Journal of the
Nigerian Mathematical Society

Vol. 35, pp. 546-559, 2016

A NEW APPROACH FOR FINDING CLOSED FORM SOLUTION OF NTH ORDER INITIAL VALUE PROBLEMS

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ABSTRACT. This article proposes a new technique for finding exact solution of N-th order, linear, nonlinear and stiff initial value problems. By recasting the problem as a system of constant coefficient polynomial ordinary differential equation, the coefficients of the power series solution is computed iteratively. The closed form solution is obtained from the truncated series solution by applying Padé and Laplace-Padé post-processing. The application of the method to various problems considered elucidated the simplicity and high accuracy of the proposed approach.

Keywords and phrases: Initial Value Problem, Analytical Solution, Power Series, Laplace-Padé series summation.
2010 Mathematical Subject Classification: 35G25

1. INTRODUCTION

Initial Value Problems (IVPs) continue to receive research attentions in the literature due to their numerous relevance in modeling real life problems emanating from physics, engineering and biology. Quite often, these IVPs do not possess the desired closed form solution. In the past years, research efforts have been devoted to finding closed form or semi-analytical or series solution of IVPs. Famous existing methods to compute power series or semi-analytic solution of IVP in the literature include Adomian Decomposition Method (ADM) [2], Variational Iteration Method (VIM) [10], Homotopy Perturbation Method (HPM) [9] and their various modifications. These methods have been successfully applied to find series solution of differential equations in general, see e.g. [8, 11, 18]. However, their application can be a bit technical and often demands a bit of extra work to be successfully applied. For instance, computation of Lagrange multiplier is a convergence criterion for VIM. Likewise, appropriate choice of initial approximation satisfying the initial or boundary conditions is a requirement in order to successfully apply both the ADM and HPM. In addition, like other perturbation

Received by the editors December 23, 2015; Revised: November 10, 2016; Accepted: December 02, 2016

methods, HPM is not applicable to problems with no small parameters. In particular, this turns out to be a serious limitation as most IVP modeling real life problems do not possess small parameters. Moreover the aforementioned methods require a fairly good computing capability for efficient application.

In view of the aforementioned challenges with existing methods, there is need to device other methods of computing series solution of IVP especially in settings where the luxury of computing capability does not exist. In this article, we propose a new, easier to implement and efficient method to compute exact solution of N-th order nonlinear initial value problem (IVP) of the form

$$y^{(N)}(t) = f(t, y(t), y'(t), y''(t), \ldots, y^{(N-1)}(t)), \quad t \geq t_0, \tag{1}$$

subject to initial conditions

$$y(t_0) = y_0,\ y'(t_0) = y'_0, \ldots, y^{(N-1)}(t_0) = y_0^{(N-1)}.$$

We assume that the function f and all its partial derivatives are continuous on some open interval containing the initial values $y_0, y'_0, \ldots, y_0^{N-1}$ so that problem (1) possess a unique solution over an open interval containing t_0.

A blend of power series methods and appropriate convergence accelerator remain an important recipe for solving IVPs. Therefore, following [16], the series solution of (1) is computed through simple recursion and using basic elementary operations. Thereafter, by subjecting the truncated series solution to Padé or Laplace-Padé post processing operation, the closed form solution of the IVP is obtained. We emphasize again that the proposed method, though simple and straightforward, is devoid of the challenges of the older existing methods. In terms of accuracy, the method competes well with famous methods in the literature.

2. SOLUTION METHODOLOGY

The first step in the proposed approach is to recast (1) as a system of constant coefficient polynomial ordinary differential equation (ODE). This is achieved by adopting the variable substitutions

$$y_1(t) = y(t),\ \ y_i(t) = y^{(i-1)}(t), \quad i = 2, \ldots, N-1,$$

which reduces (1) to a system of first order equations of the form

$$\mathbf{y}' = F(t, \mathbf{y}), \quad \mathbf{y}(t_0) = \mathbf{y}_0. \tag{2}$$

If function F is not already a constant-coefficient polynomial function at this stage, skillful choice of further variable substitutions reduces F to a constant-coefficient polynomial function $F(\mathbf{y})$, see Section 3 below.

