n-1$

$$
\begin{aligned}
&|e^{-t_2}(K_p(I-Q)Nu)^{(n-1)}(t_2) - e^{-t_1}(K_p(I-Q)Nu)^{(n-1)}(t_1)| \\
&\quad = \left| \frac{e^{-t_2}}{q(t_2)} \int_{t_2}^{\infty} (I-Q)Nu(s)ds - \frac{e^{-t_1}}{q(t_1)} \int_{t_1}^{\infty} (I-Q)Nu(s)ds \right| \\
&\quad \leq \left| \frac{e^{-t_2}}{q(t_2)} - \frac{e^{-t_1}}{q(t_1)} \right| \int_{t_2}^{\infty} |(I-Q)Nu(s)|ds \\
&\qquad + \frac{1}{q(t_1)} \int_{t_1}^{t_2} |(I-Q)Nu(s)|ds \\
&\quad \leq \| \frac{1}{q} \|_{\infty}^2 |q(t_1)e^{-t_2} - q(t_2)e^{-t_1}| \, \| (I-Q)Nu \|_1 \\
&\qquad + \| \frac{1}{q} \|_{\infty} \int_{t_1}^{t_2} |(I-Q)Nu(s)|ds \\
&\quad \leq \| \frac{1}{q} \|_{\infty}^2 |q(t_1)e^{-t_2} - q(t_2)e^{-t_1}| (\| \varphi_r \|_1 + B_n) \\
&\quad + \| \frac{1}{q} \|_{\infty} \int_{t_1}^{t_2} \Bigg[|\varphi_r(s)|ds \\
&\quad + |h(\tau)| \sum_{i=1}^{m} |\alpha_i| \int_0^{\xi_i} \int_0^{s} \int_0^{\tau_n} \cdots \int_0^{\tau_3} \frac{1}{q(\tau_2)} d\tau_2 d\tau_3 \cdots ds \Bigg] \\
&\quad \to 0 \text{ as } t_1 \to t_2.
\end{aligned}
$$

Thus $K_p(I-Q)N(D)$ is equicontinuous on every compact subset of $[0,\infty)$. We now show that $K_p(I-Q)N(D)$ is equiconvergent at infinity.

For $u \in D$, we have for $0 \leq i \leq n-2$

$$
\begin{aligned}
&|e^{-t}(K_p(I-Q)^{(i)}Nu(t)| \\
&\quad = |e^{-t} \int_0^{t} \int_0^{\tau_n} \cdots \int_0^{\tau_3} \frac{1}{q(\tau_2)} \int_{\tau_2}^{\infty} (I-Q)Nu(\tau) d\tau d\tau_2 \cdots d\tau_n \\
&\quad \leq e^{-t} t^{n-2-i} \| \frac{1}{q} \|_{\infty} [\| Nu \|_1 + \| QNu \|_1] \to 0 \text{ as } t \to \infty.
\end{aligned}
$$

For $i = n-1$

$$\begin{aligned}
&|e^{-t}(K_p(I-Q)^{(n-1)}Nu(t)| \\
&\quad = |e^{-t}\frac{1}{q(t)}\int_t^\infty (I-Q)Nu(\tau)d\tau| \\
&\quad \leq e^{-t} \parallel \frac{1}{q} \parallel_\infty [\parallel Nu \parallel_1 + \parallel QNu \parallel_1] \to 0 \text{ as } t \to \infty.
\end{aligned}$$

Therefore $K_p(I-Q)(D)$ is equiconvergent at infinity. Thus the conditions of Lemma 1 (or Theorem 2) are fulfilled. Hence $K_p(I-Q)N(D)$ is relatively compact for every bounded $D \subset X$.

Lemma 5: Let

$$E_1 = \{u \in \text{Dom } L \backslash \ker L : Lu = \lambda Nu,\ \lambda \in (0,1]\}.$$

Then E_1 is bounded.

