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ON THE STABILITY AND BOUNDEDNESS OF SOLUTIONS TO CERTAIN SECOND ORDER NONLINEAR STOCHASTIC DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. This paper emphases on stability and boundedness of solutions to certain nonlinear non autonomous second order stochastic delay differential equations. A complete Lyapunov functional is constructed and used to establish conditions, on the nonlinear functions appearing in the equations, to guarantee stability and boundedness of solutions to the second order stochastic delay differential equations considered. The obtained results are new, complement and extend the existing results on second order stochastic delay differential equations in the literature. Finally, examples together with their numerical simulations are given to confirm genuineness and assert the correctness of the obtained results.

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1. INTRODUCTION

Delay differential equation (DDE) (also known as the differential equation with retarded argument) occurs in a wide variety of natural and man made systems. The inclusion of delay effects in mathematical modelling give rise to differential equations with constant, time varying, distributed and /or state dependent delays. DDEs play a pertinent role in laser physics, environmental modelling, electronic engineering, communication system, traffic flows, control theory, population dynamics, dynamics of neuronal networks, primary infection, drug therapy, immune response, the study of chemostat

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models, circadian rhythms, epidemiology, the respiratory system, tumor growth to name few.

Accordingly, many useful techniques has been developed by authors to study delay differential equations, particularly, the development of the direct method of Lyapunov to study the behaviour of solutions of DDEs. In this direction we can mention the books of Burton [8], Hale [11], Hale and Lunel [12], Driver [13], Yoshizawa [37] which contain general background to the subject matter, and the dazzling papers of Ademola *et al.* [5], Canlon [9], Domoshnitsky [13], Ogundare *et al.* [22, 23, 24], Tunç *et al.* [29, 30, 31, 32, 33], Xianfeng and Wei [34], Yeniçerioğlu [34, 36] and the references cited therein.

Effect of noise in differential equations is increasingly a focus of investigation. This is true especially in the world of science and technology as mentioned above. Also, different approaches has been developed in this direction to study the effect of noise in stochastic differential equations (SDEs) and stochastic delay differential equations (SDDEs). See for instance the survey books of Arnold [7], Oksendal [25], Shaikihet [28], and the stupendous, papers of Abou-El-Ela *et al.* [1, 2], Ademola *et al.* [4], Caraballo *et al.* [10], Ivanov *et al.* [15], Jedrzejewski [16], Kolarova [17], Kolmanovskii and Shaikhet [18, 19], Liu and Raffoul [20], Mao [21], Raffoul [26], Rezaeyan and Farnoosh [27], Zhu *et al.* [38] and the references cited therein.

In 2014, the authors in [23] and [31] discussed conditions which guarantee boundedness and stability properties of solutions of the following second order differential equations

$$x'' + f(x)x' + g(x) = p(t, x, x')$$

and

$$x'' + c(t, x, x') + q(t)b(x) = f(t),$$

respectively.

In 2015, following authors work on different problems of second order differential equations. First, the author in [3] employed Laypunov direct method, criteria which ensure boundedness and stability (when $p(t, x(t), x'(t)) = 0$) of solutions to the second order differential equation

$$\begin{aligned}[\phi(x(t))x'(t)]' + g(t, x(t), x'(t))x'(t) + \varphi(t)h(x(t)) = \\ p(t, x(t), x'(t)),\end{aligned}$$

where ϕ, g, φ, h and p are continuous functions in their respective arguments are discussed. In [5] conditions which guarantee periodicity, stability and boundedness of solutions to the second order nonlinear delay differential equation

$$x''(t) + \phi(t)f(x(t), x(t-\tau(t)), x'(t), x'(t-\tau(t))) + g(x(t-\tau(t))) \\ = p(t, x(t), x'(t)),$$

are discussed. Furthermore, the authors in [1] and [2] established conditions for stability and boundedness of solutions of the following second order stochastic delay differential equations

$$x''(t) + ax'(t) + bx(t-h) + \sigma x(t)w'(t) = 0,$$
$$x''(t) + ax'(t) + f(x(t-h)) + \sigma x(t-\tau)w'(t) = 0$$

and

$$x''(t) + g(x'(t)) + bx(t-h) + \sigma x(t)w'(t) = p(t, x(t), x'(t), x(t-h)),$$

respectively where $h > 0, \tau > 0$ are constants delays, a, b are positive constants, f, g, p are continuous functions with $g(0) = 0$ and $w \in \mathbb{R}^m$ is an $m-$dimensional standard Brownian motion defined on the probability space.

Recently, in 2016 the authors in [4] studied criteria for stability and boundedness of solutions to a certain second order stochastic differential equation

$$x''(t) + g(x(t), x'(t))x'(t) + f(x(t)) + \sigma x(t)\omega'(t) = p(t, x(t), x'(t)),$$

where f, g, p are continuous functions and $\omega \in \mathbb{R}^m$ is an $m-$ dimensional standard Brownian motion defined on the probability space.

Nevertheless, according to our observations from the relevant literature, the problem of stability and boundedness of solutions of the nonlinear non autonomous second order stochastic delay differential equation (1.1) is still opened for discussion. Therefore the aim of this paper is to consider

$$x''(t) + \psi(t)f(x(t), x'(t))x'(t) + g(x(t-\tau)) + \sigma x(t)\omega'(t) = \\ p(t, x(t), x'(t), x(t-\tau)), \tag{1.1}$$

where the functions ψ, f, g, p are continuous functions in their respective arguments on $\mathbb{R}^+$, $\mathbb{R}^2$, $\mathbb{R}$, $\mathbb{R}^+ \times \mathbb{R}^3$ respectively with $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^+ = [0, \infty)$, $\sigma > 0$ is a constant and τ is a positive constant delay. In addition, it is assumed that the continuity and the local Lipschitz conditions on the functions f, g, p and ψ are sufficient for existence and uniqueness of continuous solution denoted by (x_t, y_t) respectively. As usual the primes stand for differentiation with respect to the independent variable $t \in \mathbb{R}^+$. If $x'(t) = y(t)$, then equation (1.1) is equivalent to the system

$$\begin{aligned} x'(t) &= y(t), \\ y'(t) &= -\psi(t)f(x(t), y(t))y(t) - g(x(t)) - \sigma x(t)\omega'(t) \\ &\qquad + p(t, x(t), y(t), x(t-\tau)) + \int_{t-\tau}^{t} g'(x(s))y(s)ds. \end{aligned} \tag{1.2}$$

The derivative of the functions g and ω (i.e., g' and ω') exist and are continuous for all x and t respectively. This paper is motivated from the works in [1, 2, 3, 4] and [22].

An equally interesting problem is the situation where the function g and the term $\sigma x(t)\omega'(t)$ in (1.1) are replaced with deviating arguments and $\sigma x(t-\tau)\omega'(t)$ respectively. This has already been considered by us and the results arising in this direction will be advertized through another outlet. The obtained results in the present paper are completely new and they extend previously known results in [3, 23, 24, 32] on second order ordinary differential equations, the works in [5, 9, 10, 14, 22, 29, 31, 33, 34, 34, 36] where the delay differential equations are considered and the recent works on second order stochastic delay differential equations in [1, 2, 4]. Some mathematical tools that will be needed in the sequel are discussed in Section 2. That is definitions of major terms used in the paper, some basic results on stability and boundedness of solutions to non autonomous $n-$dimensional stochastic differential equation (2.1) are also presented in this section. The main results of this paper and their proofs are discussed in Section 3 while examples and simulation of the numerical solutions are presented in the last section.

2. Preliminary Results

Let $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t>0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathfrak{F}_t\}_{t>0}$ satisfying the usual conditions (i.e. it is right continuous and $\{\mathfrak{F}_0\}$ contains all $\mathbb{P}-$null sets). Let $B(t) = (B_1(t), \cdots, B_m(t))^T$ be an $m-$dimensional Brownian motion defined on the probability space. Let $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by

$$|A| = \sqrt{\text{trace } (A^T A)}.$$

For more information see Arnold [7] and Mao [21]. Consider a non autonomous $n-$dimensional SDDE

$$dx(t) = F(t, x(t), x(t-\tau))dt + G(t, x(t), x(t-\tau))dB(t) \tag{2.1}$$

on $t > 0$ with initial data $\{x(\theta) : -\tau \leq \theta \leq 0\}, x_0 \in C([-\tau, 0]; \mathbb{R}^n)$. Here $F : \mathbb{R}^+ \times \mathbb{R}^{2n} \to \mathbb{R}^n$ and $G : \mathbb{R}^+ \times \mathbb{R}^{2n} \to \mathbb{R}^{n\times m}$ are measurable functions. Suppose that the functions F, G satisfy the local Lipschitz condition, given any $b > 0$, $p \geq 2$, $F(t, 0, 0) \in C^1([0, b]; \mathbb{R}^n)$ and $g(t, 0, 0) \in C^p([0, b]; \mathbb{R}^{m\times n})$. Then there must be a stopping time $\beta = \beta(\omega) > 0$ such that equation (2.1) with $x_0 \in C^p_{\mathfrak{F}_{t_0}}$ [class of $\mathfrak{F}_t$-measurable $C([-\tau, 0]; \mathbb{R}^n)$-valued random variables ξ_t and $E\|\xi_t\|^p < \infty$] has a unique maximal solution on $t \in [t_0, \beta)$ which is denoted by $x(t, x_0)$. Assume further that

$$F(t, 0, 0) = G(t, 0, 0) = 0$$

for all $t \geq 0$. Hence, the SDDE admits zero solution $x(t, 0) \equiv 0$ for any given initial value $x_0 \in C([-\tau, 0]; \mathbb{R}^n)$.

