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ALGORITHMS FOR A SYSTEM OF VARIATIONAL INEQUALITY PROBLEMS AND FIXED POINT PROBLEMS WITH DEMICONTRACTIVE MAPPINGS

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ABSTRACT. In this paper, we construct new iterative algorithms for approximating common solutions of a system of variational inequality and a fixed point problems with demicontractive mapping in real Banach spaces. Furthermore, we prove that the proposed algorithm has strong convergence. Finally, we apply our results for solving a system of convex minimization problem coupled with fixed point problems. Our technique of proof is of independent interest.

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1. INTRODUCTION

Let E be a Banach space with norm $\|\cdot\|$ and dual $E^* =$. For any $x \in E$ and $x^* \in E^*$, $\langle x^*, x\rangle$ is used to refer to $x^*(x)$.

Let $\varphi : [0, +\infty) \to [0, \infty)$ be a strictly increasing continuous function such that $\varphi(0) = 0$ and $\varphi(t) \to +\infty$ as $t \to \infty$. Such a function φ is called gauge. Associed to a gauge a duality map $J_\varphi : E \to 2^{E^*}$ defined by:

$$J_\varphi(x) := \{x^* \in E^* : \langle x, x^*\rangle = ||x||\varphi(||x||), ||x^*|| = \varphi(||x||)\}. \quad (1)$$

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If the gauge is defined by $\varphi(t) = t$, then the corresponding duality map is called the *normalized duality map* and is denoted by J. Hence the normalized duality map is given by

$$J(x) := \{x^* \in E^* : \langle x, x^* \rangle = ||x||^2 = ||x^*||^2\}, \forall\, x \in E.$$

Notice that

$$J_\varphi(x) = \frac{\varphi(||x||)}{||x||} J(x),\ \ x \neq 0. \tag{2}$$

Let E be a real normed space and let $S := \{x \in E : \|x\| = 1\}$. E is said to be *smooth* if

$$\lim_{t \to 0^+} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S$. E is said to be uniformly smooth if it is smooth and the limit is attained uniformly for each $x, y \in S$.

Let E be a normed space with dimE ≥ 2. The *modulus of smoothness* of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(\tau) := \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau \right\};\ \ \tau > 0.$$

It is known that a normed linear space E is *uniformly smooth* if

$$\lim_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0.$$

If there exists a constant $c > 0$ and a real number $q > 1$ such that $\rho_E(\tau) \leq c\tau^q$, then E is said to be *q-uniformly smooth.* Typical examples of such spaces are the L_p, ℓ_p and W_p^m spaces for $1 < p < \infty$ where, L_p *(or* l_p*) or* W_p^m is

$$\begin{array}{ll} 2 - \text{uniformly smooth and } p - \text{uniformly convex} & \text{if} \quad 2 \leq p < \infty; \\ 2 - \text{uniformly convex and } p - \text{uniformly smooth} & \text{if} \quad 1 < p < 2. \end{array}$$

Let J_q denote the generalized duality mapping from E to 2^{E^*} defined by

$$J_q(x) := \left\{ f \in E^* : \langle x, f \rangle = \|x\|^q \text{ and } \|f\| = \|x\|^{q-1} \right\}.$$

J_2 is called the normalized duality mapping and is denoted by J. It is known that E is smooth if and only if each duality map J_φ is single-valued, that E is Frechet differentiable if and only if each duality map J_φ is norm-to-norm continuous in E, and that E is uniformly smooth if and only if each duality map J_φ is norm-to-norm uniformly continuous on bounded subsets of E. Following Browder [6], we say that a Banach space has a weakly continuous duality map if there exists a gauge φ such that J_φ is single-valued and is weak-to-weak* sequentially continuous, i.e., if $(x_n) \subset E$, $x_n \xrightarrow{w} x$, then $J_\varphi(x_n) \xrightarrow{w^*} J_\varphi(x)$. It is known that l^p $(1 < p < \infty)$ has a weakly continuous duality map with gauge $\varphi(t) = t^{p-1}$ (see e.g., [9] for more details on duality maps).

