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PARAMETER UNIFORM NUMERICAL METHOD FOR SINGULARLY PERTURBED PARABOLIC DIFFERENTIAL DIFFERENCE EQUATIONS

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ABSTRACT. In this paper, a numerical study is made for solving singularly perturbed differential difference equations with small advance and delay parameters. To approximate the advance and delay terms a Taylor series expansion has been used. The resulting singularly perturbed parabolic PDE is solved by using non-standard finite difference method on uniform mesh in x-direction and implicit Runge-Kutta method is used for the resulting system of IVPs in t-direction. The method is shown to be accurate of order one. A convergence analysis has been carried out to show $\varepsilon-$ uniform convergence of the proposed scheme. Two numerical examples are considered to investigate parameter uniform convergence of the proposed method.

Keywords and phrases: Boundary layer, Differential difference equation, Method of line, Non-standard finite difference, Singular perturbation
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1. INTRODUCTION

The differential difference equations (DDEs) with delay/advance term plays an important role in modeling many real life phenomena in Control Theory, Bioscience, Economics and Engineering [7]. Some applications are the mathematical modeling of population dynamics and epidemiology [12], physiological kinetics [2], blood cell production [17] and so on. A special class of differential difference equations with at least one delay/advance term, in which the highest order derivative of the problem is multiplied by small (arbitrary) parameter is known as the singularly perturbed differential

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difference equations (SPDDEs) with delay and advance parameters. SPDDEs relate an unknown function to its derivatives evaluated at the same instance. But, singularly perturbed delay(and/or advance) partial differential equations model physical problems for which the evaluation does not only depend on the present state of the system but also on the past history. In general, when the perturbation parameter tends to zero in such case the smoothness of the solution of the SPDDEs deteriorates and it forms a boundary layers [20]. Such type of SPDDEs have variety of applications in the study of variational problems of control theory [9], in modeling of neuronal variability [27]. In [30, 29], we can find a lot of mathematical models for the determination of the behavior of a neuron to random synaptic inputs. Musila and Lansky in [19] generalized the Stein's model and come with the following mathematical model in terms of singularly perturbed parabolic differential difference equations (SPPDDEs) to consider the time evolution trajectories of the membrane potential:

$$\begin{aligned}\frac{\partial u}{\partial t} =& \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + (\mu_D - \frac{x}{\tau})\frac{\partial u}{\partial x} + \lambda_s u(x+a_s,t) + \omega_s u(x+i_s,t) \\ & - (\lambda_s+\omega_s)u(x,t)\end{aligned} \tag{1}$$

where the first derivative term is due to the exponential decay between two consecutive jumps caused by the input processes. The membrane potential decays exponentially to the resting level with a membrane time constant τ. μ_D and σ are diffusion moments of Wiener process characterizing the influence of dendritic synapses on the cell excitability. The excitatory input contributes to the membrane potential by an amplitude a_s with intensity λ_s and similarly the inhibitory input contributes by an amplitude i_s with intensity ω_s. This model makes available time evolution of the trajectories of the membrane potential. The model (1) is a differential difference equation, one can hardly derive its exact solution. In order to find a solution to such problem, one has to apply suitable numerical methods.

In the last few years, different researchers have worked on numerical treatment of solution of SPPDDEs. In papers [3, 4, 5], [13], [20], [21], [22] researchers, considered different numerical methods to study a classes of SPPDDEs and discussed the effect of delay and advance parameters on the solution behavior.

2. STATEMENT OF THE PROBLEM

A singularly perturbed 1D spatial delay and advance parabolic partial differential equation of convection diffusion type having reaction term with a delay as well as advance argument on the domain D with smooth boundary $\partial D = \bar{D}/D$ is given by:

$$\begin{aligned}\frac{\partial u}{\partial t} - \varepsilon^2 \frac{\partial^2 u}{\partial x^2} + a(x)\frac{\partial u}{\partial x} + \alpha(x)u(x-\delta,t) + \beta(x)u(x,t) \\ + \omega(x)u(x+\eta,t) = f(x,t)\end{aligned} \tag{2}$$

$(x,t) \in D = \Omega \times \Lambda = (0,1)\times(0,T]$ with some fixed positive number T, subject to the initial and interval conditions given as:

$$\begin{aligned} u(x,0) &= u_0(x), x \in D_0 = \{(x,0) : x \in \bar{\Omega}\} \\ u(x,t) &= \phi(x,t), (x,t) \in D_L = \{(x,t) : -\delta \le x \le 0, t \in \Lambda\} \\ u(x,t) &= \psi(x,t), (x,t) \in D_R = \{(x,t) : 1 \le x \le 1+\eta, t \in \Lambda\} \end{aligned} \tag{3}$$

Here, ε is a singular perturbation parameter with the condition $0 < \varepsilon \ll 1$ and δ and η are small delay and advance parameters respectively, assumed to be sufficiently small as order of $o(\varepsilon)$.
We assume the functions $a(x), \alpha(x), \beta(x), \omega(x), f(x,t), u_0(x), \phi(x,t)$ and $\psi(x,t)$ are sufficiently smooth, bounded and independent of ε. The coefficients of reaction term β, delay term α and advance term ω are assumed to satisfy the condition:

$$\alpha(x) + \beta(x) + \omega(x) \ge \theta > 0, \quad \forall x \in \bar{\Omega}$$

for some positive constant θ. This condition ensure that the solution of (2)-(3) form a boundary layer in the neighborhood of $D_L = \{(x,t) : -\delta \le x \le 0, t \in \Lambda\}$ or $D_R = \{(x,t) : 1 \le x \le 1+\eta, t \in \Lambda\}$ depending on whether $a(x) - \delta\beta(x) + \eta\omega(x) < 0$ or > 0 on $x \in \bar{\Omega}$. When the shift parameters δ, η are zero the equation in (2) changed to a singularly perturbed parabolic PDEs, which with small ε forming boundary layers depending upon the sign of the convective term coefficient $a(x)$. When $a(x) < 0$ a regular boundary layer appears in the neighborhood of D_L (left boundary layer) and $a(x) > 0$ corresponds to existence of a boundary layer near D_R (right boundary layer), in addition to that if $a(x)$ change sign then interior layer (or shock layer) will appear on the solution of the problem [10]. The layer is maintained for $\delta, \eta \ne 0$ but sufficiently small. Note that the problem in (2)-(3) reduces to the classical case of singularly perturbed parabolic PDEs when $\delta, \eta = 0$. When the shift parameters δ and η are smaller than perturbation parameter ε, the use of Taylor's series expansion for the terms containing shifts is valid [28].