Definition 2.1. The zero solution of the stochastic differential equation (2.1) is said to be *stochastically stable* or *stable in probability,* if for every pair $\epsilon \in (0, 1)$ and $r > 0$, there exists a $\delta_0 = \delta_0(\epsilon, r) > 0$ such that

$$Pr\{|x(t; x_0)| < r \text{ for all } t \geq 0\} \geq 1 - \epsilon \text{ whenever } |x_0| < \delta_0.$$

Otherwise, it is said to be *stochastically unstable*.

Definition 2.2. The zero solution of the stochastic differential equation (2.1) is said to be *stochastically asymptotically stable* if it is stochastically stable and in addition if for every $\epsilon \in (0, 1)$ and $r > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that

$$Pr\{\lim_{t\to\infty} x(t; x_0) = 0\} \geq 1 - \epsilon \text{ whenever } |x_0| < \delta.$$

Definition 2.3. A solution $x(t_0, x_0)$ of the SDDE (2.1) is said to be *stochastically bounded* or *bounded in probability,* if it satisfies

$$E^{x_0}\|x(t, x_0)\| \leq C(t_0, \|x_0\|),\ \forall\ t \geq t_0 \tag{2.2}$$

where E^{x_0} denotes the expectation operator with respect to the probability law associated with x_0, $C : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$ is a constant function depending on t_0 and x_0.

Definition 2.4. The solutions $x(t_0, x_0)$ of the SDDE (2.1) is said to be *uniformly stochastically bounded* if C in (2.2) is independent of t_0.

Let $\mathbb{K}$ denote the family of all continuous non-decreasing functions $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\rho(0) = 0$ and $\rho(r) > 0$ if $r \neq 0$. In addition, $\mathbb{K}_\infty$ denotes the family of all functions $\rho \in \mathbb{K}$ with

$$\lim_{r\to\infty} \rho(r) = \infty.$$

Suppose that $C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+)$, denotes the family of all non negative functions $V(t, x_t)$ (Lyapunov function) defined on $\mathbb{R}^+ \times \mathbb{R}^n$ which are twice continuously differentiable in x and once in t. By Itô's formula we have

$$dV(t, x_t) = LV(t, x_t)dt + V_x(t, x_t)G(t, x_t)dB(t),$$

where

$$\begin{aligned} LV(t, x_t) \; &= \frac{\partial V(t, x_t)}{\partial t} + \frac{\partial V(t, x_t)}{\partial x_i} F(t, x_t) \\ &+ \frac{1}{2} \text{trace } [G^T(t, x_t) V_{xx}(t, x_t) G(t, x_t)], \end{aligned} \tag{2.3}$$

$$V_{xx}(t, x_t) = \left(\frac{\partial^2 V(t, x_t)}{\partial x_i \partial x_j} \right)_{n \times n}, \; i, j = 1, \cdots, n,$$

with $x_t = x(t + \theta)$, $-r \leq \theta \leq 0$, $t \geq 0$. In this study we will use the diffusion operator $LV(t, x_t)$ defined in (2.3) to replace $V'(t, x_t) = \frac{d}{dt} V(t, x_t)$. We now present the basic results that will be used in the proofs of the main results.

Lemma 2.5. (See [7]) Assume that there exist $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+)$ and $\phi \in \mathbb{K}$ such that

(i) $V(t, 0) = 0$, for all $t \geq 0$;
(ii) $V(t, x_t) \geq \phi(\|x(t)\|)$, $\phi(r) \to \infty$ as $r \to \infty$; and
(iii) $LV(t, x_t) \leq 0$ for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$.

Then the zero solution of SDDE (2.1) is *stochastically stable.* If conditions (ii) and (iii) hold then (2.1) with $x_0 \in C^p_{f_0}$ has a unique global solution for $t > 0$ denoted by $x(t; t_0)$.

Lemma 2.6. (See [7]) Suppose that there exist $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+)$ and $\phi_0, \phi_1, \phi_2 \in \mathbb{K}$ such that

(i) $V(t, 0) = 0$, for all $t \geq 0$;
(ii) $\phi_0(\|x(t)\|) \leq V(t, x_t) \leq \phi_1(\|x(t)\|)$, $\phi_0(r) \to \infty$ as $r \to \infty$; and
(iii) $LV(t, x_t) \leq -\phi_2(\|x(t)\|)$ for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$.

Then the zero solution of SDDE (2.1) is *uniformly stochastically asymptotically stable in the large*

Assumption 2.7. (See [20, 26]) Let $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$, suppose that for any solutions $x(t_0, x_0)$ of SDDE (2.1) and for any fixed $0 \leq t_0 \leq T < \infty$, we have

$$E^{x_0} \left\{ \int_{t_0}^{T} V^2_{x_i}(t, x_t) G^2_{ik}(t, x_t) dt \right\} < \infty, \; 1 \leq i \leq n, \; 1 \leq k \leq m. \tag{2.4}$$

Assumption 2.8. (See [20, 26]) A special case of the general condition (2.4) is the following condition. Assume that there exits a function $\sigma(t)$ such that

$$|V_{x_i}(t, x_t) G_{ik}(t, x_t)| < \sigma(t), \; x \in \mathbb{R}^n \; 1 \leq i \leq n, \; 1 \leq k \leq m, \tag{2.5}$$

for any fixed $0 \leq t_0 \leq T < \infty$,

$$\int_{t_0}^{T} \sigma^2(t)dt < \infty. \tag{2.6}$$

Lemma 2.9. (See [20, 26]) Assume there exists a Lyapunov function $V(t, x_t) \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$, satisfying Assumption 2.7, such that for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$,

(i) $\|x(t)\|^p \leq V(t, x_t) \leq \|x(t)\|^q$,
(ii) $LV(t, x_t) \leq -\alpha(t)\|x(t)\|^r + \beta(t)$,
(iii) $V(t, x_t) - V^{r/q}(t, x_t) \leq \gamma$,

where $\alpha, \beta \in C(\mathbb{R}^+; \mathbb{R}^+)$, p, q, r are positive constants, $p \geq 1$ and γ is a non negative constant. Then all solutions of SDDE (2.1) satisfy

$$\begin{aligned} E^{x_0}\|x(t, x_0)\| \leq \Bigg\{ & V(t_0, x_0)e^{-\int_{t_0}^{t} \alpha(s)ds} \\ & + \int_{t_0}^{t} \Big(\gamma\alpha(u) + \beta(u)\Big) e^{-\int_{u}^{t} \alpha(s)ds} du \Bigg\}^{1/p}, \end{aligned} \tag{2.7}$$

for all $t \geq t_0$.

Lemma 2.10. (See [20, 26]) Assume there exists a Lyapunov function $V(t, x_t) \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$, satisfying Assumption 2.7, such that for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$,

(i) $\|x(t)\|^p \leq V(t, x_t)$,
(ii) $LV(t, x_t) \leq -\alpha(t)V^q(t, x_t) + \beta(t)$,
(iii) $V(t, x_t) - V^q(t, x_t) \leq \gamma$,

where $\alpha, \beta \in C(\mathbb{R}^+; \mathbb{R}^+)$, p, q are positive constants, $p \geq 1$ and γ is a non negative constant. Then all solutions of SDDE (2.1) satisfy (2.7) for all $t \geq t_0$.