Remark 1. Note also that a duality mapping exists in each Banach space. We recall from [2] some of the examples of this mapping in $l_p, L_p, W^{m,p}$-spaces, $1 < p < \infty$.

- (*i*) $l_p :\ Jx = \|x\|_{l_p}^{2-p} y \in l_q,\ \ x = (x_1, x_2, \cdots, x_n, \cdots),$
  $y = (x_1|x_1|^{p-2}, x_2|x_2|^{p-2}, \cdots, x_n|x_n|^{p-2}, \cdots);$
- (*ii*) $L_p :\ Ju = \|u\|_{L_p}^{2-p}|u|^{p-2}u \in L_q;$
- (*iii*) $W^{m,p} :\ Ju = \|u\|_{W^{m,p}}^{2-p} \sum_{|\alpha \le m|}(-1)^{|\alpha|} D^\alpha\Big(|D^\alpha u|^{p-2} D^\alpha u\Big) \in W^{-m,q},$

where $1 < q < \infty$ is such that $1/p + 1/q = 1$.

Finally, recall that a Banach space E satisfies Opial property (see, e.g., [23]) if: $\limsup_{n\to+\infty}\|x_n - x\| < \limsup_{n\to+\infty}\|x_n - y\|$ whenever $x_n \xrightarrow{w} x$, $x \neq y$.

Let K be a nonempty subset of E and $T : K \to K$ be a mapping. Let $Fix(T) := \{x \in D(T) : Tx = x\}$ be the set of fixed points of the mapping T. The mapping T is said to be

- (*i*) a contraction if there exists $L \in [0, 1)$ such that: $\|Tx - Ty\| \le b\|x - y\|$, for $x, y \in D(T)$. If $L = 1$, then T is called nonexpansive;
- (*ii*) quasi-nonexpansive if $Fix(T) \neq \emptyset$ and $\|Tx - p\| \le \|x - p\|$ for all $x \in D(T)$, $p \in Fix(T)$;
- (*iii*) k-strictly pseudo-contractive if there exists $j(x-y) \in J(x-y)$ and a constant $k \in (0, 1)$ such that $\langle Tx - Ty, j(x - y)\rangle \le \|x - y\|^2 - k\|(I - T)x - (I - T)y\|^2$, for all $x, y \in D(T)$;
- (*iv*) k-demicontractive if there exists $j_q(x - y) \in J_q(x - y)$ a constant $k \in (0, 1)$ such that $\langle x-Tx, j_q(x-p)\rangle \ge \dfrac{(1 - k)^{q-1}}{2^{q-1}}\|x-Tx\|^q$, for all $x \in D(T)$, $p \in Fix(T)$, $q > 1$.

Note that the following hold:
(a) Every nonexpansive mapping is strictly-pseudo-contractive;
(b) Every nonexpansive mapping is quasi-nonexpansive;
(c) Every quasi-nonexpansive mapping is 0-demicontractive;
(d) Every k-strictly pseudo-contractive mapping is k-demicontractive.

Many problems arising from different areas of mathematics, such as optimization, variational analysis and differential equations, engineering and science problems can be modeled by equations of the form :

$$x = Tx, \tag{3}$$

where T is a nonexpansive mapping. The solution set of this equation coincide to a fixed points set of T. Such operators have been studied extensively (see, e.g., Marino et al. [20], Chidume [11], Moudafi [22] and the references therein). One of the most investigated methods for approximating fixed points of nonexpansive mappings is known as viscosity approximation method, in light of Moudafi [22]. Let C be a nonempty, closed and convex subset of a real Banach space E. Let $T : C \to C$ be a nonexpansive mapping such that $Fix(T) \neq \emptyset$ and let $f : C \to C$ be a contraction. The viscosity approximation method is defined by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \end{cases} \tag{4}$$

where $\{\alpha_n\}$ is a sequence iof real numbers n $(0, 1)$. Under certain conditions, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Let C be a nonempty subset of real Banach space. Recall that an operator $A : C \to E$ is said to be accretive if there exists $j(x-y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq 0, \ \ \forall x, y \in C.$$