We use Taylor's series expansion to tackle the terms containing the shifts.
The presence of the singular perturbation parameter ε, leads to occurrences of oscillations or divergence in the computed solutions while using classical numerical methods [13]. To overcome these oscillations or divergence, a large number of mesh points are required when ε is very small. This is difficult and sometimes impossible to handle such cases. Therefore, to handle this drawback associated with classical numerical methods, we need to derive a method based on method of line (MOL) using non-standard finite difference in spatial direction together with implicit Runge-Kutta method for temporal direction, which treat the problem without creating an oscillation.
The main contribution of this paper is, to develop parameter uniform numerical scheme without any restriction on the mesh generation for the original singularly perturbed differential difference problem containing small delay and advance parameters in the space variable. A new class of parameter uniform numerical scheme is proposed by the procedure of MOL, which consist of non-standard finite difference method on x(spatial) direction and implicit Runge-Kutta order two and three in t(temporal) direction. In the proposed method, it is not required to have any adaptive property on the mesh generation. Proposed numerical scheme works well for small values of shift parameters.
This paper is organized as follows. In section 1 a brief introduction about the problem is given, in section 2 definition of the problem and the behavior of its analytical solution is given. In section 3, discretizing the spatial domain and techniques of non-standard finite difference is discussed, and the parameter uniform convergence of the semi-discrete problem is proved. Next, Runge-Kutta method used for the system of IVPs resulted from spatial discretization and discuss the convergence of the discrete scheme. In section 4, numerical results and discussion are given to validate the theoretical analysis and finally in section 5, the conclusions of the work done are presented.
Notations: Through out this paper N, M denoted for the number of mesh points in x and t direction respectively. C is denoted for positive constant independent of perturbation parameter and N. The norm $\|.\|$, $\|.\|_{\Omega^N\times\Lambda}$ and $\|.\|_{\Omega^N\times\Lambda^M}$ is used to denote maximum norm, semi-discrete maximum norm and discrete maximum norm respectively.

Estimate for the delay and advance parameters

Since $\delta, \eta < \varepsilon$, by using Taylor series expansion for $u(x-\delta,t)$ and $u(x+\eta,t)$ we obtain:

$$\begin{aligned} u(x-\delta,t) &\approx u(x,t) - \delta\frac{\partial u}{\partial x}(x,t) + \frac{\delta^2}{2}\frac{\partial^2 u}{\partial x^2}(x,t) + O(\delta^3) \\ u(x+\eta,t) &\approx u(x,t) + \eta\frac{\partial u}{\partial x}(x,t) + \frac{\eta^2}{2}\frac{\partial^2 u}{\partial x^2}(x,t) + O(\eta^3) \end{aligned} \tag{4}$$

Substituting these approximations into (2)-(3), we obtain:

$$\frac{\partial u}{\partial t} - c_\varepsilon(x)\frac{\partial^2 u}{\partial x^2} + p(x)\frac{\partial u}{\partial x} + q(x)u(x,t) = f(x,t) \tag{5}$$

where $c_\varepsilon(x) = \varepsilon^2 - \frac{\delta^2}{2}\alpha(x) - \frac{\eta^2}{2}\omega(x)$, $p(x) = a(x) - \delta\alpha(x) + \eta\omega(x)$, $q(x) = \alpha(x) + \beta(x) + \omega(x)$ with initial and boundary conditions:

$$\begin{aligned} u(x,0) &= u_0(x), \quad x \in \bar{\Omega} \\ u(0,t) &= \phi(0,t), \quad t \in \bar{\Lambda} \\ u(1,t) &= \psi(1,t), \quad t \in \bar{\Lambda} \end{aligned} \tag{6}$$

For small δ and η, Equations (2)-(3) and (5)-(6) are asymptotically equivalent, because the difference between the two equations is $O(\delta^3, \eta^3)$. We assume again $0 < c_\varepsilon(x) \leq \varepsilon^2 - \delta^2 G_1 - \eta^2 G_2 = c_\varepsilon$ where $2G_1$ and $2G_2$ are the lower bounds for $\alpha(x)$ and $\omega(x)$ respectively. It is also assumed that $p(x) = a(x) - \delta\alpha(x) + \eta\omega(x) \geq p^* > 0$, which show the existence of boundary layer on the right side of the domain D. In case $p(x) = a(x) - \delta\alpha(x) + \eta\omega(x) \leq p^* < 0$, shows the existence of the boundary layer on the left side of the domain D and we can treat it in similar manner.

We set the compatibility conditions:

$$\begin{aligned} u_0(0) &= \phi(0,0) \\ u_0(1) &= \psi(1,0) \end{aligned} \tag{7}$$

and

$$\begin{aligned} \frac{\partial \phi(0,0)}{\partial t} - c_\varepsilon\frac{\partial^2 u_0(0)}{\partial x^2} + p(0)\frac{\partial u_0(0)}{\partial x} + q(0)u_0(0) = f(0,0) \\ \frac{\partial \psi(1,0)}{\partial t} - c_\varepsilon\frac{\partial^2 u_0(1)}{\partial x^2} + p(1)\frac{\partial u_0(1)}{\partial x} + q(1)u_0(1) = f(1,0) \end{aligned} \tag{8}$$

so that, the data matches at the two corner points $(0,0)$ and $(1,0)$. In the considered case, boundary layer occurs near the right side of the rectangular domain D and hence using compatibility conditions in (7) and (8), we can have the following conditions that guarantee the existence of a constant C independent of c_ε such that for all $(x,t) \in \bar{D}$

$$|u(x,t) - u(x,0)| = |u(x,t) - u_0(x)| \leq Ct \quad \text{and}$$
$$|u(x,t) - u(0,t)| = |u(x,t) - \phi(0,t)| \leq C(1-x)$$

for the detail of this see [23] page 105 or [8].

Remark: We have to note that there does not exist a constant C independent of c_ε such that $|u(x,t) - u(1,t)| = |u(x,t) - \psi(1,t)| \leq Cx$ because a boundary layer will occur near the right side of the rectangular domain D. The reduced problem by setting $c_\varepsilon = 0$ in (5) is given by:

$$\frac{\partial u^0}{\partial t} + p(x)\frac{\partial u^0}{\partial x} + q(x)u^0(x,t) = f(x,t), \quad \forall (x,t) \in D \tag{9}$$

$$u^0(x,0) = u_0(x), \quad x \in \bar{\Omega}$$
$$u^0(0,t) = \phi_0(t), \quad 0 \leq t \leq T$$

This is a first order hyperbolic equation with initial data specified along two sides $t = 0$ and $x = 0$ of the domain $\bar{D}$. For small values of c_ε the solution $u(x,t)$ of the problem in (5)-(6) will be very close to $u^0(x,t)$. In order to obtain error bounds on the solution of the difference scheme, it is assumed that the solution of the reduced problem in (9) is sufficiently smooth.

Properties of continuous solution

In order to show on the bounds of the solutions $u(x,t)$ of (5), we assume, without loss of generality the initial condition to be zero [8]. Since $u_0(x)$ is sufficiently smooth and using the property of norm, we can prove the following lemma.