Corollary 2.11. (See [20, 26])

(i) Assume that hypotheses (i) to (iii) of Lemma 2.9 hold. In addition

$$\int_{t_0}^{t} \Big(\gamma\alpha(u) + \beta(u)\Big) e^{-\int_{u}^{t} \alpha(s)ds} du \leq M, \forall\ t \geq t_0 \geq 0, \tag{2.8}$$

for some positive constant M, then all solutions of SDDE (2.1) are *uniformly stochastically bounded.*

(ii) Assume the hypotheses (i) to (iii) of Lemma 2.10 hold. If condition (2.8) is satisfied, then all solutions of SDDE (2.1) are *stochastically bounded.*

3. Main Results

Suppose that (x_t, y_t) be any continuous solution of the SDDE (1.2) with $x(t) = x$ and $y(t) = y$. The Lyapunov functional, $V(t, x_t, y_t) = V(t, X_t)$, $X_t = (x_t, y_t) \in \mathbb{R}^2$, employed in this paper is defined as

$$\begin{aligned} 2V(t, X_t) = (a^2 + b^2)x^2 + (b+1)y^2 + xg(x) + 2axy \\ + \int_{-\tau}^{0} \int_{t+s}^{t} \lambda y^2(\theta) d\theta ds, \end{aligned} \tag{3.1}$$

where a, b are positive constants, $\lambda > 0$ is a constant which will be determined later and $\tau > 0$ is a constant delay.

Theorem 3.1. If in addition to the basic assumptions on the functions f, g, ω, ψ and p, suppose that $a, b, B, G, P, \sigma, \psi_0$ are positive constants and

(i) $a \leq f(x, y)$ for all x and y, $\psi_0 \leq \psi(t)$ for all $t \geq 0$;
(ii) $bx \leq g(x) \leq Bx$ for all $x \neq 0$, $|g'(x)| \leq G$ for all x;
(iii) $\sigma^2(b+1) < ab$, $\psi_0^{-1} < b+1$; and
(iv) $|p(t, x, y, x(t-\tau))| \leq P$, $0 \leq P < \infty$, for all $t \geq 0, x, y$ and $x(t-\tau)$.

Then the solution (x_t, y_t) of the SDDE (1.2) is uniformly stochastically bounded provided that

$$\tau < \min\left\{ \frac{ab - (b+1)\sigma^2}{aG}, \ \frac{a[(b+1)\psi_0 - 1]}{[a + 2(b+1)]G} \right\}. \tag{3.2}$$

Remark 3.2. We observe the following

(i) If the functions $\psi(t)f(x, x') = a$, $g(x(t-\tau)) = bx$, where $a > 0, b > 0$ are constants and $\sigma = 0 = p(t, x, x', x(t-\tau))$, then the SDDE (1.1) reduces to second order linear differential equation
$$x'' + ax' + bx = 0, \tag{3.3}$$
and hypotheses (i) to (iv) of Theorem 3.1 reduce to Routh-Hurwitz criteria $a > 0$, $b > 0$ for asymptotic stability of the second order linear differential equation (3.3).

(ii) The term $\sigma x(t)\omega'(t)$ when $\tau = 0$ extends all results on second order ordinary differential equations discussed recently in [3, 23, 24, 32] and the references cited therein.

(iii) The term $\sigma x(t)\omega'(t)$ is an extension to all results on stability and boundedness of solutions of second order delay differential equations studied in [5, 9, 10, 14, 22, 29, 31, 33, 34, 34, 36] and the references cited therein.

(iv) Whenever $\tau = 0$ and $\psi(t) = 1$ equation (1.1) specializes to stochastic differential equation discussed in [4]. Thus, the stability and boundedness results obtained in this investigation include and extend the results in [4].

(v) When $\psi(t)f(x,x') = a$, $g(x(t-\tau)) = bx$, where a, b are positive constants and $p(t,x,x',x(t-\tau)) = 0$, equation (1.1) reduces to equation (1.1) considered in [1]. Thus the stability results of this paper include and improve the stability result obtained in Theorem 2.3 of [1].

(vi) If $\psi(t)f(x,x') = g(x')$ and $g(x(t-\tau)) = bx(t-\tau)$, equation (1.1) reduces to that studied in [2], hence our stability and boundedness results extend the results in [2].

Next, we will state and proof a result that will be useful in the proofs of our results.

Lemma 3.3. Subject to the hypotheses of Theorem 3.1, there exist positive constants $D_0 = D_0(a,b)$ and $D_1 = D_1(a,b,B,\lambda,\tau)$ such that

$$D_0(x^2(t) + y^2(t)) \leq V(t,X_t) \leq D_1(x^2(t) + y^2(t)), \tag{3.4}$$

for all $t \geq 0$, x and y. In addition, there exist positive constants $D_2 = D_2(a,b,G,\sigma,\psi_0)$ and $D_3 = D_3(a,b)$ such that equation (3.1) and the system (1.2) using Itô's formula (2.3) gives

$$\begin{aligned} LV(t,X_t) \leq -D_2(x^2(t) + y^2(t)) + D_3(|x(t)| + |y(t)|)\times \\ |p(t,x,y,x(t-\tau))|, \end{aligned} \tag{3.5}$$

for all $t \geq 0, x$ and y.

Proof. Let (x_t, y_t) be any solution of SDDE (1.2). It is clear from equation (3.1) that

$$V(t,\mathbf{0}) = 0, \forall\ t \geq 0, \mathbf{0} = (0,0) \in \mathbb{R}^2. \tag{3.6}$$

Furthermore, the functional $V(t,X_t)$ defined in equation (3.1) can be represented as

$$\begin{aligned} V(t,X_t) = \frac{1}{2}(b^2 + x^{-1}g(x))x^2 + \frac{1}{2}by^2 + \frac{1}{2}(ax+y)^2 \\ + \frac{1}{2}\int_{-\tau}^{0}\int_{t+s}^{t} \lambda y^2(\theta)d\theta ds. \end{aligned} \tag{3.7}$$

Since the double integrals in (3.7) is non-negative and the fact that $g(x) \geq bx$ for all $x \neq 0$, it follows that from equation (3.7) the existence of a positive constant $\delta_0 = \delta_0(a,b)$ such that

$$V(t,X_t) \geq \delta_0(x^2 + y^2), \tag{3.8}$$

for all $t \geq 0, x$ and y, where

$$\delta_0 := \frac{1}{2}\min\left\{b^2 + b + \min\{a,1\},\ b + \min\{a,1\}\right\}.$$

From equation (3.6) and the inequality (3.8) the functional $V(t,X_t)$ defined by (3.1) is positive semi-definite. Also, from the inequality (3.8), we find that

$$V(t, X_t) = 0 \text{ if and only if } x^2 + y^2 = 0, \tag{3.9}$$

$$V(t, X_t) > 0 \text{ if and only if } x^2 + y^2 \neq 0 \tag{3.10}$$

and

$$V(t, X_t) \to +\infty \text{ as } x^2 + y^2 \to \infty. \tag{3.11}$$

Estimate (3.11) shows that the functional $V(t, X_t)$ is radially unbounded. Next, since $g(x) \leq Bx$ for all $x \neq 0$ and the fact that the inequality $2xy \leq x^2 + y^2$ holds for all $(x, y) \in \mathbb{R}^2$, there exists a positive constant $\delta_1 = \delta_1(a, b, B, \lambda, \tau)$ such that

$$V(t, X_t) \leq \delta_1(x^2 + y^2) \tag{3.12}$$

for all $t \geq 0, x$ and y, where

$$\delta_1 := \frac{1}{2} \max \left\{ a^2 + b^2 + a + B,\ a + b + \lambda\tau + 1 \right\}.$$

By inequality (3.12) the functional $V(t, X_t)$ of (3.1) is decrescent. Combining inequalities (3.8) and (3.12), we obtain

$$\delta_0(x^2 + y^2) \leq V(t, X_t) \leq \delta_1(x^2 + y^2) \tag{3.13}$$

for $t \geq 0, x$ and y. Estimate (3.13) establish the inequality (3.4) with δ_0 and δ_1 equivalent to D_0 and D_1 respectively.