It is said to be α-inverse strongly accretive if, for some $\alpha > 0$,

$$\langle Ax - Ay, j(x - y)\rangle \geq \alpha\|Ax - Ay\|^2, \ \ \forall x, y \in C.$$

Note that in a Hilbert spaces, the normalized duality map is the identity map. Hence, in Hilbert spaces, *monotonicity* and *accretivity* coincide.

We now introduce the following variational inequality in Banach space E: find a point $x^* \in C$ such that, for some $j(x - x^*) \in J(x - y)$,

$$\langle Ax^*, j(x - x^*)\rangle \geq 0, \ \ \forall x \in C. \tag{5}$$

This general variational inequality was considered by Aoyama et al. [3]. The solution set of variational inequality (5) is denoted by $VI(C, A)$, that is,

$$VI(C, A) := \{x^* \in C, \ \ \langle Ax^*, j(x - x^*)\rangle \geq 0, \ \ \forall x \in C\}.$$

For a lot of real-life problems, such as, in signal processing, resource allocation, image recovery and so on, the constraints can be expressed as the variational inequality problem. Hence, the problem of finding solutions of variational inequality has became a flourishing area of contemporary research for numerous mathematicians working in nonlinear operator theory; (see, for example, [8, 28, 4, 25, 26] and the references contained in them). For solving the above variational inequality (5), Aoyama et al. [3] introduced an iterative algorithm:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Q_C(I - \lambda_n A)x_n \ n \geq 0, \tag{6}$$

where Q_C is a sunny nonexpansive retraction from E onto C and $\{\alpha_n\} \subset (0, 1)$, $\{\lambda_n\} \subset (0, \infty)$ are two real number sequences. Aoyama et al. [3] proved the following weak convergence theorem for solving variational inequality (5).

Theorem 1 (Aoyama et. al, [3]). *Let C be a nonempty, closed and convex subset of a uniformly convex and 2-uniformly smooth real Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C. Let $A : C \to E$ be an α-inverse strongly accretive operator with $VI(C, A) \neq \emptyset$. If $\{\lambda_n\}$ and $\{\alpha_n\}$ are chosen so that $\lambda_n \in \left[a, \frac{\alpha}{K^2}\right]$ for some $a > 0$ and $\alpha_n \in [b, c]$ for some b, c with $0 < b < c < 1$, then the sequence $\{x_n\}$ given by (6) converges weakly to z, a solution of the variational inequality (5), where here, the real number K is the 2-uniformly smoothness constant of the Banach space E.*

Recently, many authors studied the following convex feasibility problem (for short, CFP):

$$\textit{finding} \quad an \ x^* \in \bigcap_{i=1}^{m} K_i, \tag{7}$$

where $m \geq 1$ is an integer and each i, K_i is a nonempty, closed and convex subset of a real Hilbert space H. There is a considerable investigation on the CFP in the setting of Hilbert spaces which captures applications in various disciplines such as image restoration [15, 16, 7], computer tomography and radiation therapy treatment planning [8]. In this paper, we shall consider the case where, for each i, K_i is the solution set of a finite family of variational inequalities and fixed point problems involving nonlinear mapping in real Banach spaces. Very recently, Zhang et al. studied the convex feasibility problem (7) (where $K_i = VI(C, Ai)$ for $i = 1, 2, ..., m$) by considering a finite family of inverse-strongly monotone mappings $A_i : C \to H$ and a strict pseudocontraction. They established a strong convergence theorem which extends the corresponding results in [17, 14, 24].