We assume without loss of generality that $t_0 = 0$. Note that this choice is not a restriction on the applicability of the method as the scaling $t \to t - t_0$ can always bring problems with $t_0 \neq 0$ to this case.

In the next step we assume power series expansion for all the dependent variables in (2), that is,

$$\mathbf{y}(t) = \sum_{i=0}^{N} \mathbf{y}_i t^i; \tag{3}$$

$$\mathbf{y}'(t) = \sum_{i=0}^{N} \mathbf{y}'_i t^i. \tag{4}$$

We shall adopt the modified Picard iteration of [16] to compute the coefficients $\mathbf{y}_i$. Therefore, differentiating (3) and appropriately shifting indexes, we obtained

$$\mathbf{y}'(t) = \frac{d}{dt}\left(\sum_{i=0}^{N} \mathbf{y}_i t^i\right) = \sum_{i=0}^{N} i\mathbf{y}_i t^{i-1} = \sum_{i=0}^{N} (i+1)\mathbf{y}_{i+1} t^i.$$

Hence comparing the above with (4) it holds $\sum_{i=0}^{N} \mathbf{y}'_i t^i = \sum_{i=0}^{N} (i+1)\mathbf{y}_{i+1} t^i$, which in turn yields a recurrence relation for the coefficients

$$\mathbf{y}_{i+1} = \frac{\mathbf{y'_i}}{i+1} = \frac{F(\mathbf{y_i})}{i+1}, \quad i = 0, 1, 2, \cdots \tag{5}$$

with $\mathbf{y}_0 = \mathbf{y}(0)$. Finally, using the above coefficients in (3), the truncated series solution

$$\mathbf{y}(t) = y_0 + \sum_{i=1}^{N} \frac{F(\mathbf{y}_{i-1})}{i} t^i \tag{6}$$

is obtained.

For nonlinear problems, based on the explicit form of $F(\mathbf{y})$ in (5), one often have to perform basic arithmetic operations on power series in order to successfully apply the above procedure. For operations of addition, subtraction, multiplication and division, the results are summarized below, see e.g. [5] and [14].

Proposition 1: If $f, g, h : t \in \mathbb{R} \mapsto \mathbb{R}$ are functions of class $\mathcal{C}^n$ and denoting by h_j the jth power series coefficient of $h(t) = \sum_{j \geq 0} h_j t^j$, we have

i. if $h(t) = f(t) \pm g(t)$ then
$$h_j = f_j \pm g_j;$$
ii. if $h(t) = f(t) \cdot g(t)$ then
$$h_j = \sum_{i=0}^{j} f_{j-i} g_i;$$

iii. if $h(t) = \frac{f(t)}{g(t)}$ then for $g_0 \neq 0$

$$h_j = \frac{1}{g_0}\left(f_j - \sum_{i=1}^{j} h_{j-i}g_i\right);$$

iv. if $h(t) = f(t)^\alpha$ then $h_0 = (f_0)^\alpha$ and for $j > 0$

$$h_j = \frac{1}{jf_0}\sum_{i=0}^{j-1}\Big(j\alpha - i(\alpha+1)\Big)f_{j-i}h_i.$$

For cases where division by zero arise, by appropriately 'shifting' the series until the first non-zero $g_0(t)$ or $f_0(t)$, the problem is circumvented, see [19].