Besides, using Itô's formula defined by equation (2.3) with system (1.2), we find that

$$\begin{aligned} LV(t, X_t) &= -ax^{-1}g(x)x^2 - [(b+1)\psi(t)f(x, y) - a]y^2 \\ &- \frac{1}{2}\sum_{i=1}^{2} W_i + (b+1)\sigma^2 x^2 + W_3 + 2[ax + (b+1)y]\times \\ &p(t, x, y, x(t-\tau)) + \lambda\tau y^2 - \lambda\int_{t-\tau}^{t} y^2(\mu)d\mu, \end{aligned} \tag{3.14}$$

where

$$W_1 := ax^{-1}g(x)x^2 + 4[a\psi(t)f(x, y) - (a^2+b^2)]xy + [(b+1)\psi(t)f(x, y) - a]y^2,$$

$$W_2 := ax^{-1}g(x)x^2 + 2[(2b+1)x^{-1}g(x) - g'(x)]xy + [(b+1)\psi(t)f(x, y) - a]y^2$$

and

$$W_3 := 2[ax + (b+1)y]\int_{t-\tau}^{t} g'(x(\mu))y(\mu)d\mu.$$

Engaging the following inequalities

$$4[a\psi(t)f(x, y) - (a^2 + b^2)]^2 < ax^{-1}g(x)[(b+1)\psi(t)f(x, y) - a]$$

and

$$[(2b+1)x^{-1}g(x) - g'(x)]^2 < ax^{-1}g(x)[(b+1)\psi(t)f(x, y) - a]$$

in W_1 and W_2 respectively, we find

$$W_1 = W_2 \geq \Bigg[\sqrt{ax^{-1}g(x)}|x| - \sqrt{[(b+1)\psi(t)f(x,y)-a]}|y|\Bigg]^2 \geq 0 \tag{3.15}$$

for $t \geq 0, x$ and y. Moreover, since $g'(x) \leq G$ for all x, we have

$$W_3 \leq G\tau(ax^2 + (b+1)y^2) + (a+b+1)G\int_{t-\tau}^{t} y^2(\mu)d\mu \tag{3.16}$$

for all $t \geq 0, x$ and y. Using inequalities (3.15) and (3.16) in equation (3.14), we obtain

$$\begin{aligned} LV(t,X_t) \leq\ & -ax^{-1}g(x)x^2 - [(b+1)\psi(t)f(x,y)-a]y^2 \\ & - [\lambda - (a+b+1)G]\int_{t-\tau}^{t} y^2(\mu)d\mu \\ & + (b+1)\sigma^2x^2 + G\tau[ax^2 + (b+1)y^2] + \lambda\tau y^2 \\ & + 2[ax + (b+1)y]p(t,x,y,x(t-\tau)), \end{aligned} \tag{3.17a}$$

for all $t \geq 0, x$ and y. From hypothesis (iii) of Theorem 3.1 and choose $\lambda := (a+b+1)G > 0$, there exist positive constants $\delta_2 = \delta_2(a,b,G,\sigma,\psi_0)$ and $\delta_3 = \delta_3(a,b)$ such that

$$LV(t,X_t) \leq -\delta_2(x^2+y^2) + \delta_3(|x|+|y|)|p(t,x,y,x(t-\tau))|, \tag{3.17b}$$

for all $t \geq 0, x$ and y, where

$$\delta_2 := \min\Big\{[ab-(b+1)\sigma^2] - aG\tau,\ a[(b+1)\psi_0 - 1] - [a+2(b+1)]G\tau\Big\}$$

and

$$\delta_3 := 2\max\{a, b+1\}.$$

From inequality (3.17b), inequality (3.5) of Lemma 3.3 holds with δ_2 and δ_3 equivalent to D_2 and D_3 respectively. This completes the proof of Lemma 3.3. □

Proof of Theorem 3.1. Let (x_t, y_t) be any solution of of the SDDE (1.2). To prove the theorem, we first show that condition of Assumptions 2.7 and 2.8 of Section 2 hold. To see this, from system (1.2), formula (2.3) and equation (3.1) there exists a positive constant $\delta_4 = \delta_4(a,b,\sigma)$ such that

$$\Big|V_{x_i}(t,X_t)G_{ik}(t,X_t)\Big| \leq \delta_4(x^2+y^2),\ 1 \leq i \leq 2,\ 1 \leq k \leq 2, \tag{3.18}$$

for all $t \in \mathbb{R}^+$, $X_t \in \mathbb{R}^2$ where

$$\delta_4 := \sigma\max\{2a+b+1,\ b+1\} = (2a+b+1)\sigma.$$

Inequality (3.18) satisfies inequality (2.5) of Assumption 2.8 with $\sigma(t) = \delta_4(x^2+y^2)$.

What's more, for any fixed $0 \le t_0 \le T < \infty$, we have

$$\int_{t_0}^{T} \delta_4^2(x^2+y^2)^2 dt < \infty, \tag{3.19}$$

so that inequality (3.19) fulfills estimate (2.6). Estimates (3.18) and (3.19) satisfy conditions of Assumptions 2.8, and therefore Assumption 2.7 follows immediately. Next, we show that hypotheses (i) to (iii) of Lemma 2.9 hold true. To see these, comparing inequality (3.13) with item (i) of Lemma 2.9, we find that $p = q = 2$. Also, using hypothesis (iv) of Theorem 3.1 in estimate (3.17b), since $\delta_2 > 0$,

$$\left[|x| - \delta_2^{-1}\delta_3 P\right]^2 \ge 0 \text{ and } \left[|y| - \delta_2^{-1}\delta_3 P\right]^2 \ge 0,\ \forall\ x, y,$$

there exist positive constants $\delta_5 = \delta_5(\delta_2)$ and $\delta_6 = \delta_6(\delta_2, \delta_3, P)$ such that

$$LV(t, X_t) \le -\delta_5(x^2+y^2) + \delta_6, \tag{3.20}$$

for all $t \in \mathbb{R}^+$, $X_t \in \mathbb{R}^2$, where

$$\delta_5 := \frac{1}{2}\delta_2 \text{ and } \delta_6 := \delta_2^{-1}\delta_3^2 P^2.$$

Inequality (3.20) established the second hypothesis of Lemma 2.9 with $\alpha(t) = \delta_5$, $r = 2$ and $\beta(t) = \delta_6$. Since $r = q = 2$, hypothesis (iii) of Lemma 2.9 follows with $\gamma = 0$.

Furthermore, if inequality (3.12) holds and the fact that $e^{-\delta_5(t-t_0)}$ is decaying rapidly for all $t \ge t_0$ it follows that

$$V(t_0, X_0)e^{-\int_{t_0}^{t}\alpha(s)ds} \le \delta_1(x_0^2+y_0^2)e^{-\delta_5(t-t_0)} \le \delta_1 X_0^2 \tag{3.21}$$

for all $t \ge t_0 \ge 0$, $X_0 \in \mathbb{R}^2$ and $X_0^2 := x_0^2 + y_0^2$. Also,

$$\begin{aligned}\int_{t_0}^{t}\left[\Big(\gamma\alpha(u) + \beta(u)\Big)e^{-\int_u^t \alpha(s)ds}\right]du &= \delta_5^{-1}\delta_6\left[1 - e^{-\delta_5(t-t_0)}\right] \\ &\le \delta_5^{-1}\delta_6,\end{aligned} \tag{3.22}$$

for all $t \ge t_0$. Now from inequalities (3.21) and (3.22) all solutions of SDDE (1.2) satisfies

$$E^{X_0}\|X(t, X_0)\| \le \left(\delta_1 X_0^2 + \delta_5^{-1}\delta_6\right)^{1/2},\ \forall\ t \ge t_0, \tag{3.23}$$

thus inequality (2.7) holds.

Finally, in view of inequality (3.22), estimate (2.8) of Corollary 2.11 (i) holds true with $M := \delta_5^{-1}\delta_6$. Thus by Corollary 2.11 (i) all solutions of the SDDE (1.2) are uniformly stochastically bounded. This completes the proof of Theorem 3.1. □

Theorem 3.4. Suppose that hypotheses of Theorem 3.1 and inequality (3.2) hold, then all solutions of SDDE (1.2) are stochastically bounded.

Proof. Let (x_t, y_t) be any solution of SDDE (1.2). From the proof of Theorem 3.1, Assumption 2.7 hold. Next, from inequality (3.8) hypothesis (i) of Lemma 2.10 holds with $p = 2$. Also, from estimates (3.12) and (3.20), we find

$$LV(t, X_t) \leq -\delta_1^{-1}\delta_5 V(t, X_t) + \delta_6, \tag{3.24}$$

for all $t \in \mathbb{R}^+$, $X_t \in \mathbb{R}^2$. Estimate (3.24) establish hypothesis (ii) of Lemma 2.10 with $\alpha(t) := \delta_1^{-1}\delta_5$, $q = 1$ and $\beta(t) = \delta_6$.

What's more, since $q = 1$, it follows from hypothesis (iii) of Lemma 2.10 that $\gamma = 0$. Therefore, all hypotheses of Lemma 2.10 are satisfied. Moreover,

$$\int_{t_0}^{t} \left[\Big(\gamma\alpha(u) + \beta(u)\Big)e^{-\int_u^t \alpha(s)ds}\right] du \leq \delta_1\delta_5^{-1}\delta_6, \tag{3.25}$$

for all $t \geq t_0 \in \mathbb{R}^+$, where $M := \delta_1\delta_5^{-1}\delta_6$, and that inequality (2.8) holds. Assumptions of Corollary 2.11 (ii) hold, thus by Corollary 2.11 (ii) all solutions of SDDE (1.2) are stochastically bounded. This completes the proof of Theorem 3.4. □

Next, when $p(t, x, x', x(t-\tau)) = 0$ and $p(t, x, y, x(t-\tau)) = 0$ in SDDEs (1.1) and (1.2) respectively, we have the following equations.

$$x''(t) + \psi(t)f(x(t), x'(t))x'(t) + g(x(t-\tau)) + \sigma x(t)\omega'(t) = 0, \tag{3.26}$$

where the functions ψ, f, g, p are continuous functions defined in Section 1. Equation (3.26) when transformed to system of first order is

$$\begin{aligned} x'(t) &= y(t), \\ y'(t) &= -\psi(t)f(x(t), y(t))y(t) - g(x(t)) - \sigma x(t)\omega'(t) \\ &\quad + \int_{t-\tau}^{t} g'(x(s))y(s)ds. \end{aligned} \tag{3.27}$$

We have the following results.