Proof: For $u \in E_1$, $u \notin \ker L$, $\lambda \neq 0$ and hence $Nu \in \text{Im } L$. Since $\ker Q = \text{Im } L$, we obtain

$$\sum_{i=1}^m \alpha_i \int_0^{\xi_i}\int_0^{\tau_n}\cdots\int_0^{\tau_3}\frac{1}{q(t_2)}\int_{t_2}^\infty y(\tau_1)d\tau_1 d\tau_2\cdots d\tau_n = 0,\ y \in Z.$$

From (A1) we derive a $t_1 \in [0,\infty)$ such that $|u^{(n-2)}(t_1)| \leq M_1$. Then

$$\begin{aligned}
|u^{(n-2)}(0)| &= \left| u^{(n-2)}(t_1) - \int_0^{t_1} u^{(n-1)}(s)ds \right| \\
&\leq |u^{(n-2)}(t_1)| + \int_0^{t_1} |u^{n-1}(s)|ds \\
&\leq M_1 + \parallel u^{(n-1)} \parallel_1 .
\end{aligned} \tag{18}$$

From (6) one obtains

$$u^{(n-1)}(t) = -\frac{1}{q(t)}\int_t^\infty (q(s)u^{(n-1)})'ds = -\frac{1}{q(t)}\int_t^\infty Lu(s)ds$$

and hence

$$\parallel u^{(n-1)} \parallel_1 = \int_0^\infty \left| -\frac{1}{q(t)}\int_t^\infty Lu(s)ds \right| dt \leq \parallel \frac{1}{q} \parallel_1 \parallel Nu \parallel_1 . \tag{19}$$

From (18) and (19) we get

$$|u^{(n-2)}(0) < M_1+ \| \frac{1}{q} \|_1 \| Nu \|_1 . \tag{20}$$

For $u \in E_1$, $(I-P)u \in \operatorname{Dom} L \cap \ker P$. Therefore using (14), we obtain

$$\begin{aligned} \| (I-P)u \| &= \| K_p L(I-P)u \| \leq D_n \| \frac{1}{q} \|_1 \| (L(I-P)u \|_1 \\ &\leq D_n \| \frac{1}{q} \|_1 \| Lu \|_1 \leq D_n \| \frac{1}{q} \|_1 \| Nu \|_1 . \end{aligned} \tag{21}$$

From the definition of P, and (3.13), we derive

$$Pu(t) = \frac{u^{(n-2)}(0)}{(n-2)!} t^{n-2}, (Pu)^{(i)}(t) = \frac{u^{(n-2)}(0)}{(n-2-i)!} t^{n-2-i},\ 0 \leq i \leq n-2.$$

$$\begin{aligned} \| Pu \| &= \left\{ \max_{0 \leq i \leq n-2} \sup_{t \in [0,\infty)} e^{-t} \frac{t^{n-2-i}}{(n-2-i)!} \right\} |u^{(n-2)}(0)| < D_n |u^{(n-2)}(0)| \\ &\leq D_n [M_1 + \| \frac{1}{q} \|_1 \| Nu \|_1]. \end{aligned} \tag{22}$$

$$\begin{aligned} \| u \| &= \| Pu + (I-P)u \| \leq \| Pu \| + \| (I-P)u \| \\ &\leq D_n [M_1 + \| \frac{1}{q} \|_1 \| Nu \|_1] + D_n \| \frac{1}{q} \|_1 \| Nu \|_1 \\ &= M_1 D_n + 2D_n \| \frac{1}{q} \|_1 \| Nu \|_1 . \end{aligned} \tag{23}$$

From (8) we obtain

$$\begin{aligned} \| Nu \|_1 &\leq \int_0^\infty |f(s, u(s), \cdots, u^{(n-1)}(s)| ds \\ &\leq \sum_{i=1}^{n-1} \| a_i \| \| u_i \| + \| a_n(t) \|_1 \| u_n(t) \|^\theta + \| r \|_1 . \end{aligned} \tag{24}$$

Hence, (23) and (24) yields

$$\| u \| \le M_1 D_n + 2D_n \| \frac{1}{q} \|_1 \left[\sum_{i=1}^{n-1} \| a_i \|_1 \| u \| + \| a_n \|_1 \| u \|^{\theta} + \| r \|_1 \right]$$

or

$$\| u \| \le \frac{M_1 D_n}{1 - 2D_n \| \frac{1}{q} \|_1 \sum\limits_{i=1}^{n-1} \| a_i \|_1} + \frac{2D_n \| \frac{1}{q} \|_1 [\| a_n \|_1 \| u \|^{\theta} + \| r \|_1]}{1 - 2D_n \|_1 \sum\limits_{i=1}^{n-1} \| a_i \|_1}.$$

Since $\theta \in [0,1)$ and (3.4), there exists $M_2 > 0$ such that

$$\| u \| \le M_2, \tag{25}$$

therefore E_1 is bounded.

Lemma 6: The set

$$E_2 = \{u \in \ker L : Nu \in \mathrm{Im}\ L\},$$

is bounded.