2.1. PADÉ AND LAPLACE-PADÉ POST PROCESSING

It is well known that series solution of nonlinear differential equations often have finite radius and short interval of convergence [3]. This is the situation where series solution to a problem fails to converge in the entire domain of the problem. For instance, the series approximation $1 + x + x^2 + \cdots + x^n + \cdots = \sum_{i=0}^{\infty} x^i$ to the rational function $1/(1-x)$ has radius of convergence $R = 1$, see e.g [1][Theorem 17]. Consequently, the series approximation has interval of convergence $(-1, 1)$, in which the series approximation is valid. Therefore, the truncated series solution obtained through the simple iteration described in Section 2 above, though may be accurate, might still have a short interval of convergence to be able to capture some nice properties of the solution e.g. transition phase in stiff problems. The need to improve the convergence interval of the truncated series therefore becomes limpid. Known applicable methods to improve convergence of series solution include the Padé approximant [13], Continuous Analytic Continuation [4] and Wynn-epsilon convergence [21]. In some cases, a combination of these methods are required to obtain the closed form solution or to extend the validity of the truncated series solution. In this article, we apply the Padé approximant (see [3, 13]), and where the need arises, we follow the idea of [7] to combine the Laplace transform approach with the Padé approximation to obtain exact solution to (1). The latter approach is described below.

Suppose that the power series solution $y_P(t)$ of (1) has been computed following the discussion of Section 2, the exact solution $y(t)$ can be obtained by applying Laplace-Padé post processing to $y_P(t)$ via the following steps:

(1) Apply t-Laplace transform to $y_P(t)$,
(2) Replace s by $\frac{1}{t}$,
(3) Apply t-Padé approximation to the resulting series,
(4) Replace t by $\frac{1}{s}$,

(5) Finally, apply inverse Laplace transform to obtain a closed form solution (if it exists), otherwise a better approximation than $y_P(t)$ is obtained.

The above procedure can easily be automated in a few lines of code in any symbolic computation platform.

3. NUMERICAL EXAMPLES

Example 1: We consider a first order non-homogeneous, nonlinear problem [12]

$$y'(x) - y(x)^2 = 1, \quad 0 \le x \le \frac{\pi}{4}, \quad y(0) = 0$$

with closed form solution $y(x) = \tan x$.

In line with the discussion of Section 2,

$$y(x) = \sum_{i \ge 0} y_i x^i,$$

and taking $F(y) = y^2$, we obtain the recursion

$$y_{i+1} = \frac{(y^2)_i}{i+1}, \quad i = 1, 2, \dots; \ y_0 = 0, \quad y_1 = 1 + y_0^2.$$

By choosing $f = g = y$ in Proposition (ii.), the derived variable $(y^2)_i$ can be written as $(y^2)_i = \sum_{j=0}^{i} y_j y_{i-j}$. Thus the coefficients y_i are obtained iteratively through the difference scheme

$$y_{i+1} = \frac{\sum_{j=0}^{i} y_j y_{i-j}}{i+1}, \quad i = 1, 2, \dots; \quad y_0 = 0, \quad y_1 = 1 + y_0^2$$

as

$$y_0 = y_2 = y_4 = y_6 = \cdots = 0, \ y_1 = 1, y_3 = \frac{1}{3}, y_5 = \frac{2}{15}, y_7 = \frac{17}{315}, \dots$$

Hence,

$$y(x) = \sum_{i \ge 0} y_i x^i = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \dots \qquad (7)$$

$$= \sum_{i \ge 0} B_{2n} \frac{-2^{2n}(1 - 2^{2n})}{(2n)!} x^{2n-1} \qquad (8)$$

$$= \tan x, \qquad (9)$$

which is the exact solution. In the above, the constants B_k denote the Bernoulli numbers.

Example 2: We consider a variable-coefficient first order nonlinear problem [12]

$$y'(x) + (2x - 1)y(x)^2 = 0, \ \ 0 \le x \le 1, \ y(0) = 1,$$

with known closed form solution $y(x) = (x^2 - x + 1)^{-1}$.