Lemma 1: The bound on the solution $u(x,t)$ of the continuous problem (5)-(6) is given by:

$$|u(x,t)| \leq C, \quad \forall (x,t) \in \bar{D}$$

Proof: From the inequality $|u(x,t) - u(x,0)| = |u(x,t) - u_0(x)| \leq Ct$, we have

$$|u(x,t)| - |u_0(x)| \leq |u(x,t) - u_0(x)| \leq Ct$$
$$\Rightarrow \quad |u(x,t)| \leq Ct + |u_0(x)|, \quad \forall (x,t) \in \bar{D}$$

since $t \in [0, T]$ and $u_0(x)$ is bounded it implies $|u(x,t)| \leq C$

Lemma 2: Continuous maximum principle:
Let z be a sufficiently smooth function defined on D which satisfies $z(x,t) \geq 0, \ \ \forall (x,t) \in \partial D$. Then $Lz(x,t) > 0, \ \ \forall (x,t) \in D$ implies that $z(x,t) \geq 0, \ \ \forall (x,t) \in \bar{D}$. where $Lz = z_t - c_\varepsilon z_{xx} + p(x) z_x + q(x) z$
Proof: Let (x^*, t^*) be such that $z(x^*, t^*) = \min_{(x,t) \in \bar{D}} z(x,t)$, and suppose that $z(x^*, t^*) < 0$. It is clear that $z(x^*, t^*) \notin \partial D$. So we have

$$Lz(x^*, t^*) = z_t(x^*, t^*) - c_\varepsilon z_{xx}(x^*, t^*) + p(x) z_x(x^*, t^*) + q(x) z(x^*, t^*)$$

Since $z(x^*, t^*) = \min_{(x,t) \in \bar{D}} z(x,t)$ which implies $z_x(x^*, t^*) = 0$, $z_t(x^*, t^*) = 0$ and $z_{xx}(x^*, t^*) \geq 0$ and implies that $Lz(x^*, t^*) < 0$ which contradict the assumption made above $Lz(x^*, t^*) > 0, \ \forall (x,t) \in D$.
Therefore

$$z(x,t) \geq 0, \quad \forall (x,t) \in \bar{D}$$

Lemma 3: Stability Estimate: Let $u(x,t)$ be the solution of the continuous problem in (5)-(6). Then we have the bound

$$u \leq q^{-1} ||f|| + \max\{u_0(x), \max\{\phi(x,t), \psi(x,t)\}\}$$

Proof: We define two barrier functions $\vartheta^{\pm}$ as

$$\vartheta^{\pm}(x,t) = q^{-1} ||f|| + \max\{u_0(x), \max\{\phi(x,t), \psi(x,t)\}\} \pm u(x,t)$$

At the initial stage we have

$$\begin{aligned}
\vartheta^{\pm}(x,0) &= q^{-1} ||f|| + \max\{u_0(x), \max\{\phi(x,t), \psi(x,t)\}\} \pm u(x,0) \\
&= q^{-1} ||f|| + \max\{u_0(x), \max\{\phi(x,t), \psi(x,t)\}\} \pm u_0(x) \\
&\geq 0.
\end{aligned}$$

at the boundaries we obtain:

$$\begin{aligned}
\vartheta^{\pm}(0,t) &= q^{-1} ||f|| + \max\{u_0(0), \max\{\phi(0,t), \psi(0,t)\}\} \pm u(0,t) \\
&= q^{-1} ||f|| + \max\{u_0(0), \max\{\phi(0,t), \psi(0,t)\}\} \pm u_0(0) \\
&\geq 0. \\
\vartheta^{\pm}(1,t) &= q^{-1} ||f|| + \max\{u_0(1), \max\{\phi(1,t), \psi(1,t)\}\} \pm u(1,t) \\
&= q^{-1} ||f|| + \max\{u_0(1), \max\{\phi(1,t), \psi(1,t)\}\} \pm u_0(1) \\
&\geq 0. \ \text{ and}
\end{aligned}$$

$$\begin{aligned}L\vartheta^{\pm}(x,t) &= \vartheta_t^{\pm}(x,t) - c_\varepsilon \vartheta_{xx}^{\pm}(x,t) + p(x)\vartheta_x^{\pm}(x,t) + q(x)\vartheta^{\pm}(x,t)\\ &= (\max\{\phi_t(x,t), \psi_t(x,t)\} \pm u_t(x,t)) - c_\varepsilon(\max\{\phi_{xx}(x,t),\\ &u_{0xx}(x), \psi_{xx}(x,t)\} \pm u_{xx}(x,t)) +\\ &p(x)\big(\max\{u_{0x}(x,t), \max\{\phi_x(x,t), \psi_x(x,t)\}\} \pm u_x(x,t)\big) +\\ &q(x)\big(q^{-1}||f|| + \max\{u_0(x), \max\{\phi(x,t), \psi(x,t)\}\} \pm u(x,t)\big)\\ &\geq 0\end{aligned}$$

since $c_\varepsilon \geq 0,\ p(x) \geq p^* > 0$ and $q(x) \geq q > 0$. which implies that

$$L\vartheta^{\pm}(x,t) \geq 0$$

Hence by maximum principle we obtain:

$$\vartheta^{\pm}(x,t) \geq 0, \quad \forall (x,t) \in \bar{D}$$

which gives the required stability estimate

$$u(x,t) \leq q^{-1}||f|| + \max\{u_0(x), \max\{\phi(x,t), \psi(x,t)\}\}$$

Lemma 4: The bound on the derivative of the solution $u(x,t)$ of the problem in (5)-(6) with respect to x is given by:

$$\left|\frac{\partial^i u(x,t)}{\partial x^i}\right| \leq C\big(1 + c_\varepsilon^{-i} e^{-p^*(1-x)/c_\varepsilon}\big), \quad \forall (x,t) \in \bar{D},\ i = 0,1,2,3,4.$$

Proof: See on [11].

3. FORMULATION OF NUMERICAL SCHEME

3.1. **SPATIAL DISCRETIZATION.** The theoretical basis of non-standard discrete modeling method is based on the concept of 'exact' and 'best' finite difference schemes. In [18], Mickens presented techniques for constructing non-standard finite difference methods. According to Mickens's rules, to construct a discrete scheme, denominator function for the discrete derivatives must be expressed in terms of more complicated functions of step sizes than those used in the standard or classical procedure. These complicated denominator functions constitutes a general property of these schemes, which is useful for constructing the reliable scheme for such problems. On the spatial domain $[0,1]$, we introduce the equidistant meshes with uniform mesh length $\Delta x = h$ such that $\Omega_x^N = \{x_i = x_0 + ih,\ i = 0,1,...,N, \ \ x_0 = 0, x_N = 1,\ h = 1/N\}$ where N is the number of mesh points in spatial direction. For the problem in (5), we consider the sub-equation which is influenced by the spatial derivative i.e.