Theorem 3.5. Suppose that hypotheses (i) to (iii) of Theorem 3.1 are satisfied, then the trivial solution of the SDDE (3.27) is uniformly stochastically asymptotically stable in the large provided that inequality (3.2) holds.

Proof. To prove this result, it is enough to show that conditions (i) to (iii) of Lemma 2.6 hold. To see this let $(x_t, y_t) = X_t \in \mathbb{R}^2$ be any solution of the SDDE (3.27). From equation (3.6), estimates (3.8), (3.11) and (3.12), hypotheses (i) and (ii) of Lemma 2.6 hold. In addition, from (3.1) and (3.27) using Itô's formula (2.3), we find that

$$LV(t, X_t) \leq -\delta_5(x^2 + y^2) \tag{3.28}$$

for all $t \in \mathbb{R}^+$ and $X_t \in \mathbb{R}^2$. Estimate (3.28) established hypothesis (iii) of Lemma 2.6. Hypotheses of Lemma 2.6 hold, thus by Lemma 2.6 the trivial solution $X_t \equiv 0$ of (3.27) is uniformly stochastically asymptotically stable in the large. This completes the proof of Theorem 3.5. □

Next, we shall state and proof stability and uniqueness of a global solution of the SDDE (3.27). We obtain the following result.

Theorem 3.6. Suppose that b, and G are positive constants:

(i) If hypotheses (i) and (iii) of Theorem 3.1 hold, $g(x) \geq bx$ for all $x \neq 0$ and $|g'(x)| \leq G$ for all x then the trivial solution of SDDE (3.27) is stochastically stable provided that inequality (3.2) holds; and

(ii) If hypotheses of Theorem 3.6 (i) hold, then the system (3.27) possesses a unique global solution for $t > 0$.

Proof. (i) To prove Theorem 3.6 (i), we will show that hypotheses (i) to (iii) of Lemma 2.5 hold true. To see this let $(x_t, y_t) = X_t \in \mathbb{R}^2$ be any solution of the SDDE (3.27). From equation (3.6), inequalities (3.8) and (3.28), hypotheses of Lemma 2.5 hold, thus by Lemma 2.5 the trivial solution $X_t \equiv 0$ of (3.27) is stochastically stable.

(ii) Here, we shall show that conditions (ii) and (iii) of Lemma 2.5 hold true. From inequalities (3.8) and (3.28) items (ii) and (iii) of Lemma 2.5 are satisfied, hence by Lemma 2.5 the solution of the SDDE (3.27) possesses a unique global solution for $t > 0$. This completes the proof of Theorem 3.6. □

If the forcing term $p(t, x, x', x(t-\tau))$ is replaced by $p(t)$ defined on $\mathbb{R}^+$, we have a special case of (1.1) as follows.

$$x''(t) + \psi(t)f(x(t), x'(t))x'(t) + g(x(t-\tau)) + \sigma x(t)\omega'(t) = p(t), \tag{3.29}$$

where the functions ψ, f, g, p are continuous functions defined in Section 1. Equation (3.29) when transformed to system of first order is

$$\begin{aligned} x'(t) &= y(t), \\ y'(t) &= -\psi(t)f(x(t), y(t))y(t) - g(x(t)) - \sigma x(t)\omega'(t) \\ &\quad + \int_{t-\tau}^{t} g'(x(s))y(s)ds + p(t). \end{aligned} \tag{3.30}$$

We have the following results.

Corollary 3.7. If hypotheses (i) to (iii) of Theorem 3.1 and inequality (3.2) hold, if the function $p(t)$ is bounded by a finite constant, then the solutions $(x_t, y_t) = X_t \in \mathbb{R}^2$ of SDDE (3.30) are

(i) uniformly stochastically bounded; and

(ii) stochastically bounded.

Proof. See the proofs of Theorem 3.1 and Theorem 3.4 respectively. This completes the proof of Corollary 3.7. □

4. Examples

In this section we will discuss some special cases to show the correctness of the results in Section 3.

Example 4.1. Consider the second order SDDE

$$\begin{aligned} & x'' + (3+\cos 4t)\left(\frac{4+x^2+(x')^2}{1+x^2+(x')^2}\right)x' \\ & + \left[2x(t-\tau)+\frac{1}{2}\sin(x(t-\tau))\right] \\ & + \sigma x(t)\omega'(t) = \frac{2+t^2+x^2+(x')^2+|x(t-\tau)|}{1+t^2+x^2+(x')^2+|x(t-\tau)|}, \end{aligned} \tag{4.1}$$

where $\sigma > 0$ is a constant and $\tau > 0$ is a constant delay which will be determined later. If $x' = y$, equation (4.1) becomes

$$\begin{aligned} x' &= y, \\ y' &= -(3+\cos 4t)\left(\frac{4+x^2+y^2}{1+x^2+y^2}\right)y - \left[2x+\frac{1}{2}\sin x\right] \\ & \quad - \sigma x(t)\omega'(t) + \frac{1}{2}\int_{t-\tau}^{t}\left[4+\cos(x(s))\right]ds \\ & \quad + \frac{2+t^2+x^2+y^2+|x(t-\tau)|}{1+t^2+x^2+y^2+|x(t-\tau)|}. \end{aligned} \tag{4.2}$$

From equations (1.2) and (4.2) we obtain the following relations

(i) the function $\psi(t)$ is defined as

$$\psi(t) := 3+\cos 4t.$$

since

$$-1 \le \cos 4t \le 1$$

for all t, it follows that

$$\psi(t) = 3+\cos 4t \ge \psi_0 = 2 > 0 \;\forall\; t.$$

The behaviour or path of $\psi(t)$ for $t \in [-6\pi, 6\pi]$ is shown in Figure 1

(ii) The function

$$f(x,y) := 3 + \frac{1}{1+x^2+y^2}.$$

Since $1+x^2+y^2$ is monotonically increasing for all x and y, thus

$$f(x,y) \ge a = 3 > 0,$$

for all x and y. The behaviour of this non linear function is shown in Figure 2.

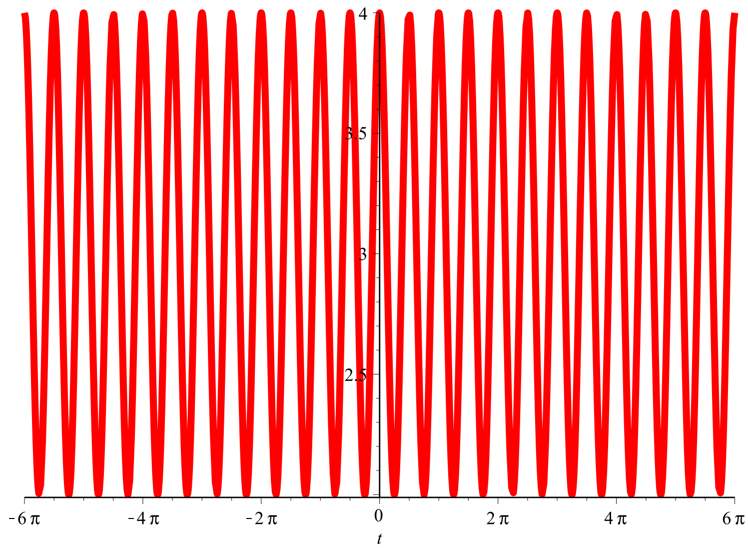

FIGURE 1. The behaviour of the function $\psi(t)$ for $t \in [-6\pi, 6\pi]$.

(iii) The function

$$g(x) := 2x + \frac{\sin x}{2}$$

so that

$$\frac{g(x)}{x} = 2 + \frac{\sin x}{2x}.$$

Since

$$G(x) = \frac{\sin x}{2x}$$

lies in the closed interval $[-0.1, 0.5]$, it follows that

$$0 < 2 = b \leq \frac{g(x)}{x} = 2 + \frac{\sin x}{2x} \leq B = 2.5$$

for all $x \neq 0$. Furthermore, the derivative of g with respect to x is

$$g'(x) := 2 + \frac{1}{2}\cos x$$

for all x, hence

$$|g'(x)| \leq G = 2.5$$

for all x. The paths of $x^{-1}g(x)$, $\frac{\sin x}{2x}$ and $|g'(x)|$ are depicted in Figure 3.