Theorem 2 (Zhang and Yuan, [29])**.** *Let C be a nonempty, closed and convex subset of a real Hilbert space H. For m a positive integer and i, $1 \leq i \leq m$,, let $A_i : C \to H$ be a α_i-inverse-strongly monotone mapping. Let $S : C \to C$ be a k-strict pseudocontraction with a fixed point and and le $f : C \to C$ be a fixed α-contractive mapping.*

Assume that $\Omega := \Big(\bigcap_{i=1}^{m} VI(C, A_i)\Big) \bigcap Fix(S) \neq \emptyset$. Let $\{\lambda_i\}$ be real numbers in $(0, 2\alpha_i)$ and let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $(0, 1)$. Let $\{x_n\}$ be a sequence generated iteratively by the algorithm:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S \displaystyle\sum_{i=1}^{m} \eta_i y_{n,i}, \end{cases} \tag{8}$$

where the criterion for the approximate computation of $y_{n,i}$ in C is $\|y_{n,i} - P_C(I - \lambda_i A_i)x_n\| \leq e_{n,i}$, where $\lim_{n\to\infty} \|e_{n,i}\| = 0$ for each $1 \leq i \leq m$. Assume that the above control sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:

(a) $\alpha_n + \beta_n + \gamma_n = \displaystyle\sum_{i=1}^{m} \eta_i = 1$;

(b) $\lim_{n\to\infty} \alpha_n = 0$, $\displaystyle\sum_{n=0}^{\infty} \alpha_n = \infty$;

(c) $1 > \limsup_{n\to\infty} \beta_n \geq \liminf_{n\to\infty} \beta_n > 0$.

Then, the sequences $\{x_n\}$ *generated by* (8) *converges strongly to* $x^* \in \Omega$, *which is a unique solution of the following variational inequality:*

$$\langle x^* - f(x^*), x^* - p\rangle \leq 0, \quad \forall p \in \Omega. \tag{9}$$

In this work, we study the problem of finding an element of $\Gamma := \Big(\bigcap_{i=1}^{m} VI(C, A_i)\Big) \bigcap Fix(T)$, where $A_i : C \to E$ is α_i-inverse strongly accretive for $i = 1, 2, ..., m,\ T : C \to C$ is a k-demicontractive mapping. Based on the well-known extragradient method and viscosity approximation method, we introduce the following iterative:

$$\begin{cases} x_0 \in C, \ \textit{choosen arbitrarily,} \\ \\ z_n = \theta_n x_n + (1-\theta_n)Tx_n, \\ \\ y_n = \lambda_0 z_n + \displaystyle\sum_{i=1}^{m} \lambda_i Q_C(I - \beta_i A_i) z_n, \\ \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) y_n, \end{cases} \tag{10}$$

where $\sum_{i=0}^{m} \lambda_i = 1,\ \ \theta_n \in [c, d] \subset \Big(0,\ \min\{1, \frac{1-k}{d_2}\}\Big)$, $\beta_i > 0$ and $\{\alpha_n\} \subset (0, 1)$ satisfies the following conditions:

$$\lim_{n\to\infty} \alpha_n = 0; \ \text{ and } \ \sum_{n=0}^{\infty} \alpha_n = \infty.$$

Under these assumptions, we proved that the sequence $\{x_n\}$ generated by the algorithm (10) converges strongly to an element of $\Big(\bigcap_{i=1}^{m} VI(C, A_i)\Big) \bigcap Fix(T)$. The results obtained here, represent an extension and an important improvement of Theorem 1 and Theorem 2 in the following aspects:

- (i) the problem of finding an element of $\Big(\bigcap_{i=1}^{m} VI(C, A_i)\Big) \bigcap Fix(T)$ is more general and more complex than the problem of finding an element of $VI(C, A)\Big)$ as in Theorem 1;
- (ii) the problem studied here extends and generalizes the one considering in Theorem 2 from Hilbert spaces to Banach spaces;

(*iii*) the algorithm (10) is different to algorithm (6), considered in Theorem 1. It is also different to algorithm (8) considered in Theorem 2.