$$-c_\varepsilon \frac{d^2u(x)}{dx^2} + p(x)\frac{du(x)}{dx} = 0 \tag{10}$$

Then using the finite difference scheme as

$$-c_\varepsilon \frac{U_{i+1} - 2U_i + U_{i-1}}{\gamma_i^2} + p(x_i)\frac{U_i - U_{i-1}}{h} = 0 \tag{11}$$

we calculate for the denominator function γ_i^2 using the following procedures. First we rewrite the equation (10) equivalently as a system of two first order coupled differential equations as

$$\frac{du}{dx} = y, \tag{12}$$

$$\frac{dy}{dx} = \frac{p(x)}{c_\varepsilon}y \tag{13}$$

which implies $y = \exp\big(\frac{p(x)}{c_\varepsilon}x\big)$, so we have $y_i = \exp\big(\frac{p(x_i)}{c_\varepsilon}x_i\big)$.
By applying first order difference scheme for (12) as:

$$y_i = \frac{U_{i+1} - U_i}{h} \tag{14}$$

then solving for γ_i^2 from (11) we obtain:

$$\gamma_i^2 = \frac{hc_\varepsilon}{p(x_i)}\big(\exp\big(\frac{hp(x_i)}{c_\varepsilon}\big) - 1\big)$$

By using the denominator function γ_i^2 in to the main scheme, we obtain the difference scheme as:

$$U_t(x_i,t) - c_\varepsilon \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{\gamma_i^2} + p(x_i)\frac{U_i(t) - U_{i-1}(t)}{h} + q(x_i)U_i(t) = f(x_i,t)$$

where γ_i^2 is defined above and it is a function of $c_\varepsilon, p(x_i)$ and h. At this stage the problem in (5)-(6) reduces to semi discrete form as

$$\begin{aligned} L^hU_i(t) \equiv \frac{dU_i(t)}{dt} - c_\varepsilon \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{\gamma_i^2} \\ + p(x_i)\frac{U_i(t) - U_{i-1}(t)}{h} + q(x_i)U_i(t) = f_i(t) \end{aligned} \tag{15}$$

together with the semi-discrete boundary conditions and initial condition:

$$U_0(t) = \phi(0,t), \quad U_N(t) = \psi(1,t) \ \text{ and } \ U_i(0) = u_0(x_i) \tag{16}$$

The above system of equations of initial value problems in (15)-(16) can be written in compact form as:

$$\frac{dU_i(t)}{dt} + AU_i(t) = F_i(t) \tag{17}$$

Here A is a tridiagonal matrix of $N-1 \times N-1$ and $U_i(t)$ and $F_i(t)$ are $N-1$ column vectors. The entries of A and F are given as:

$$\begin{aligned} A_{ii} &= \frac{2c_\varepsilon}{\gamma_i^2} + \frac{p(x_i)}{h} + q(x_i), \quad i = 1, 2, ..., N-1 \\ A_{ii+1} &= -\frac{c_\varepsilon}{\gamma_i^2}, \qquad i = 1, 2, ..., N-2 \\ A_{ii-1} &= -\frac{c_\varepsilon}{\gamma_i^2} - \frac{p(x_i)}{h}, \qquad i = 2, ..., N-1 \end{aligned}$$

and

$$\begin{aligned} F_1(t) &= f_1(t) + \left(\frac{c_\varepsilon}{\gamma_1^2} + \frac{p(x_1)}{h}\right)\phi(0, t), \\ F_i(t) &= f_i(t), \qquad i = 2, 3, ..., N-2 \\ F_{N-1}(t) &= f_{N-1}(t) + \left(\frac{c_\varepsilon}{\gamma_{N-1}^2}\right)\psi(1, t) \end{aligned}$$

respectively.
Now we need to show the semi-discrete operator L^h also satisfies the maximum principle and the uniform stability estimate.

Theorem 3.1: The operator defined by the discrete scheme in (17) satisfies a semi-discrete maximum principle.i.e. Suppose $U_0(t) \geq 0,\ U_N(t) \geq 0$. Then $L^hU_i(t) \geq 0, \forall i = 1, 2, ..., N-1$ implies that $U_i(t) \geq 0, \forall i = 0, 1, ..., N$.

Proof: Suppose there exist $p \in \{0, 1, ..., N\}$ such that $U_p(t) = \min_{0 \leq i \leq N} U_i(t)$. Suppose that $U_p(t) < 0$ which implies $p \neq 0, N$. Also we have $U_{p+1} - U_p > 0$ and $U_p - U_{p-1} < 0$. Now we have

$$\begin{aligned} L^hU_p(t) =& \frac{dU_p(t)}{dt} - c_\varepsilon \frac{U_{p+1}(t) - U_p(t) - (U_p(t) - U_{p-1}(t))}{\gamma_p^2} \\ &+ p_p \frac{U_p(t) - U_{p-1}(t)}{h} + q_pU_p(t) < 0 \end{aligned}$$

using the assumption, we obtain $L^hU_i(t) < 0$ for $i = 1, 2, ..., N-1$. Thus the supposition $U_i(t) < 0, i = 0, 1, ..., N$ is wrong.
Hence $U_i(t) \geq 0, \forall i = 0, 1, ..., N$

Lemma 5: The solution $U_i(t)$ of the semi-discrete problem in (17) satisfy the following bound.

$$|U_i(t)| \leq q^{-1}\max|L^hU_i(t)| + \max\{|u_0(x_i)|, \max\{\phi(x_i,t), \psi(x_i,t)\}\}$$

Proof: Let $p = q^{-1}\max|L^hU_i(t)| + \max\{|u_0(x_i)|, \max\{\phi(x_i,t), \psi(x_i,t)\}\}$ and define the barrier function $\vartheta_i^{\pm}(t)$ by

$$\vartheta_i^{\pm}(t) = p \pm U_i(t) \tag{18}$$

At the boundary points we have

$$\begin{aligned}\vartheta_0^{\pm}(t) &= p \pm U_0(t) = p \pm \phi(0,t) \geq 0\\ \vartheta_N^{\pm}(t) &= p \pm U_N(t) = p \pm \psi(1,t) \geq 0\end{aligned}$$

On the discretized domain $0 < i < N$, we have

$$\begin{aligned}L^h\vartheta_i^{\pm}(t) = {}& \frac{d\big(p \pm U_i(t)\big)}{dt} - c_\varepsilon\Big(\frac{p \pm U_{i+1}(t) - 2(p \pm U_i(t)) + p \pm U_{i-1}(t)}{\gamma^2}\Big)\\ &+ p_i\Big(\frac{p \pm U_i(t) - p \pm U_{i-1}(t)}{h}\Big) + q_i(p \pm U_i(t))\\ &= q_ip \pm L^hU_i(t)\\ &= q_i\big(q^{-1}\max|L^hU_i(t)| + \max\{|u_0(x_i)|, \max\{\phi(x_i,t), \psi(x_i,t)\}\}\big)\\ &\pm f_i(t) \geq 0, \quad \text{since } \ q_i \geq q.\end{aligned}$$

From theorem (3.1), we obtain $\vartheta_i^{\pm}(t) \geq 0, \quad \forall (x_i,t) \in \bar{\Omega}^N \times \Lambda$.
The proof is completed.