If we let $v = (2x-1)y(x)$, then the problem becomes

$$y' = -yv, \quad v' = -v^2 + 2y.$$

Thus, the coefficients y_i in the assumed series solution

$$y(x) = \sum_{i\geq 0} y_i x^i \tag{10}$$

are obtained iteratively through the recursions

$$y_{i+1} = -\frac{\sum_{j=0}^{i} y_j v_{i-j}}{i+1}; \quad v_{i+1} = \frac{-\sum_{j=0}^{i} v_j v_{i-j} + 2y_i}{i+1} \tag{11}$$

subject to initial conditions $y_0 = 1, \;\; v_0 = -1$. The first few solutions to (11) are

$$\begin{pmatrix} y_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} y_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} y_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} y_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix},$$
$$\begin{pmatrix} y_5 \\ v_5 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \begin{pmatrix} y_6 \\ v_6 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \ldots$$

Hence, substituting the values of y_i in (10), we obtain the series solution

$$y(x) = 1 + x - x^3 - x^4 + x^6 + x^7 - x^9 - x^{10} + \ldots$$

Let $y_P(x) = 1 + x - x^3 - x^4 + x^6 + x^7 - x^9 - x^{10}$. The [L/M] $x-$Padé approximation applied to $y_P(x)$ for $L \geq 1, M > 1$ yields the exact solution $1/(x^2 - x + 1)$.

Example 3: We consider the periodic or oscillatory linear initial value problem [17]

$$y''(t) = -100y(t) + 99\sin t, \; 0 < t < 1000, \quad y(0) = 1, \; y'(0) = 11$$

with closed form solution $y(t) = \cos 10t + \sin 10t + \sin t$.

Through the variable substitutions $u = y, \; v = y', \; w = \sin t, \; z = \cos t$ the above problem is transformed to a system of first order constant-coefficient equation

$$u' = v; \;\; v' = -100u + 99w; \;\; w' = z; \;\; z' = -w. \tag{12}$$

As before, we assume series solution for all the variables

$$y(t) = u(t) = \sum_{i\geq 0} u_i t^i, \;\; v(t) = \sum_{i\geq 0} v_i t^i, \;\; w(t) = \sum_{i\geq 0} w_i t^i, \;\; z(t) = \sum_{i\geq 0} z_i t^i, \tag{13}$$

then the coefficients u_i, v_i, w_i, z_i are obtained from the recursions

$$u_{i+1} = \frac{v_i}{i+1}, \quad v_{i+1} = \frac{-100u_i + 99w_i}{i+1}, \tag{14}$$

$$w_{i+1} = \frac{z_i}{i+1}, \quad z_{i+1} = \frac{-w_i}{i+1}, \tag{15}$$

with initial conditions $u_0 = 1, \; v_0 = 11, \; w_0 = 0, \; z_0 = 1$.

The first few coefficients are computed as

$$\begin{pmatrix} u_1 \\ v_1 \\ w_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 11 \\ -100 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \\ w_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} -50 \\ -1001/2 \\ 0 \\ -1/2 \end{pmatrix}, \begin{pmatrix} u_3 \\ v_3 \\ w_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} -1001/6 \\ 5000/3 \\ -1/6 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} u_4 \\ v_4 \\ w_4 \\ z_4 \end{pmatrix} = \begin{pmatrix} 1250/3 \\ 100001/24 \\ 0 \\ 1/24 \end{pmatrix}, \dots$$

Substituting the obtained coefficients in (13) we obtain in particular the series

$$u(t) = 1 + 11\,t - 50\,t^2 - \frac{1001}{6}\,t^3 + \frac{1250}{3}\,t^4 + \frac{100001}{120}\,t^5 - \frac{12500}{9}\,t^6 - \frac{10000001}{5040}\,t^7 + \frac{156250}{63}\,t^8 + \frac{142857143}{51840}\,t^9 - \frac{1562500}{567}\,t^{10} + \dots \tag{16}$$

Observe that (16) is already an accurate approximation to the solution $y(t)$. However, the exact solution can be obtained by applying the Laplace-Padé post processing procedure of Section 2.1.