(*iv*) the problem of solving variational inequality coupled with fixed point problem become the special case to the problem considered in this paper.

2. PRELIMINARY

In the sequel, we shall need the following notions and results.

Let C be a nonempty subset of a smooth real Banach space E. A mapping $Q_C : E \to C$ is said to be sunny if

$$Q_C(Q_C x + t(x - Q_C x)) = Q_C x$$

for each $x \in E$ and $t \geq 0$. It is said to be a retraction if $Q_C x = x$ for each $x \in C$.

Lemma 1 (Halpern, [13]). *Let C and D be nonempty subsets of a smooth real Banach space E with $D \subset C$ and let $Q_D : C \to D$ be a retraction from C into D. Then Q_D is sunny and nonexpansive if and only if*

$$\langle z - Q_D z, J(y - Q_D z) \rangle \leq 0$$

for all $z \in C$ and $y \in D$.

Noted that from (2), Lemma 1 still holds if the normalized duality map is replaced by the general duality map J_φ, where φ is gauge function.

Remark 2. If K is a nonempty, closed and convex subset of a Hilbert space H, then the nearest point projection P_K from H to K is a sunny nonexpansive retraction.

The demiclosedness of a nonlinear operator T usually plays an important role in dealing with the convergence of fixed point iterative algorithms.

Definition 3. Let K be a nonempty, closed and convex subset of a real Hilbert space H and let $T : K \to K$ be a single-valued mapping. The map $I - T$ is said to be demiclosed at 0 if for any sequence $\{x_n\} \subset D(T)$ such that $\{x_n\}$ converges weakly to p and $\|x_n - Tx_n\|$ converges to zero, then $p \in Fix(T)$.

Lemma 2 (Goebel and Kirk, [12]). *Let E be a real Banach space satisfying the Opial's property. Let K be a closed convex subset of E, and $T : K \to K$ be a nonexpansive mapping such that $Fix(T) \neq \emptyset$. Then $I - T$ is demiclosed.*

Lemma 3 (Marino and Xu,[20] (Proposition 2.1)). *Let K ibe a closed convex subset of a Hilbert space H. and $T : K \to K$ be a self-mapping of C. If T is k-demicontractive, then the fixed point set $Fix(T)$ is closed and convex.*

Theorem 4 (Chidume,[11]). *Let $q > 1$ be a fixed real number and E be a smooth Banach space. Then the following are equivalent:*
(i) E is q-uniformly smooth;
(ii) there is a constant $d_q > 0$ such that for all $x, y \in E$

$$\|x + y\|^q \leq \|x\|^q + q\langle y\ ,\ J_q(x)\rangle + d_q\|y\|^q;$$

(iii) there is a constant $c_1 > 0$ such that

$$\langle x - y\ ,\ J_q(x) - J_q(y)\rangle\ \leq c_1\|x - y\|^q\ \ \forall\ \ x, y \in E.$$

Observing that L_p-spaces, $2 \leq p < \infty$ are 2-uniformly smooth, we have the following.

Corollary 5. *Assume that $E = l_p,\ \ 2 \leq p < \infty$. For each $x, y \in E$,*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y\ ,\ J(x)\rangle + (p - 1)\|y\|^2.$$

Lemma 4 (Marino, Xu, [20]). *Let K be a nonempty closed convex subset of a real Hilbert space H and $T : K \to K$ be a mapping.*
(i) If T is k-strictly pseudo-contractive, then T satisfies the Lipschitzian condition

$$\|Tx - Ty\| \leq \frac{1 + k}{1 - k}\|x - y\|,\ \forall\, x, y \in K.$$

(ii) If T is k-strictly pseudo-contractive, then the mapping $I - T$ is demiclosed at 0.