3.2. **ERROR ESTIMATE FOR SEMI-DISCRETE SCHEME.**
Let $u(x_i,t)$ and $U_i(t)$ are denoted for the exact and the spatial direction approximate semi-discrete solution respectively.
Next let us analyze these spatial discretization for convergence. we proved above the semi-discrete operator L^h satisfy the maximum principle and the uniform stability estimate.
Let define the forward and backward finite differences in space as:

$$D^+v(x_i,t) = \frac{v(x_{i+1},t) - v(x_i,t)}{h}, \ \ D^-v(x_i,t) = \frac{v(x_i,t) - v(x_{i-1},t)}{h}$$

respectively and the second order finite difference operator as:

$$\delta^2 v(x_i,t) = D^+D^-v(x_i,t) = \frac{D^+v(x_i,t) - D^-v(x_i,t)}{h}$$

Theorem 3.2: Let the coefficients functions $p(x), q(x)$ and $f(x,t)$ in (5) be sufficiently smooth functions so that $u(x,t) \in C^4[0,1] \times$

$[0,T]$. Then the semi-discrete solution $U_i(t)$ of the problem (5) - (6) satisfies

$$|L^h\big(u(x_i,t)-U_i(t)\big)| \leq Ch\big(1+\sup_{0\leq i\leq N}\frac{\exp\big(-p^*(1-x_i)/c_\varepsilon\big)}{c_\varepsilon^3}\big) \quad (19)$$

Proof: We consider first

$$\begin{aligned}
\big|L^h\big(u(x_i,t)-U_i(t)\big)\big| &= \big|L^h u(x_i,t)-L^h U_i(t)\big| \\
&\leq C\Big|-c_\varepsilon\big(u_{xx}(x_i,t)-\frac{D_x^+D_x^-h^2}{\gamma_i^2}u(x_i,t)\big) \\
&+p_i\big(u_x(x_i,t)-D_x^-u(x_i,t)\big)\Big| \\
&\leq Cc_\varepsilon\Big|u_{xx}(x_i,t)-D_x^+D_x^-u(x_i,t)\Big| \\
&+Cc_\varepsilon\Big|\big(\frac{h^2}{\gamma_i^2}-1\big)D_x^+D_x^-u(x_i,t)\Big|+Ch\big|u_{xx}(x_i,t)\big| \\
&\leq Cc_\varepsilon h^2\big|u_{xxxx}(x_i,t)\big|+Ch\big|u_{xx}(x_i,t)\big|
\end{aligned}$$

Above used estimate $c_\varepsilon\big|\frac{h^2}{\gamma_i^2}-1\big|\leq Ch$ is based on the non-standard denominator function behavior used in [3].
Let define $\rho=p_ih/c_\varepsilon,\ \rho\in(0,\infty)$. Then

$$c_\varepsilon\Big|\frac{h^2}{\gamma_i^2}-1\Big| = p_ih\Big|\frac{1}{\exp(\rho)-1}-\frac{1}{\rho}\Big| =: p_ihQ(\rho).$$

then we have

$$Q(\rho)=\frac{\exp(\rho)-1-\rho}{\rho(\exp(\rho)-1)}$$

and from this we have

$$\lim_{\rho\to 0}Q(\rho)=\frac{1}{2},\quad \lim_{\rho\to\infty}Q(\rho)=0.$$

Therefore

$$Q(\rho)\leq C,\quad \rho\in(0,\infty)$$

So, the error estimate becomes

$$|L^h\big(u(x_i,t)-U_i(t)\big)| \leq Cc_\varepsilon h^2|u_{xxxx}(x_i,t)|+Ch|u_{xx}(x_i,t)| \quad (20)$$

From (20) and boundedness of derivatives of solution in lemma (4), we obtain:

$$\begin{aligned}\left|L^h\big(u(x_i,t)-U_i(t)\big)\right| &\leq Cc_\varepsilon h^2\left|1+c_\varepsilon^{-4}\exp\big(\frac{-p^*(1-x_i)}{c_\varepsilon}\big)\right|\\ &\quad + Ch\left|1+c_\varepsilon^{-2}\exp\big(\frac{-p^*(1-x_i)}{c_\varepsilon}\big)\right|\\ &\leq Ch\Big(1+\sup_{x\in(0,1)}\frac{\exp\big(\frac{-p^*(1-x)}{c_\varepsilon}\big)}{c_\varepsilon^3}\Big),\\ &\text{since } \ c_\varepsilon^{-2}\leq c_\varepsilon^{-3}\end{aligned}$$

Lemma 6: For a fixed mesh and for $\varepsilon \to 0$, it holds

$$\lim_{c_\varepsilon\to 0}\max_{1\leq j\leq N-1}\frac{\exp\big(\frac{-\alpha(1-x_j)}{c_\varepsilon}\big)}{c_\varepsilon^m}=0, \quad m=1,2,3,... \tag{21}$$

where $x_j = jh, h = 1/N, \forall j = 1, 2, ..., N-1$

Proof: Consider the partition $[0,1] := \{0 = x_0 < x_1 < ... < x_{n-1} < x_n = 1\}$ for the interior grid points, we have

$$\max_{1\leq j\leq n-1}\frac{\exp(-\alpha x_j/c_\varepsilon)}{c_\varepsilon^m}\leq\frac{\exp(-\alpha x_1/c_\varepsilon)}{c_\varepsilon^m}=\frac{\exp(-\alpha h/c_\varepsilon)}{c_\varepsilon^m}$$

and

$$\begin{aligned}\max_{1\leq j\leq n-1}\frac{\exp(-\alpha(1-x_j)/c_\varepsilon)}{c_\varepsilon^m}&\leq\frac{\exp(-\alpha(1-x_{n-1})/c_\varepsilon)}{c_\varepsilon^m}\\ &=\frac{\exp(-\alpha h/c_\varepsilon)}{c_\varepsilon^m},\end{aligned}$$

since $x_1 = h, 1 - x_{n-1} = h$ then application of L'Hospital's rule gives

$$\begin{aligned}\lim_{c_\varepsilon\to 0}\frac{\exp(-\alpha h/c_\varepsilon)}{c_\varepsilon^m}&=\lim_{p=1/c_\varepsilon\to\infty}\frac{p^m}{\exp(\alpha hp)}\\ &=\lim_{p=1/c_\varepsilon\to\infty}\frac{m!}{(\alpha h)^m\exp(\alpha hp)}=0\end{aligned}$$

this shows the proof is completed.

Theorem 3.3: Under the hypothesis of boundedness of semi-discrete solution, lemma 6 and theorem 3.2 above, the semi-discrete solution satisfy the following bound.

$$\sup_{0<c_\varepsilon\ll 1}\|u(x_i,t)-U_i(t)\|_{\Omega^N\times\Lambda}\leq CN^{-1} \tag{22}$$

Proof: Immediate result from boundedness of solution, lemma (6) and theorem (3.2) will give the required estimates.