Consider the truncated series

$$u_P(t) = 1 + 11\,t - 50\,t^2 - \frac{1001}{6}\,t^3 + \frac{1250}{3}\,t^4 + \frac{100001}{120}\,t^5 - \frac{12500}{9}\,t^6 - \frac{10000001}{5040}\,t^7 + \frac{156250}{63}\,t^8 + \frac{142857143}{51840}\,t^9 - \frac{1562500}{567}\,t^{10}. \tag{17}$$

Applying Laplace transform to (17) gives

$$\mathcal{L}[u_P(t)] = \frac{1}{s} + \frac{11}{s^2} - \frac{100}{s^3} - \frac{1001}{s^4} + \frac{10000}{s^5} + \frac{100001}{s^6} - \frac{1000000}{s^7} \tag{18}$$

$$- \frac{10000001}{s^8} + \frac{100000000}{s^9} + \frac{1000000001}{s^{10}} - \frac{10000000000}{s^{11}}. \tag{19}$$

Substituting $s = 1/t$ in the above, we obtain

$$\begin{aligned}\mathcal{L}[u_P(t)] &= t + 11\,t^2 - 100\,t^3 - 1001\,t^4 + 10000\,t^5 + 100001\,t^6 \\ &\quad - 1000000\,t^7 - 10000001\,t^8 + 100000000\,t^9 \\ &\quad + 1000000001\,t^{10} - 10000000000\,t^{11}\end{aligned}$$

Next, all [L/M] $t-$Padé approximation with $L, M \geq 4$ and $L + M \leq 11$ applied to $\mathcal{L}[u_P(t)]$ yield

$$[L/M]_{u_P} = \frac{110t^4 + t^3 + 11t^2 + t}{1 + 101t^2 + 100t^4}.$$

Now replacing t by $1/s$ in the above to obtain

$$[L/M]_{u_P} = \frac{110 + s + 11s^2 + s^3}{s^4 + 101s^2 + 100}.$$

Finally,

$$y(t) = u(t) = \mathcal{L}^{-1}\left(\frac{110 + s + 11s^2 + s^3}{s^4 + 101s^2 + 100}\right) = \cos(10\,t) + \sin(10\,t) + \sin(t)$$

the exact solution.

Example 4: We consider the nonlinear Vanderpole Oscillator problem [6]

$$y''(t) + y'(t) + y(t) + y(t)^2 y'(t) = 2\cos t - \cos^3 t, \quad y(0) = 0, \quad y'(0) = 1$$

with exact solution $y(t) = \sin t$.

Setting $v = y'$, $w = \cos t$, $z = \sin t$ reduces the problem to the first order constant-coefficient ODE system

$$y' = v; \quad v' = 2w - w^3 - v - y - y^2 v; \quad w' = -z; \quad z' = w.$$

The above system then yields the recursions

$$y_{i+1} = \frac{v_i}{i+1}, \quad y_0 = 0 \tag{20}$$

$$v_{i+1} = \frac{2w_i - (w^3)_i - v_i - y_i - (y^2 v)_i}{i+1}, \quad v_0 = 1 \tag{21}$$

$$w_{i+1} = -\frac{z_i}{i+1}, \quad w_0 = 1 \tag{22}$$

$$z_{i+1} = \frac{w_i}{i+1}, \quad z_0 = 0 \tag{23}$$

where the quantities $(w^3)_i$ and $(y^2 v)_i$ are computed by repeated use of Proposition 1. (ii) or (iv). Solving these difference equations give

$$\begin{pmatrix} y_1 \\ v_1 \\ w_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} y_2 \\ v_2 \\ w_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1/2 \\ -1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} y_3 \\ v_3 \\ w_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} -1/6 \\ 0 \\ 0 \\ -1/6 \end{pmatrix},$$

$$\begin{pmatrix} y_4 \\ v_4 \\ w_4 \\ z_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/24 \\ 1/24 \\ 0 \end{pmatrix}, \begin{pmatrix} y_5 \\ v_5 \\ w_5 \\ z_5 \end{pmatrix} = \begin{pmatrix} 1/120 \\ 0 \\ 0 \\ 1/120 \end{pmatrix}, \ldots$$

Using the coefficients y_i above, we obtain the series solution

$$y(t) = \sum_{i \geq 0} y_i t^i = t - \frac{1}{6}t^3 + \frac{1}{120}t^5 + \ldots$$