Lemma 5 (Lim, Xu, [18]). *Let E be a Banach space with a weakly continous duality mapping J_φ for some jauge φ. Then, we have*

$$\Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, J_\varphi(x + y)\rangle \tag{11}$$

for all x,y $\in E$, where $\Phi(t) = \displaystyle\int_0^t \varphi(\sigma)d\sigma,\ \forall t \geq 0$. In particular, for the normilized duality mapping, we have the important special version of (11)

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y)\rangle. \tag{12}$$

for all x,y $\in E$.

Lemma 6 (Xu, [27])**.** *Assume that* $\{a_n\}$ *is a sequence of nonnegative real numbers such that* $a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n$ *for all* $n \geq 0$*, where* $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *and* $\{\sigma_n\}$ *is a sequence in* $\mathbb{R}$ *such that*
(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$*,* (b) $\limsup_{n\to\infty} \sigma_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\sigma_n\alpha_n| < \infty$*. Then* $\lim_{n\to\infty} a_n = 0$*.*

Lemma 7 (Chang et al. [10])**.** *Let* E *be a uniformly convex real Banach space. For arbitrary* $r > 0$*, let* $B_r(0) := \{x \in E : ||x|| \leq r\}$*, be the closed ball with center* 0 *and radius* $r > 0$*. For any given sequence* $\{u_1, u_2, ..., u_m\} \subset B_r(0)$ *and any positive real numbers* $\{\lambda_1, \lambda_2, ..., \lambda_m\}$ *with* $\sum_{k=1}^{m} \lambda_k = 1$*, then there exists a continuous, strictly increasing and convex function*

$$g : [0, 2r] \to \mathbb{R}^+,\ g(0) = 0,$$

such that for any integer i, j *with* $i < j$*,*

$$\|\sum_{k=1}^{m} \lambda_k u_k\|^2 \leq \sum_{k=1}^{m} \lambda_k \|u_k\|^2 - \lambda_i\lambda_j g(\|u_i - u_j\|).$$

Lemma 8 (Mainge,[21])**.** *Let* $\{t_n\}$ *be a sequence of real numbers that does not decrease at infinity in a sense that there exists a subsequence* $\{t_{n_i}\}$ *of* t_n *such that* t_{n_i} *such that* $t_{n_i} \leq t_{n_{i+1}}$ *for all* $i \geq 0$*. For sufficiently large numbers* $n \in \mathbb{N}$*, an sequence of integers,* $\{\tau(n)\}$ *is defined as follows:*

$$\tau(n) = \max\{k \leq n : \ t_k \leq t_{k+1}\}.$$

Then, $\tau(n) \to \infty$ *as* $n \to \infty$ *and*

$$\max\{t_{\tau(n)},\ \ t_n\} \leq t_{\tau(n)+1}.$$

Lemma 9 (Aoyama *et al.*,[3])**.** *Let* C *be a nonempty, closed and convex subset of a smooth real Banach space* E*. Let* Q_C *be a sunny nonexpansive retraction from* E *onto* C *and let* A *be an accretive operator of* C *into* E*. Then for all* $\lambda > 0$*,*

$$VI(C, A) = Fix(Q_C(I - \lambda A)). \tag{13}$$

Lemma 10 (Aoyama *et al.*,[3])**.** *Let* C *be a nonempty, closed and convex subset of a 2-uniformly smooth real Banach space* E*. Let* $\alpha > 0$ *and* A *be an* α*-inverse strongly accretive operator of* C *into* E*. If* $0 < \lambda \leq \frac{\alpha}{K^2}$*, then* $I - \lambda A$ *is a nonexpansive mapping of* C *into* E*, where* K *is the 2-uniformly smoothness constant of* E*.*

3. STRONG CONVERGENCE THEOREMS

We now prove the following result.