3.3. **DISCRITIZATION IN TEMPORAL DIRECTION.**

On the time domain $[0, T]$, we introduce the discretization with time step $\Delta t_j = t_{j+1} - t_j, j = 0, 1, 2, ..., M$ such that $\Lambda^M = \Omega_t^M$ where M denotes the number of mesh in time direction. Let $U_{i,j}$ is denoted for the approximate solution of the problem. At this stage we use low order numerical method to discretize the system of initial value problems in (17) using special type of Runge-Kutta method developed by Bogacki and Shampine in 1989 with order two and three implicit given in [24],[25]. First rewrite (17) as the form:

$$\frac{dU_i(t)}{dt} = f\big(t, U_i(t)\big)$$

with the initial condition $U(x_i, 0) = u_0(x_i),\ i = 0, 1, 2, ..., N$, here $f\big(t, U_i(t)\big) = -AU_i(t) + F_i(t)$ so for each $j = 1, 2, ..., M$ we construct the scheme as:

$$\begin{aligned}
k_1 &= f(t_j, U_{i,j}),\\
k_2 &= f(t_j + \frac{1}{2}\Delta t_j, U_{i,j} + \frac{1}{2}\Delta t_j k_1),\\
k_3 &= f(t_j + \frac{3}{4}\Delta t_j, U_{i,j} + \frac{3}{4}\Delta t_j k_2),\\
U^*_{i,j+1} &= U_{i,j} + \frac{2}{9}\Delta t_j k_1 + \frac{1}{3}\Delta t_j k_2 + \frac{4}{9}\Delta t_j k_3,\\
k_4 &= f(t_j + \Delta t_j, U^*_{i,j+1}),\\
U_{i,j+1} &= U_{i,j} + \frac{7}{24}\Delta t_j k_1 + \frac{1}{4}\Delta t_j k_2 + \frac{1}{3}\Delta t_j k_3 + \frac{1}{8}\Delta t_j k_4,
\end{aligned}$$

for $i = 1, 2, ..., N-1$.

The difference between $U^*_{i,j+1}$ and $U_{i,j+1}$ can be used to adapt the step size. It is stated in [14],[25] that, for $j = 1, 2, ..., M$ the local approximation $U_{i,j+1}$ to $U_i(t_{j+1})$ has third order (i.e. $O(\Delta t_j)^3$) accuracy.

Let $\Delta t = \max_{0 \le j \le M} \Delta t_j$ then, we have the following lemma.

Lemma 7: From the above approximation method in temporal direction, the global error estimates in this direction are given by

$$\|E_{j+1}\|_\infty = \|U_i(t_{j+1}) - U_{i,j+1}\|_{\Omega^N \times \Lambda^M} \le C(\Delta t)^2$$

where E_{j+1} is the global error in the temporal direction at $(j+1)^{th}$ time level.

Proof: Using the local error estimate upto jth time step, we obtain the global error estimate at $(j+1)th$ time step.

$$
\begin{aligned}
\|E_{j+1}\|_\infty &= \sum_{l=1}^{j} \|e_l\|_\infty, \quad j \leq M, \\
&\leq \|e_1\|_\infty + \|e_2\|_\infty + ... + \|e_j\|_\infty, \quad \|e_l\|_\infty = C_l(\Delta t_j)^3 \\
&\leq C_1(j\Delta t)(\Delta t)^2 \\
&\leq C_1 T(\Delta t)^2, \quad \text{since} j\Delta t \leq T \\
&\leq C(\Delta t)^2
\end{aligned}
$$

Assume that $C(\Delta t)^2 \leq N^{-1}$ for comparable number of mesh points, then using the boundedness of the solution and lemma (3.3) implies

$$
\sup_{0<c_\varepsilon \ll 1} \|U_i(t_{j+1}) - U_{i,j+1}\|_{\Omega^N \times \Lambda^M} \leq CN^{-1} \tag{23}
$$

this shows that the discretization in temporal direction is consistent and global error is bounded.

Now we use (23) to prove the parameter uniform convergence of the fully discrete scheme as:

$$
\begin{aligned}
\sup_{0<c_\varepsilon \ll 1} \|u(x_i, t_j) - U_{i,j}\|_{\Omega^N \times \Lambda^M} \leq & \sup_{0<c_\varepsilon \ll 1} \|u(x_i, t_j) - U_i(t_j)\|_{\Omega^N \times \Lambda^M} \\
& + \sup_{0<c_\varepsilon \ll 1} \|U_i(t_j) - U_{i,j}\|_{\Omega^N \times \Lambda^M}
\end{aligned} \tag{24}
$$

Using boundedness of the solution, theorem 3.3, lemma 7 and equation (23) we obtain:

$$
\sup_{0<c_\varepsilon \ll 1} \|u(x_i, t_j) - U_{i,j}\|_{\Omega^N \times \Lambda^M} \leq CN^{-1} \tag{25}
$$

Remark: The inequality in (25) shows the parameter uniform convergence of the proposed scheme with order $O(h)$, for $h = N^{-1}$.

4. NUMERICAL RESULTS AND DISCUSSION

To confirm the established theoretical results in this study, we perform some experiments using the proposed numerical scheme on the problem of the form given in (2) - (3).

Example 1: In this example, we take the functions $a(x) = 2 - x^2$, $\alpha(x) = 2$, $\beta(x) = x - 3$, $\omega(x) = 1$, $f(x,t) = 10t^2 \exp(t)x(1 -$

TABLE 1. Example 1's maximum absolute error for the proposed method and results in [20] and [13] for $\delta = 0.6\varepsilon$, and $\eta = 0.5\varepsilon$.

ε $\downarrow$	N=32 M=60	N=64 M=120	N=128 M=240	N=256 M=480	N=512 M=960
Our Scheme					
2^{-6}	9.7515e-03	4.8801e-03	2.4414e-03	1.2211e-03	6.1064e-04
2^{-8}	9.7071e-03	4.8578e-03	2.4303e-03	1.2155e-03	6.0785e-04
2^{-10}	9.6961e-03	4.8523e-03	2.4275e-03	1.2141e-03	6.0716e-04
2^{-12}	9.6933e-03	4.8509e-03	2.4268e-03	1.2138e-03	6.0698e-04
2^{-14}	9.6927e-03	4.8506e-03	2.4266e-03	1.2137e-03	6.0694e-04
2^{-16}	9.6925e-03	4.8505e-03	2.4266e-03	1.2137e-03	6.0693e-04
2^{-18}	9.6924e-03	4.8505e-03	2.4266e-03	1.2137e-03	6.0693e-04
2^{-20}	9.6924e-03	4.8505e-03	2.4266e-03	1.2137e-03	6.0693e-04
Result in[20]					
2^{-6}	1.0012e-02	6.0741e-03	3.5186e-03	1.9947e-03	1.1196e-03
2^{-8}	9.9898e-03	6.0284e-03	3.4715e-03	1.9503e-03	1.0808e-03
2^{-10}	9.9962e-03	6.0278e-03	3.4695e-03	1.9481e-03	1.0781e-03
2^{-12}	9.9986e-03	6.0283e-03	3.4697e-03	1.9481e-03	1.0781e-03
2^{-14}	9.9992e-03	6.0284e-03	3.4698e-03	1.9482e-03	1.0781e-03
2^{-16}	9.9993e-03	6.0285e-03	3.4698e-03	1.9482e-03	1.0781e-03
2^{-18}	9.9994e-03	6.0285e-03	3.4698e-03	1.9482e-03	1.0781e-03
2^{-20}	1.0000e-02	6.0282e-03	3.4696e-03	1.9483e-03	1.0781e-03
Result in [13]					
2^{-6}	7.3811e-03	4.3778e-03	2.1211e-03	1.0929e-03	5.5057e-04
2^{-8}	7.4985e-03	4.4456e-03	2.3806e-03	1.2446e-03	5.7770e-04
2^{-10}	7.5020e-03	4.4954e-03	2.4363e-03	1.2530e-03	6.2975e-04
2^{-12}	7.4982e-03	4.4966e-03	2.4448e-03	1.2716e-03	6.4618e-04
2^{-14}	7.4970e-03	4.4961e-03	2.4450e-03	1.2728e-03	6.4893e-04
2^{-16}	7.4966e-03	4.4959e-03	2.4450e-03	1.2728e-03	6.4909e-04
2^{-18}	7.4966e-03	4.4958e-03	2.4449e-03	1.2728e-03	6.4909e-04
2^{-20}	7.4965e-03	4.4958e-03	2.4449e-03	1.2728e-03	6.4909e-04