To compute the closed form solution, let $y_P(t) = t - \frac{1}{6}t^3 + \frac{1}{120}t^5$. Applying Laplace transform on $y_P(t)$ and subsequently substituting $t =$

$1/s$, we obtain

$$\mathcal{L}[y_P(t)] = t^2 - t^4 + t^6.$$

Applying [L/M] t- Padé approximant with $L, M \geq 2$ yields

$$[L/M]_{y_P} = \frac{t^2}{1+t^2}$$

which on replacing t by $1/s$ gives

$$\frac{1}{s^2+1}.$$

Lastly the closed form solution is obtained as

$$y(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin t.$$

Example 5: We consider the nonlinear stiff system of IVPs [20]

$$u'(t) = -1002u(t) + 1000v(t)^2, \quad u(0) = 1, \tag{24}$$

$$v'(t) = u(t) - v(t)(1 + v(t)), \quad v(0) = 1 \tag{25}$$

whose closed form solution is known to be $u(t) = e^{-2t},\ v(t) = e^{-t}$.

In line with our proposed method, to obtain power series solution of the form

$$u(t) = \sum_{i\geq 0} u_i t^i, \quad v(t) = \sum_{i\geq 0} v_i t^i, \tag{26}$$

we set up an iterative scheme

$$u_{i+1} = \frac{-1002u_i + 1000(v^2)_i}{i+1}, \quad u_0 = 1, \tag{27}$$

$$v_{i+1} = \frac{u_i - v_i - (v^2)_i}{i+1}, \quad v_0 = 1. \tag{28}$$

From the above recursions the coefficients u_i and v_i are computed as

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1/2 \end{pmatrix}, \begin{pmatrix} u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} -4/3 \\ -1/6 \end{pmatrix}, \begin{pmatrix} u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/24 \end{pmatrix},$$

$$\begin{pmatrix} u_5 \\ v_5 \end{pmatrix} = \begin{pmatrix} -4/15 \\ -1/120 \end{pmatrix}, \ldots$$

Using the above coefficients in (26), we obtain the series solution

$$u(t) = 1 - 2t + 2t^2 - \frac{4}{3}t^3 + \frac{2}{3}t^4 - \frac{4}{15}t^5 + \ldots, \tag{29}$$

$$v(t) = 1 - t + \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{24}t^4 - \frac{1}{120}t^5 + \ldots \tag{30}$$

The Laplace-Padé post processing procedure are now applied to the equations (29)-(30). Let $u_P(t) = 1 - 2t + 2t^2 - \frac{4}{3}t^3 + \frac{2}{3}t^4 - \frac{4}{15}t^5; \quad v_P(t) =$

$1 - t + \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{24}t^4 - \frac{1}{120}t^5$. Applying Laplace transform on $u_P(t)$ gives

$$\mathcal{L}[u_P(t)] = \frac{1}{s} - \frac{2}{s^2} + \frac{4}{s^3} - \frac{16}{s^5} - \frac{32}{s^6}$$

which on substituting $s = 1/t$ implies

$$\mathcal{L}[u_P(t)] = t - 2t^2 + 4t^3 - 8t^4 + 16t^5 - 32t^6.$$

On the next step [L/M] $t-$Padé approximation applied to the obtained series $\mathcal{L}[u_P(t)]$ gives (for all $L, M \geq 1$ and $L + M \leq 6$)

$$[L/M]_{u_P} = \frac{t}{1+2t}.$$

On substituting $t = 1/s$ we obtained

$$[L/M]_{u_P} = \frac{1}{s+2}.$$

Finally, the exact solution $u(t)$ is obtained by taking the inverse Laplace transform

$$u(t) = \mathcal{L}^{-1}\left(\frac{1}{s+2}\right) = e^{-2t}.$$

In a similar manner, the Laplace-Padé post processing procedure applied to $v_P(t)$ yields

$$v(t) = \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) = e^{-t}.$$

Example 6: Consider the third order IVP

$$y''' = 3\sin x, \quad y(0) = 1, \;\; y'(0) = 0, \;\; y''(0) = -2$$

with closed form solution $y(x) = 3\cos x + \frac{x^2}{2} - 2$.