(iv) The function

$$p(t, x, y, x(t-\tau)) := 1 + \frac{1}{1 + t^2 + x^2 + y^2 + |x(t-\tau)|}.$$

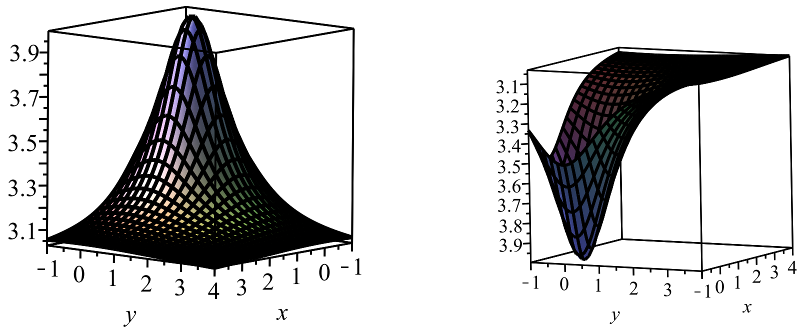

FIGURE 2. Different views of the function $f(x, y)$ in 3D.

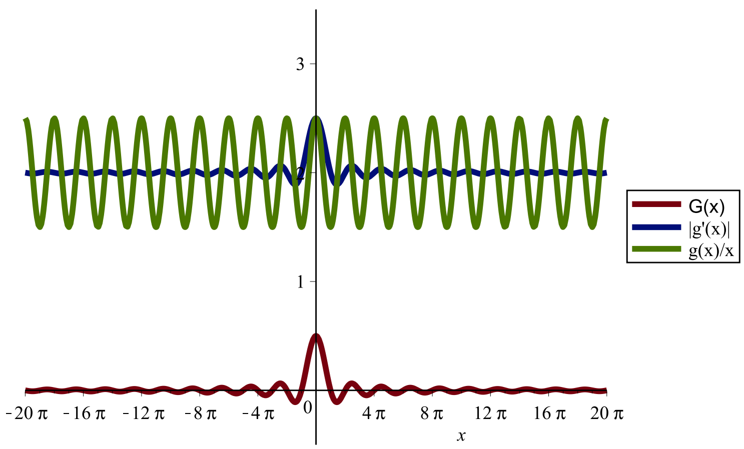

FIGURE 3. The Paths of Functions $\dfrac{g(x)}{x}$, $G(x)$ and $|g'(x)|$.

The fact that $1 + t^2 + x^2 + y^2 + |x(t - \tau)|$ is increasing for all $t \geq 0, x, y$ and $x(t - \tau)$, we have

$$|p(t, x, y, x(t - \tau))| \leq P = 2 < \infty,$$

for all $t \geq 0, x, y$ and $x(t - \tau)$.

(v) Using the calculated values of constants a, b, G, ψ_0 in item (iii) of Theorem 3.1, we find that

$$\sigma < \sqrt{2}$$

and we choose

$$\sigma = 1.3.$$

We also verified that

$$(b + 1)\psi_0 > 1.$$

Finally, we calculate the value of τ defined by inequality (3.2) as

$$\tau < \min\{1.24, 0.67\} = 0.67 \approx 0.7,$$

and we choose

$$\tau = 0.1.$$

All hypotheses of Theorem 3.1 and Theorem 3.4 hold, thus by Theorems 3.1 and 3.4 the solutions $X_t = (x_t, y_t) \in \mathbb{R}^2$ of the SDDE (4.2) are

(i) uniformly stochastically bounded; and
(ii) stochastically bounded.

Alternatively, we can show that hypotheses of Assumption 2.7, Lemmas 2.9, 2.10 and Corollary 2.11 hold. To see this, the functional V defined in equation (3.1) becomes

$$\begin{aligned} 2V(t, X_t) = 13x^2 + 3y^2 + (2x + \frac{1}{2}\sin x)x + 6xy \\ + 15\int_{-0.1}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds. \end{aligned} \tag{4.3}$$

The behaviour of $V(t, X_t)$ is shown in Figure 4. Clearly, from equation

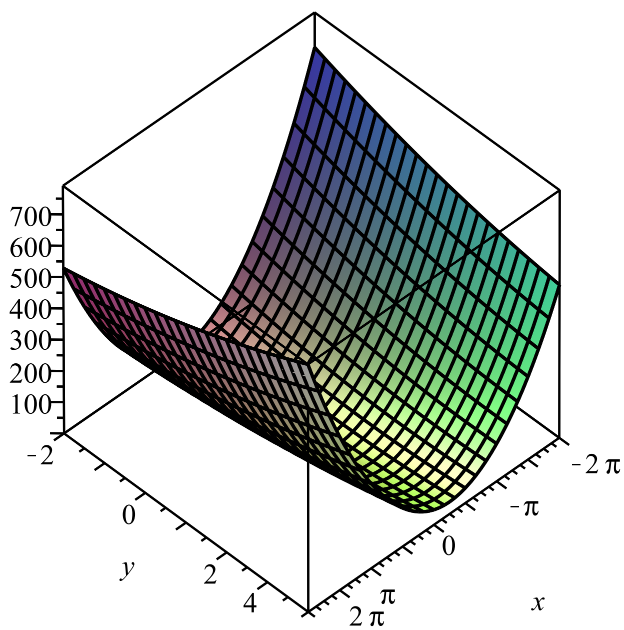

FIGURE 4. The Behaviour of $V(t, X_t)$ for $t = 1/15$.

(4.3) that

$$V(t, 0) = 0, \ \forall t \in \mathbb{R}^+. \tag{4.4}$$

We also confirmed that the double integrals

$$\int_{-0.1}^{0}\int_{t+s}^{t} y(\theta)d\theta ds = \frac{600t^2 - 40t + 1}{120000}$$

is non negative for all $t \in \mathbb{R}^+$. Furthermore, since

$$2 + \frac{\sin x}{2x} \geq 2$$

for all $x \neq 0$, we find that

$$V(t, X_t) \geq \frac{3}{2}(x^2 + y^2),\ \forall t \in \mathbb{R}^+, X_t \in \mathbb{R}^2 \tag{4.5}$$

where $3/2 = \delta_0$ of inequality (3.8). We also find (from inequality (4.5)) that

$$V(t, X_t) = 0 \Leftrightarrow x^2 + y^2 = 0, \tag{4.6}$$

$$V(t, X_t) > 0 \Leftrightarrow x^2 + y^2 \neq 0 \tag{4.7}$$

and

$$V(t, X_t) \to +\infty \text{ as } x^2 + y^2 \to \infty. \tag{4.8}$$

In addition, since

$$2 + \frac{\sin x}{2x} \leq 2.5$$

for all $x \neq 0$, $\tau = 0.1$ and $\lambda = 15$, equation (4.3) becomes

$$V(t, X_t) \leq 9.3(x^2 + y^2),\ \forall t \in \mathbb{R}^+, X_t \in \mathbb{R}^2 \tag{4.9}$$

where $9.3 = \delta_1$ of inequality (3.12). From inequalities (4.5) and (4.9) we have

$$1.5(x^2 + y^2) \leq V(t, X_t) \leq 9.3(x^2 + y^2),\ \forall t \in \mathbb{R}^+, X_t \in \mathbb{R}^2 \tag{4.10}$$

Next, using Itô's formula (2.3) with system (4.2), we obtain

$$\begin{aligned}
LV(t, X_t) &= -6\left[(3 + \cos 4t)\left(\frac{4 + x^2 + y^2}{1 + x^2 + y^2}\right) - 1\right]y^2 \\
&- 4\left[3(3 + \cos 4t)\left(\frac{4 + x^2 + y^2}{1 + x^2 + y^2}\right) - 13\right]xy - 2\left[3\left(2 + \frac{\sin x}{2x}\right)\right. \\
&- \left.\left(2 + \frac{\cos x}{2}\right)\right]xy + 6(x + y)\int_{t-0.1}^{t}\left(2 + \frac{\cos x}{2}\right)ds + 5.07x^2 \\
&+ 6(x + y)\left(\frac{2 + t^2 + x^2 + y^2 + |x(t - \tau)|}{1 + t^2 + x^2 + y^2 + |x(t - \tau)|}\right) \\
&- 6\left(2 + \frac{\sin x}{2x}\right)x^2 + 1.5y^2 - 15\int_{t-0.1}^{t} y^2(\mu)d\mu.
\end{aligned} \tag{4.11}$$

Using the following inequalities

$$4\left[3(3+\cos 4t)\left(\frac{4+x^2+y^2}{1+x^2+y^2}\right)-13\right]^2 < 9\left(2+\frac{\sin x}{2x}\right)\left[(3+\cos 4t)\left(\frac{4+x^2+y^2}{1+x^2+y^2}\right)-1\right],$$

$$\left[5\left(2+\frac{\sin x}{2x}\right)\left(2+\frac{\cos x}{2}\right)\right]^2 < 9\left(2+\frac{\sin x}{2x}\right)\times \left[(3+\cos 4t)\left(\frac{4+x^2+y^2}{1+x^2+y^2}\right)-1\right]$$

and

$$\left|2+\frac{\cos x}{2}\right| < 2.5,$$

equation (4.11) becomes

$$LV(t,X_t) \le -0.15(x^2+y^2)+6(|x|+|y|)\times \left|\left(\frac{2+t^2+x^2+y^2+|x(t-\tau)|}{1+t^2+x^2+y^2+|x(t-\tau)|}\right)\right|, \tag{4.12}$$

for all $t \in \mathbb{R}^+$ and $X_t = (x_t, y_t) \in \mathbb{R}^2$, where $0.15 = \delta_2$ and $6 = \delta_3$.