$x)$, $u_0(x) = 0$, $\phi(x) = \psi(x) = 0$ and $T = 3$.

Example 2: In this example, we take the functions $a(x) = 2 - x^2$, $\alpha(x) = 1 + x$, $\beta(x) = x^2 + 1 + \cos(\pi x)$, $\omega(x) = 3$, $f(x,t) = \sin(\pi x)$, $u_0(x) = 0$, $\phi(x) = \psi(x) = 0$ and $T = 3$.

Exact solution is not available for these two problems, so the maximum point-wise errors are calculated by using the double mesh principle given by:

$$E_{\varepsilon,\delta,\eta}^{N,M} = \max_{1\le i\le N-1, 1\le j\le M-1} \left| U_{i,j}^{N,M} - U_{i,j}^{2N,2M} \right|$$

Table 2. Comparison of results for Example 1's maximum absolute error and rate of convergence.

Scheme $\downarrow$	N=64 M=120	N=128 M=240	N=256 M=480	N=512 M=960
Our Scheme				
$E^{N,M}$	4.8801e-03	2.4414e-03	1.2211e-03	6.1064e-04
$r^{N,M}$	0.9992	0.9995	0.9998	0.9999
Result in[20] by upwind				
$E^{N,M}$	9.2021e-03	4.9863e-03	2.6885e-03	1.4245e-03
$r^{M,N}$	0.8840	0.8912	0.9163	0.9178
Result in[20] by Midpt.				
$E^{N,M}$	6.5942e-03	3.8199e-03	2.1180e-03	1.1399e-03
$r^{N,M}$	0.7877	0.8508	0.8938	0.9107
Result in[13] by B-spline				
$E^{N,M}$	4.4966e-03	2.4450e-03	1.2728e-03	6.4609e-04
$r^{N,M}$	0.8791	0.9418	0.9715	0.9859

Table 3. Example 2's maximum absolute error for the proposed method and result in [13] for $\delta = 0.6\varepsilon$, and $\eta = 0.5\varepsilon$.

ε $\downarrow$	N=16 M=30	N=32 M=60	N=64 M=120	N=128 M=240	N=256 M=480
Our Scheme					
2^{-6}	9.2896e-03	5.7917e-03	3.1887e-03	1.6682e-03	8.5261e-04
2^{-8}	9.3200e-03	5.8078e-03	3.1969e-03	1.6723e-03	8.5467e-04
2^{-10}	9.3277e-03	5.8119e-03	3.1990e-03	1.6733e-03	8.5519e-04
2^{-12}	9.3296e-03	5.8129e-03	3.1995e-03	1.6736e-03	8.5532e-04
2^{-14}	9.3300e-03	5.8132e-03	3.1996e-03	1.6737e-03	8.5535e-04
2^{-16}	9.3302e-03	5.8132e-03	3.1996e-03	1.6737e-03	8.5536e-04
2^{-18}	9.3302e-03	5.8132e-03	3.1997e-03	1.6737e-03	8.5536e-04
2^{-20}	9.3302e-03	5.8132e-03	3.1997e-03	1.6737e-03	8.5536e-04
Result in [13]					
2^{-6}	1.5094e-02	7.5604e-03	3.8074e-03	1.9127e-03	9.5839e-04
2^{-8}	1.5222e-02	7.6300e-03	3.8341e-03	1.9259e-03	9.6541e-04
2^{-10}	1.5237e-02	7.6373e-03	3.8377e-03	1.9274e-03	9.6613e-04
2^{-12}	1.5240e-02	7.6385e-03	3.8382e-03	1.9277e-03	9.6624e-04
2^{-14}	1.5240e-02	7.6387e-03	3.8383e-03	1.9277e-03	9.6627e-04
2^{-16}	1.5241e-02	7.6388e-03	3.8384e-03	1.9277e-03	9.6627e-04
2^{-18}	1.5241e-02	7.6388e-03	3.8384e-03	1.9277e-03	9.6627e-04
2^{-20}	1.5241e-02	7.6388e-03	3.8384e-03	1.9277e-03	9.6627e-04

TABLE 4. Example 2's maximum absolute error and rate of convergence by our scheme and result in [13].

ε $\downarrow$	N=16 M=30	N=32 M=60	N=64 M=120	N=128 M=240	N=256 M=480
Our Scheme					
$E^{N,M}$	9.3302e-03	5.8132e-03	3.1997e-03	1.6737e-03	8.5536e-04
$r^{N,M}$	0.6826	0.8614	0.9349	0.9685	0.9844
Result in[13]					
$E^{N,M}$	1.5241e-02	7.6388e-03	3.8384e-03	1.9277e-03	9.6627e-04
$r^{N,M}$	0.9924	0.9925	0.9925	0.9964	0.9981

where N and M are the number of mesh points in x and t direction respectively. $U_i^{N,M}$ are the computed solution of the problem using N, M mesh numbers and $U_{2i,2j}^{2N,2M}$ are computed solution on double number of mesh points $2N, 2M$ by adding the mid points $x_{i+1/2} = \frac{x_{i+1}+x_i}{2}$ and $t_{j+1/2} = \frac{t_{j+1}+t_j}{2}$ into the mesh points . For any value of the mesh point N and M the ε-uniform error estimate are calculated using the formula

$$E^{N,M} = \max_{\varepsilon,\delta,\eta} |E_{\varepsilon,\delta,\eta}^{N,M}|$$

The rate of convergence of the method is calculated using the formula given by

$$r_{\varepsilon,\delta,\eta}^{N,M} = \log_2 \left(E_{\varepsilon,\delta,\eta}^{N,M} / E_{\varepsilon,\delta,\eta}^{2N,2M}\right) = \frac{\log\left(E_{\varepsilon,\delta,\eta}^{N,M}\right) - \log\left(E_{\varepsilon,\delta,\eta}^{2N,2M}\right)}{\log 2}$$

and the $\varepsilon-$ uniform rate of convergence is calculated using the formula given by

$$r^{N,M} = \log_2 \left(E^{N,M} / E^{2N,2M}\right) = \frac{\log\left(E^{N,M}\right) - \log\left(E^{2N,2M}\right)}{\log 2}$$