Proof: For $u \in E_2$, $u(t) = dt^{n-2}$ where $d \in \mathbb{R}$, $t \in [0,\infty)$ and $Nu \in \mathrm{Im}\ L$ implies $Nu = \ker Q$. Hence

$$\sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} \int_0^{\tau_n} \cdots \int_0^{\tau_3} \frac{1}{q(t_2)} \int_{t_2}^{\infty} f(\tau_1, d\tau_1^{n-2}, \cdots, (n-2)!d, 0) d\tau_1 d\tau_2 \cdots d\tau_n = 0.$$

By (A3) there exists $t_1 \in [0,\infty)$ such that $|u^{(n-2)}(t_1)| \le M_2$, that is $(n-2)!|d| \le M_2$ or $|d| < \frac{M_2}{(n-2)!}$. On the other hand

$$\| u \| \le \left[\max_{0 \le i \le n-2} \sup_{t \in [0,\infty)} e^{-t} t^{n-2-i} \right] |d| \le M_2 D_n.$$

This shows that E_2 is bounded.

We shall now prove theorem 3.

Proof: Let $T : \ker L \to \mathrm{Im}\ Q$ be the isomorphism defined by

$$T(dt^{n-2}) = dh(t),\ t \in [0,\infty),$$

where $h(t)$ is as in (7). If (9) holds, Let

$$E_3 = \{u \in \ker L : \lambda T u + (1-\lambda) Q N u = 0,\ \lambda \in [0, i]\},$$

then

$$-\lambda T u = (1-\lambda) Q N u$$

that is

$$-\lambda d h(t) = (1-\lambda) h(t) \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} \int_0^{t} \int_0^{\tau_n} \cdots \int_0^{\tau_3}$$

$$\frac{1}{q(\tau_2)} \int_{\tau_2}^{\infty} f(\tau_1, d t_1^{n-2}, \cdots, (n-2)! d, 0) d\tau_1 d\tau_2 \cdots d\tau_n dt.$$

If $\lambda = 1$ then $d = 0$ and if $|d| > M_2$ then in view of (9) we have

$$-\lambda d^2 = (1-\lambda) d \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} \int_0^{t} \int_0^{\tau_n} \cdots \int_0^{\tau_3}$$

$$\frac{1}{q(\tau_2)} \int_{\tau_2}^{\infty} f(\tau_1, d t_1^{n-2}, \cdots, (n-2)! d, 0) d\tau_1 d\tau_2 \cdots d\tau_n dt > 0$$

which is a contradiction. Hence E_3 is bounded.
If (10) holds, we set

$$E_3 = \{u \in \ker L : -\lambda T u + (1-\lambda) Q N u = 0,\ \lambda \in [0, 1]\}.$$

Using a similar argument we derive that E_3 is bounded.

Let E be an open bounded set such that $\cup_{i=1}^{3} \bar{E} \subset E$. Then it is easily seen that assumption (i) and (ii) of theorem 2.1 are satisfied. Lemmas 2.3 and 2.4 have established that L is Fredholm mapping of index zero and the mapping N is L-compact on E.

To verify the third assumption we define

$$H(u, \lambda) = \pm \lambda T u + (1-\lambda) Q N u.$$

It is easily seen that $H(u, \lambda) \neq = 0$ for every $u \in \partial E \cap \ker L$. Therefore

$$\begin{aligned} \deg(QN|_{\ker L}, E \cap \ker L, 0) &= \deg(H(\cdot, 0), E \cap \ker L) \\ &= \deg(H(\cdot, 1), E \cap \ker L, 0) \\ &= \deg(\pm T, E \cap \ker L, 0) \neq 0. \end{aligned}$$

Therefore from theorem 1, $Lu = Nu$ has at least one solution in Dom $L \cap E$.

4. EXAMPLE

Consider the boundary value problem

$$(q(t)u^{(n-1)}(t))' = e^{-t}(1+\sum_{j=0}^{n-2}\frac{|u^{(j)}(t)|}{2^j(i+t)^5}+|\cos t||u^{(n-1)}(t)|^{\frac{1}{16}}), \quad (26)$$

$$\begin{aligned} &u^{(n-2)}(0) = \sum_{i=1}^{(n-1)!}\alpha_i\int_0^{\xi_i}u(t)dt, \\ &u^{(i)}(0) = 0,\ i = 0,1,2,\cdots,(n-3), \lim_{t\to\infty} q(t)u^{(n-1)}(t) = 0, \end{aligned} \quad (27)$$

where $q(t) = e^t,\ t \in [0,\infty),\ \xi_i = \dfrac{1}{(1+i)^{\frac{1}{n-1}}},\ \alpha_i = (1+i),$