From system (4.2) and equation (4.3), we also find that

$$|V_{x_i}(t,X_t)G_{ik}(t,X_t)| \le 11.7(x^2+y^2), 1 \le i \le 2, 1 \le k \le 2, \tag{4.13}$$

for all $t \in \mathbb{R}^+$ and $X_t = (x_t, y_t) \in \mathbb{R}^2$, where $11.7 = \delta_4$. What's more, for any $0 \le t_0 \le T < \infty$,

$$\int_{t_0}^{T} 136.89(x^2+y^2)dt < \infty. \tag{4.14}$$

Now, from inequalities (4.13) and (4.14) Assumption 2.8 hold and Assumption 2.7 follows immediately. Next we show that hypotheses (i) to (iii) of Lemma 2.9 hold. From the inequality (4.10) hypothesis (i) of Lemma 2.9 holds with $p = 2, q = 2$. Next, since

$$\left|\left(\frac{2+t^2+x^2+y^2+|x(t-\tau)|}{1+t^2+x^2+y^2+|x(t-\tau)|}\right)\right| \le 2 < \infty,\ (|x|-80)^2 \ge 0,$$

and

$$(|y|-80)^2 \ge 0$$

for all $t \in \mathbb{R}^+$ and $X_t = (x_t, y_t) \in \mathbb{R}^2$, inequality (4.12) becomes

$$LV(t,X_t) \le -0.075(x^2+y^2)+960 \tag{4.15}$$

for all $t \in \mathbb{R}^+$ and $X_t = (x_t, y_t) \in \mathbb{R}^2$, where $0.075 = \delta_5$ and $960 = \delta_6$. Comparing inequality (4.15) with item (ii) of Lemma 2.9 we find that $\alpha(t) = 0.075$, $r = 2$ and $\beta(t) = 960$. Since $r = 2 = q$ item (iii) of Lemma 2.9 follows with $\gamma = 0$. Furthermore,

$$V(t_0,X_0)e^{-\int_{t_0}^{t}\alpha(s)ds} \le 9.3(x_0^2+y_0^2) = 9.3X_0^2. \tag{4.16}$$

Also,

$$\int_{t_0}^{t} \left[\left(\gamma\alpha(u) + \beta(u) \right) e^{-\int_u^t \alpha(s)ds} \right] du \leq 12800, \tag{4.17}$$

for all $t \geq t_0 \geq 0$. From estimates (4.16) and (4.17) we have

$$E^{X_0} \|X(t, X_0)\| \leq \left(9.3X_0^2 + 12800 \right)^{1/2}, \tag{4.18}$$

for all $t_0 \in \mathbb{R}^+$ and $X_0 = (x_0, y_0) \in \mathbb{R}^2$. Inequality (4.18) fulfills that of (2.7). Finally hypotheses (i) and (ii) of Lemma 2.9 hold and from estimate (4.17) hypotheses of Corollary 2.11 (i) hold, thus by Corollary 2.11 (i) all solutions of SDDE (4.2) are uniformly stochastically bounded.

Next, we shall establish the boundedness of solution of SDDE (4.2). Let (x_t, y_t) be any solution of SDDE (4.2), from inequality (4.5) item (i) of Lemma 2.10 holds with $p = 2$. Moreover from estimate (4.9) we have

$$0.11V(t, X_t) \leq x^2 + y^2, \tag{4.19}$$

for all $t \in \mathbb{R}^+$ and $X_t \in \mathbb{R}^2$, where $0.11 = \delta_1^{-1}$. Using inequality (4.19) in (4.15), we have

$$LV(t, X_t) \leq -0.008V(t, X_t) + 960, \tag{4.20}$$

for all $t \in \mathbb{R}^+$ and $X_t \in \mathbb{R}^2$, where $0.008 = \delta_1^{-1}\delta_5$ and $960 = \delta_6$. Inequality (4.20) fulfills item (ii) of Lemma 2.10 with $q = 1$, $\alpha(t) = 0.008$ and $\beta(t) = 960$. Moreover, since $q = 1$, we have $\gamma = 0$ so that item (iii) of Lemma 2.10 holds. In addition,

$$\int_{t_0}^{t} \left[\left(\gamma\alpha(u) + \beta(u) \right) e^{-\int_u^t \alpha(s)ds} \right] du \leq 119040, \tag{4.21}$$

for all $t \geq t_0 \geq 0$, where $\delta_1\delta_5^{-1}\delta_6 = 119040 = M > 0$. From inequalities (4.16) and (4.21) all solutions of SDDE (4.2) satisfy estimate (2.7). Finally, inequality (4.21) satisfies condition (2.8) of Corollary 2.11 (ii), hence by Corollary 2.11 (ii) solutions of SSDE (4.2) are stochastically bounded.

Example 4.2. Consider the SSDE

$$\begin{aligned} x'' + (3 + \cos 4t)\left(\frac{4 + x^2 + (x')^2}{1 + x^2 + (x')^2} \right) x' + \Big[2x(t - \tau) \\ + \frac{1}{2}\sin(x(t - \tau)) \Big] + \sigma x(t)\omega'(t) = 0, \end{aligned} \tag{4.22}$$

its equivalent system is

$$\begin{aligned} x' &= y, \\ y' &= -(3+\cos 4t)\left(\frac{4+x^2+y^2}{1+x^2+y^2}\right)y - \left[2x + \frac{1}{2}\sin x\right] \\ &\quad - \sigma x(t)\omega'(t) + \frac{1}{2}\int_{t-\tau}^{t}\left[4+\cos(x(s))\right]ds. \end{aligned} \tag{4.23}$$

Let (x_t, y_t) be any solution of system (4.23), the Lyapunov functional (4.3) is still valid for system (4.23) so that items (i) (ii) (iii) and (vi) of Theorem 3.5 hold. Thus by Theorem 3.5 the trivial solution of (4.23) is uniformly stochastically asymptotically stable in the large.

Alternatively, using the background results of Section 2 for system (4.23), equation (4.4), estimates (4.5) and (4.8) so that items (i) and (ii) of Lemma 2.5 hold. Moreover, from inequality (4.12), we have

$$LV(t, X_t) \leq -0.15(x^2 + y^2) \leq 0, \tag{4.24}$$

for all $t \in \mathbb{R}^+$ and $X_t \in \mathbb{R}^2$. Inequality (4.24) justifies item (iii) of Lemma 2.5. Hypotheses of Lemma 2.5 hold, hence by Lemma 2.5 the trivial solution $X_t \equiv 0$ of the SDDE (4.23) is stochastically stable. Besides, from estimates (4.5), (4.8) and (4.24) system (4.23) has a unique global solution for $t > 0$.

Finally, from equation (4.4), inequalities (4.5), (4.8) and (4.9), items (i) and (ii) of Lemma 2.6 hold. Furthermore, from inequality (4.24) we have

$$LV(t, X_t) \leq -0.15(x^2 + y^2), \tag{4.25}$$

for all $t \in \mathbb{R}^+$ and $X_t \in \mathbb{R}^2$. Inequality (4.25) bears out item (iii) of Lemma 2.6, hence by Lemma 2.6 the trivial solution $X_t \equiv 0$ of (4.23) is uniformly stochastically asymptotically stable in the large.