The solution of Examples 1 and 2 above has a boundary layer at the right side of the x-domain (see figures in $1, 2$ and 3). The computed solutions $U_{i,j}$ for different values of perturbation parameters are also shown in figures 2 and 3, and the effect of delay and advance parameters is shown in figures 4 by using different values for delay and advance parameters to our test problems. The numerical results displayed in tables 1 and 4 clearly indicate that the proposed method based on MOL by using a non-standard finite difference method in x direction with implicit Runge-Kutta method in t direction is parameter-uniformly convergent. From the results in tables 1 and 2, we observe that the maximum point-wise error

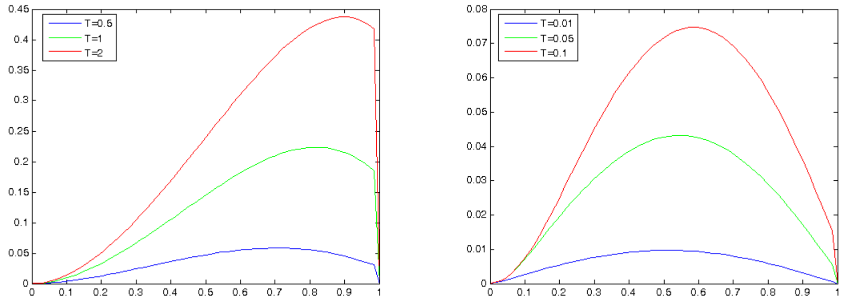

FIGURE 1. Numerical solution at different time step on left hand side Example 1 and Example 2 on right hand side, when $\varepsilon = 2^{-4}, \delta = 0.6\varepsilon, \eta = 0.5\varepsilon$.

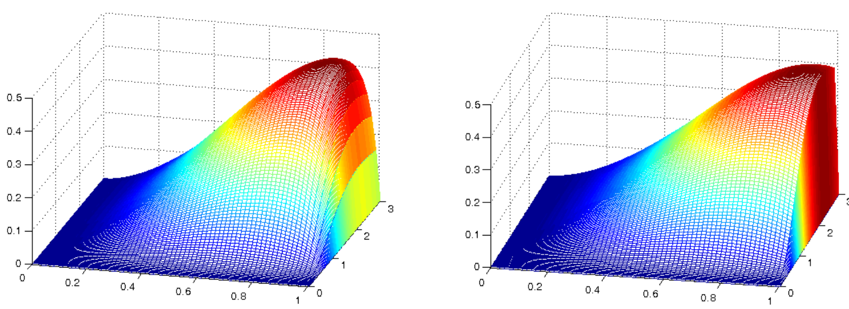

FIGURE 2. The numerical solution of Example 1 when $\varepsilon = 2^{-2}$, on left hand side and when $\varepsilon = 2^{-20}$, on right hand side for $\delta = 0.6\varepsilon, \eta = 0.5\varepsilon$ resp. exhibiting a layer on the right side of the domain.

$E^{N,M}_{\varepsilon,\delta,\eta}$ decreases as N increases for each value of ε. We see that the maximum point-wise error is stable as $\varepsilon \to 0$ for each N, M. Using these two examples we confirm that the proposed numerical method is more accurate, stable and $\varepsilon-$ uniform convergent with rate of convergence almost one.

Numerical results shows the parameter-uniformness of the proposed

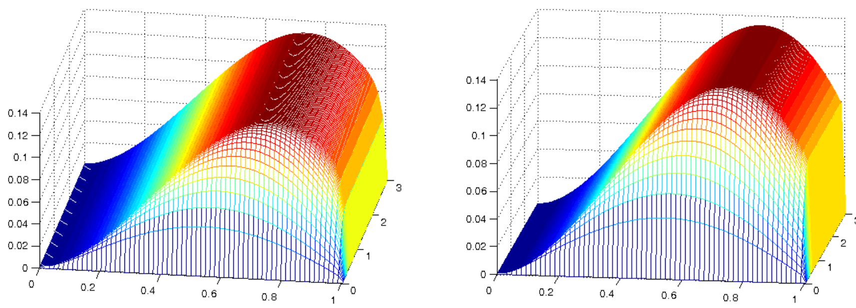

FIGURE 3. The numerical solution of Example 2 when $\varepsilon = 2^{-2}$, on left hand side and when $\varepsilon = 2^{-20}$, on right hand side for $\delta = 0.6\varepsilon, \eta = 0.5\varepsilon$ resp. exhibiting a layer on the right side of the domain.

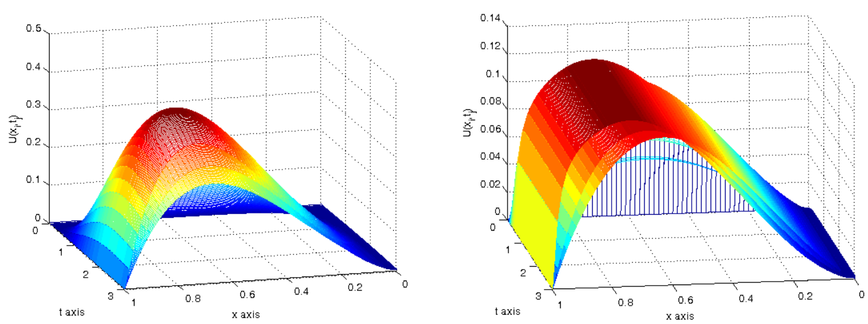

FIGURE 4. On the left hand side, numerical solution of Example 1 for $\varepsilon = 2^{-1}$ and for different $\delta = \eta = 0, 0.4\varepsilon, 0.6\varepsilon$ in increasing order and on the right hand side Example 2's numerical solution for $\varepsilon = 2^{-1}$ and for different $\delta = \eta = 0, 0.4\varepsilon, 0.6\varepsilon$ in increasing order on the figures.

scheme on uniform mesh. The results by this method is better than that obtained in [13], [20] (see table $1 - 4$).

5. CONCLUSIONS

A numerical method is developed to solve a singularly perturbed parabolic differential-difference equation with both delay and advance parameters that the solution exhibit a boundary layer. This method is based on method of line that constitute the non-standard finite difference for the spatial discretization and the Runge-Kutta order 2 and 3 implicit method in the temporal direction for the system of initial value problem resulting from the spatial discretization. Stability and convergence analysis of the proposed scheme is proved. The applicability of the proposed scheme is investigated by taking two examples. The effect of the perturbation parameter, the delay and advance parameters on the solution of the problem are shown by using figures. The method is shown to be uniformly convergent i.e., independent of perturbation parameter with order of convergence one. The performance of the proposed scheme is investigated by comparing the results with prior studies. It has been found that the proposed method gives more accurate and stable numerical results.

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