$f(t,u(t),u'(t),\cdots,u^{(n-1)}(t))$

$= e^{-t}\left(1+\sum_{j=0}^{n-2}\dfrac{|u^{(j)}(t)|}{2^j(1+t)^5}+|\cos t||u^{(n-1)}(t)|^{\frac{1}{16}}\right),$

$\sum_{i=1}^{(n-1)!}\alpha_i\xi_i^{n-1} = \sum_{i=1}^{(n-1)!}\dfrac{(1+i)}{(1+i)} = (n-1)!,$

$\sum_{i=1}^{(n-1)!}\alpha_i\int_0^{\xi_i}\int_0^{t}\int_0^{\tau_n}\cdots\int_0^{\tau_3}\dfrac{e^{-s}}{e^s)}ds d\tau_3\cdots d\tau_n dt \neq 0.$

Assumption (A0) is therefore satisfied. It is easily observed that since $f(t,u_1,u_2,\cdots,u_n) > 0$ for all $(t,u_1,u_2,\cdots,u_n) \in [0,\infty)\times\mathbb{R}^n$ then $QNu(t) \neq 0$ on $[0,\infty)$ for all $u \in \text{Dom } L\setminus\ker L$. This satisfies assumption (A1)

$|f(t,u(t),u'(t),\cdots,u^{(n-1)}(t)| \leq e^{-t}\left(1+\sum_{j=0}^{n-2}\frac{|u^{(j)}(t)|}{2^j(1+t)^5}+|u^{(n-1)}(t)|^{\frac{1}{16}}\right).$

Here $a_i(t) = \frac{1}{2^i(1+t)^5}$. Hence assumption (A2) is satisfied.

Assumption (A3) can be computed as follows

$$d \cdot \sum_{i=1}^{(n-1)!} \alpha_i \int_0^{\xi_i} \int_0^{\tau_n} \cdots \int_0^{\tau_3}$$

$$\frac{1}{q(\tau_2)} \int_{\tau_2}^{\infty} f(\tau, dt_1^{n-1}, \cdots, (n-2)!d) d\tau_1, d\tau_2 \cdots d\tau_n$$

$$= d \sum_{i=1}^{(n-1)!} (1+i) \int_0^{\overline{(1+i)^{\overline{(n-1}}}} \int_0^{\tau_n} \cdots \int_0^{\tau_3} e^{-s}$$

$$\int_s^{\infty} e^{-\tau_1} \left(1 + \sum_{j=0}^{n-2} \frac{|(d\tau_1^{n-2})^{(j)}|}{2^j(1+t)^5} \right) d\tau_1 ds d\tau_3 \cdots d\tau_n$$

$$\geq d \sum_{i=1}^{(n-1)!} (1+i) \int_0^{\overline{(1+i)^{\overline{(n-1}}}} \int_0^{\tau_n} \cdots \int_0^{\tau_3} e^{-2s} ds d\tau_3 \cdots d\tau_n$$

and satisfies (8) or (9) respectively if $|d| > M_2 = 1$.

Now $D_n = \max \left[\max\limits_{0 \leq i \leq n-1} \sup\limits_{t \in [0,\infty)} e^{-t} t^{n-2-i}, \ \frac{\| \frac{1}{q} \|_\infty}{\| \frac{1}{q} \|_1} \right]$,

$\| \frac{1}{q} \|_\infty = \| \frac{1}{q} \|_1 = 1$ and since

$\lim\limits_{t \to \infty} e^{-t} t^{n-2-i} \to 0$, $D_n = \left[\max\limits_{0 \leq i \leq n-1} \sup\limits_{t \in [0,\infty)} e^{-t} t^{n-2-i}, 1 \right] = 1$.

$\| a_i \|_1 = \frac{1}{2} i \int\limits_0^\infty \frac{1}{(1+t)^5} dt = \frac{1}{2^i} \cdot \frac{1}{4}$,

$\sum\limits_{i=1}^{(n-1)!} \| a_i \|_1 = \frac{1}{4} \sum\limits_{i=1}^{(n-1)!} \frac{1}{2^i} = \frac{1}{4}(1 - (\frac{1}{2})^{(n-1)!})$,

$\sum\limits_{i=1}^{(n-1)!} \| a_i \|_i = \frac{1}{4}(1 - (\frac{1}{2})^{(n-1)!}) < \frac{1}{2D_n \| \frac{1}{q} \|_1} = \frac{1}{2}$.

Therefore all the assumption of Theorem 3.1 are satisfied. Therefore (26) - (27) possess at least one solution.

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