5. Simulation of Solutions

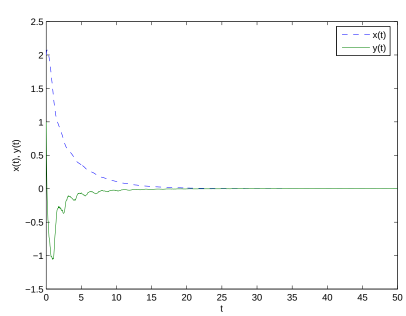

Figure 5. The Behaviour of X_t

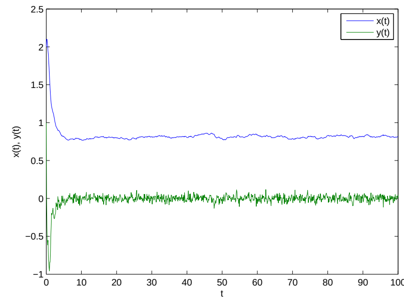

FIGURE 6. The Behaviour of X_t

6. Concluding Remarks

This paper emphases on stability and boundedness of solutions to certain nonlinear non autonomous second order stochastic delay differential equations. By employing the second method of Lyapunov, a complete Lyapunov functional is constructed and used to establish conditions, on the nonlinear functions appearing in the equations, to guarantee stability and boundedness of solutions to the second order stochastic delay differential equations considered.

REFERENCES

[1] A.M.A. Abou-El-Ela, A.I. Sadek and A.M. Mahmoud, *On the stability of solutions for certain second-order stochastic delay differential equations,* Differential Equations and Control Processes, **2015,** N 2, 2015.

[2] A.M.A. Abou-El-Ela, A.I. Sadek, A.M. Mahmoud and R.O.A. Taie, *On the stochastic stability and boundedness of solutions for stochastic delay differential equation of the second order, Chinese Journal of Mathematics,* **2015,** Article ID 358936, 8 pages, 2015.

[3] A.T. Ademola, *Boundedness and stability of solutions to certain second order differential equations,* Differential Equations and Control Processes, **2015,** N 3, 38 - 50, 2015.

[4] A.T. Ademola, S. Moyo, B.S. Ogundare, M.O. Ogundiran and O.A. Adesina, *Stability and boundedness of solutions to a certain second order non autonomous stochastic differential equation,* International Journal of Analysis, **2016,** Article ID 2012315, 11 pages, 2016.

[5] A.T. Ademola, B.S. Ogundare, M.O. Ogundiran and O.A. Adesina, *Periodicity, stability, and boundedness of solutions to certain second order delay differential equations,* International Journal of Differential Equations, **2016,** Article ID 2843709, 10 pages, 2016.

[6] T.A. Ademola, and M.O. Ogundiran, *On the existence and uniqueness of solutions of a generalized Lipschitz ordinary differential equations,* Ife Journal of Science, **9,** (2), 241 - 246, 2007.

[7] L. Arnold, *Stochastic differential equations: Theory and applications,* John Wiley & Sons, 1974.

[8] T.A. Burton, *Stability and periodic solutions of ordinary and functional differential equations,* Mathematics in Science and Engineering, **178** Academic Press. Inc., Orlando, FL, 1985.

[9] C. Cahlon and D. Schmidt, *Stability criteria for certain second order delay differential equations with mixed coefficients,* J. Comput. Appl. Math., **170,** 79 - 102, 2004.

[10] T. Caraballo, M.A. Diop and A.S. Ndoye, *Fixed points and exponential stability for stochastic partial integro-differential equations with delays,* Advances in Dynamical Systems and Applications, **9,** (2), 133 - 147, 2014.

[11] J.K. Hale, *Theory of functional differential equations,* Applied Mathematical Sciences 3. Springer-Verlag New York, 1977.

[12] J.K. Hale and S.M.V. Lunel, *Introduction to functional differential equations, Applied Mathematical Sciences* **99,** 1993.

[13] R.D. Driver, *Ordinary and delay differential equations,* Applied Mathematical Sciences 20. Springer-Verlag New York, Heidelberg Berlin 1977.

[14] A. Domoshnitsky, *Non oscillation, maximum principles, and exponential stability of second order delay differential equations without damping term,* Journal of Inequalities and Applications, **2014,** (361), 1 - 26, 2014.

[15] A.F. Ivanov, Y.I. Kazmerchuk and A.V. Swishchuk, *Theory, stochastic stability and applications of stochastic delay differential equations: a survey of recent results,* Differential Equations and Dynamical Systems **11** (1), Jan., 2003.

[16] F. Jedrzejewski and D. Brochard, *Lyapounv exponents and stability stochastic dynamical structures,* Irsn/Dend/Sate 23/07/2000.

[17] E. Kolarova, *An application of stochastic integral equations to electrical networks,* Acta Electrotechnica et Informatica, **8,** (3), 14 - 17, 2008.

[18] V.B. Kolmanovskii and L.E. Shaikhet, *A method for constructing Lyapunov functionals for stochastic systems with after effect,* Differentsial′nye Uravneniya, **29,** (11), 1909 - 1920, 1993.

[19] V.B. Kolmanovskii and L.E. Shaikhet, *Construction of Lyapunov functionals for stochastic hereditary systems: A survey of some recent results,* Mathematical and Computer Modelling, **36,** 691 - 716, 2002.

[20] R. Liu and Y. Raffoul, *Boundedness and exponential stability of highly nonlinear stochastic differential equations,* Electronic Journal of Differential Equations, **2009,** (143), 1 - 10, 2009.

[21] X. Mao, *Some contributions to stochastic asymptotic stability and boundedness via multiple Lyapunov functions,* Journal of Mathematical Analysis and Applications, **260,** 325 - 340, 2001.

[22] B.S. Ogundare, A.T. Ademola, M.O. Ogundiran and O.A. Adesina, *On the qualitative behaviour of solutions to certain second order nonlinear differential equation with delay,* Ann Univ Ferrara, 1 - 21, 2016.

[23] B.S. Ogundare and A.U. Afuwape, *Boundedness and stability properties of solutions of generalized Liénard equation,* Kochi J. Math., **9,** 97 - 108, 2014.

[24] B.S. Ogundare and G.E. Okecha, *Boundedness, periodicity and stability of solution* $x'' + a(t)g(x') + b(t)h(x) = p(t; x, x')$, Math. Sci. Res. J., **11** (5), 432 - 443, 2007.

[25] B. Oksendal, *Stochastic differential equations, an introduction with applications,* Springer-Verlage, 2000.

[26] Y.N. Raffoul, *Boundedness and exponential asymptotic stability in dynamical systems with applications to nonlinear differential equations with unbounded terms,* Advances in Dynamical Systems and Applications, **2,** (1), 107 - 121, 2007.

[27] R. Rezaeyan and R. Farnoosh, *Stochastic differential equations and application of the Kalman-Bucy filter in the modeling of RC circuit,* Applied Mathematical Sciences, **4,** (23), 1119 - 1127, 2010.

[28] L. Shaikihet, *Lyapunov functionals and stability of stochastic differential equations,* Springer, http://www.springer.com/978-3-319-00100-5.

[29] C. Tunç and T. Ayhan, *Global existence and boundedness of solutions of a certain nonlinear integro-differential equation of second order with multiple deviating arguments,* Journal of Inequalities and Applications, **2016,** (46) DOI 10.1186/s13660-016-0987-2, 2016.

[30] C. Tunç, *A note on the stability and boundedness of non-autonomous differential equations of second order with a variable deviating argument,* Afr. Math., **25**(2), 417 - 425, 2014.

[31] C. Tunç, *A note on the bounded solutions to $x'' + c(t,x,x') + q(t)b(x) = f(t)$,* Appl. Math. Inf. Sci., **8**, (1), 393 - 399, 2014.

[32] C. Tunç, *Boundedness of solutions to certain system of differential equations with multiple delays,* Mathematical Modeling and Applications in Nonlinear Dynamics, Springer Book Series, **Chapter 5**, 109-123, 2016.

[33] C. Tunç, *New Results on the existence of periodic solutions for Rayleigh equation with state-dependent delay,* J. Math. Fund. Sci., **45,** (2), 154 - 162, 2013.

[34] Z. Xianfeng and J. Wei, *Stability and boundedness of a retarded Liénard-type equation,* Chin. Q. J. Math., **18,** (1), 7 - 12, 2013.

[34] A.F. Yeniçerioğlu, *The behavior of solutions of second order delay differential equations,* J. Math. Anal. Appl., **332,** 1278 - 1290, 2007.

[36] A.F. Yeniçerioğlu, *Stability properties of second order delay integro-differential equations,* Computers and Mathematics with Applications, **56,** 3109 - 3117, 2008.

[37] T. Yoshizawa, *Stability theory by Liapunov's second method, The Mathematical Society of Japan,* 1966.

[38] W. Zhu, J. Huang, X. Ruan and Z. Zhao, *Exponential stability of stochastic differential equation with mixed delay,* Journal of Applied Mathematics, **2014,** Article ID 187037, 11 pages, 2014.

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