With variable substitutions $v = y', w = y'', t = \sin x, z = \cos x$, the problem is recast as a first order system

$$y' = v; \;\; v' = w; \;\; t' = z; \;\; z' = -t$$

which in line with our proposed method leads to the recursions

$$y_{i+1} = \frac{v_i}{i+1}; \;\; v_{i+1} = \frac{w_i}{i+1}; \;\; t_{i+1} = \frac{z_i}{i+1}; \;\; z_{i+1} = -\frac{t_i}{i+1}$$

subject to initial conditions $y_0 = 1,\ \ v_0 = 0,\ \ w_0 = -2,\ \ t_0 = 0,\ \ z_0 = 1.$ The first few iterates to the above recursions are obtained as

$$\begin{pmatrix} y_1 \\ v_1 \\ w_1 \\ t_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_2 \\ v_2 \\ w_2 \\ t_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 3/2 \\ 0 \\ -1/2 \end{pmatrix}, \begin{pmatrix} y_3 \\ v_3 \\ w_3 \\ t_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \\ 0 \\ -1/6 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} y_4 \\ v_4 \\ w_4 \\ t_4 \\ z_4 \end{pmatrix} = \begin{pmatrix} 1/8 \\ 0 \\ -1/8 \\ 0 \\ 1/24 \end{pmatrix}, \begin{pmatrix} y_5 \\ v_5 \\ w_5 \\ t_5 \\ z_5 \end{pmatrix} = \begin{pmatrix} 0 \\ -1/40 \\ 0 \\ 1/120 \\ 0 \end{pmatrix}, \ldots$$

Hence, the power series solution is obtained using the computed coefficients y_i as

$$y(x) = \sum_{i \geq 0} y_i x^i = 1 - x^2 + \frac{1}{8}\,x^4 - \frac{1}{240}\,x^6 + \frac{1}{13440}\,x^8 - \frac{1}{1209600}\,x^{10} + \ldots$$

Let $y_P(x) = 1-x^2+\frac{1}{8}\,x^4-\frac{1}{240}\,x^6+\frac{1}{13440}\,x^8-\frac{1}{1209600}\,x^{10}$. We now apply the Laplace-Padé post processing on $y_P(x)$. Laplace transform applied to $y_P(x)$ yields

$$\mathcal{L}[y_P(x)] = \frac{1}{s} - \frac{2}{s^3} + \frac{3}{s^5} - \frac{3}{s^7} + \frac{3}{s^9} - \frac{3}{s^{11}}. \tag{31}$$

Substituting $s = 1/x$ in the above, we obtain

$$\mathcal{L}[y_P(x)] = x - 2x^3 + 3x^5 - 3x^7 + 3x^9 - 3x^{11}. \tag{32}$$

Applying [L/M] $x-$Padé approximation with $L, M \geq 5$ to (32) yields

$$[L/M]_u = \frac{x^5 - x^3 + x}{1 + x^2}$$

and on replacing x by $1/s$, we get

$$[L/M]_u = \frac{1 - s^2 + s^4}{s^3(s^2 + 1)}.$$

Lastly, the closed form solution is obtained as

$$y(x) = \mathcal{L}^{-1}\left(\frac{1 - s^2 + s^4}{s^3(s^2 + 1)}\right) = 3\cos x + \frac{x^2}{2} - 2.$$

Example 7: We consider the fourth order IVP

$$y^{iv} = -\sin x + \cos x, \quad y(0) = 0,\ \ y'(0) = y''(0) = -1,\ \ y'''(0) = 7$$

with closed form solution $y(x) = -\sin x + \cos x + x^3 - 1.$

The above problem reduces to a system of first order equations by letting $v = y',\ \ w = y'',\ \ t = y''',\ \ p = \sin x,\ \ q = \cos x$ from which the recursions

$$y_{i+1} = \frac{v_i}{i+1};\ \ v_{i+1} = \frac{w_i}{i+1};\ \ w_{i+1} = \frac{t_i}{i+1};\ \ t_{i+1} = \frac{-p_i + q_i}{i+1};$$
$$p_{i+1} = \frac{q_i}{i+1};\ \ q_{i+1} = \frac{-p_i}{i+1}$$

follow, subject to initial conditions $y_0 = 0; v_0 = -1; w_0 = -1; t_0 = 7; p_0 = 0; q_0 = 1$. The unknowns are computed as

$$\begin{pmatrix} y_1 \\ v_1 \\ w_1 \\ t_1 \\ p_1 \\ q_1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 7 \\ 1 \\ 1 \\ 0 \end{pmatrix},\ \begin{pmatrix} y_2 \\ v_2 \\ w_2 \\ t_2 \\ p_2 \\ q_2 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 7/2 \\ 1/2 \\ -1/2 \\ 0 \\ 1/2 \end{pmatrix},\ \begin{pmatrix} y_3 \\ v_3 \\ w_3 \\ t_3 \\ p_3 \\ q_3 \end{pmatrix} = \begin{pmatrix} 7/6 \\ 1/6 \\ -1/6 \\ -1/6 \\ -1/6 \\ 0 \end{pmatrix},$$
$$\begin{pmatrix} y_4 \\ v_4 \\ w_4 \\ t_4 \\ p_4 \\ q_4 \end{pmatrix} = \begin{pmatrix} 1/24 \\ -1/24 \\ -1/24 \\ 1/24 \\ 0 \\ 1/24 \end{pmatrix},\ \begin{pmatrix} y_5 \\ v_5 \\ w_5 \\ t_5 \\ p_5 \\ q_5 \end{pmatrix} = \begin{pmatrix} -1/120 \\ -1/120 \\ 1/120 \\ 1/120 \\ 1/120 \\ 0 \end{pmatrix}, \ldots$$

Hence, using the coefficients y_i, we form an approximate series solution

$$y(x) = \sum_{i\geq 0} y_i x^i = -x - \frac{1}{2}x^2 + \frac{7}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5 - \frac{1}{720}x^6 + \frac{1}{5040}x^7 + \frac{1}{40320}x^8 - \frac{1}{362880}x^9 + \ldots$$

Finally to obtain a closed form solution, we truncate the above series solution, set

$$y_P(x) = \sum_{i\geq 0} y_i x^i = -x - \frac{1}{2}x^2 + \frac{7}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5 - \frac{1}{720}x^6 + \frac{1}{5040}x^7 + \frac{1}{40320}x^8 - \frac{1}{362880}x^9$$

and apply the post processing procedure. Applying Laplace transform to $y_P(x)$, and subsequently replacing s by $1/x$ to obtain

$$\mathcal{L}[y_P(x)] = -x^2 - x^3 + 7x^4 + x^5 - x^6 - x^7 + x^8 + x^9 - x^{10} - x^{11}.$$

Next, [6/5] $x-$ Padé approximation applied to $\mathcal{L}[y_P(x)]$ above yields

$$\mathcal{L}[y_P(x)] = \frac{6x^6 + 6x^4 - x^3 - x^2}{1 + x^2}$$

which on replacing $x = 1/s$ implies

$$\mathcal{L}[y_P(x)] = -\frac{-6 - 6s^2 + s^3 + s^4}{s^4(s^2+1)}.$$

Hence,

$$y(x) = \mathcal{L}^{-1}\left(-\frac{-6 - 6s^2 + s^3 + s^4}{s^4(s^2+1)}\right) = -\sin x + \cos x + x^3 - 1.$$

4. CONCLUDING REMARKS

In this article, we have proposed a new, simple and yet highly accurate procedure for finding closed form solution of initial value problems. The method has been applied to linear, nonlinear, stiff and system of stiff initial value problems. In all cases, the procedure yielded the closed form solution of the problems considered.

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