Theorem 6. *Let E be a 2-uniformly smooth and uniformly convex real Banach space with a weakly continuous duality map J_φ. Let C be a nonempty, closed convex subset of E. Let Q_C be a sunny nonexpansive retraction from E onto C and $f : C \to C$ be a b-contraction. for $i = 1, 2, ..., m$, let $A_i : C \to E$ be a α_i-inverse strongly accretive mapping and let $T : C \to C$ be a k-demicontractive mapping such that $\Gamma := \Big(\bigcap_{i=1}^{m} VI(C, A_i)\Big) \bigcap Fix(T) \neq \emptyset$. Assume that $I - T$ is demiclosed at the origin and $\beta_i \in \Big[a, \frac{\alpha_i}{K^2}\Big]$ for some $a > 0$, where K is the 2-uniformly smoothness constant of E. Then, the sequence $\{x_n\}$ generated by (10) converges strongly to $x^* \in \Gamma$, which is the unique solution of the following variational inequality:*

$$\langle x^* - f(x^*), J_\varphi(x^* - p)\rangle \leq 0, \quad \forall p \in \Gamma. \tag{14}$$

Proof. We first show the uniqueness of the solution of the variational inequality (14). Assume that $x^* \in \Gamma$ and $x^{**} \in \Gamma$ are both solutions to (14). Then, it follows

$$\langle x^* - f(x^*), J_\varphi(x^* - x^{**})\rangle \leq 0 \tag{15}$$

and

$$\langle x^{**} - f(x^{**}), J_\varphi(x^{**} - x^*)\rangle \leq 0. \tag{16}$$

Adding up (15) and (16) yields

$$\langle x^{**} - x^* + f(x^*) - f(x^{**}), J_\varphi(x^{**} - x^*)\rangle \leq 0. \tag{17}$$

Noting that

$$\langle x^{**} - x^* + f(x^*) - f(x^{**}), J_\varphi(x^{**} - x^*)\rangle \geq (1-b)\varphi\Big(\|x^* - x^{**}\|\Big)\|x^* - x^{**}\|,$$

it follows that $x^* = x^{**}$, which proves the uniqueness. In what follows, x^* denotes the unique solution of (14).

We prove that the sequence $\{x_n\}$ is bounded. For this, Let $p \in \Gamma$. Using (10), inequality (ii) of Theorem 4 and the fact that T is

k-demicontractive, we have

$$\begin{aligned}
\|z_n - p\|^2 &= \Big\|(1-\theta_n)(x_n - p) + \theta_n(Tx_n - p)\Big\|^2 \\
&= \Big\|(1-\theta_n)(x_n - p) + \theta_n(Tx_n - x_n) + \theta_n(x_n - p)\Big\|^2 \\
&= \Big\|x_n - p + \theta_n(Tx_n - x_n)\Big\|^2 \\
&\leq \|x_n - p\|^2 - 2\theta_n\langle x_n - Tx_n, J(x_n - p)\rangle + d_2\Big\|\theta_n(Tx_n - x_n)\Big\|^2 \\
&\leq \|x_n - p\|^2 - \theta_n(1-k)\|x_n - Tx_n\|^2 + d_2\Big\|\theta_n(Tx_n - x_n)\Big\|^2 \\
&\leq \|x_n - p\|^2 - \theta_n(1 - k - d_2\theta_n)\|Tx_n - x_n\|^2. \qquad (18)
\end{aligned}$$

Since $1 - k - d_2\theta_n > 0$, it follows that

$$\|z_n - p\Big\| \leq \|x_n - p\Big\|. \qquad (19)$$

From (10), Lemmas 9 and 10, we obtain

$$\begin{aligned}
\|y_n - p\| &= \|\lambda_0 z_n + \sum_{i=1}^{m} \lambda_i Q_C(I - \beta_i A_i)z_n - p\| \\
&\leq \lambda_0\|z_n - p\| + \sum_{i=1}^{m} \lambda_i \|Q_C(I - \beta_i A_i)z_n - p\| \\
&\leq \|z_n - p\|.
\end{aligned}$$

Therefore,

$$\|y_n - p\| \leq \|z_n - p\| \leq \|x_n - p\|. \qquad (20)$$

Hence,

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n f(x_n) + (1-\alpha_n)y_n - p\| \\
&\leq \alpha_n\|f(x_n) - f(p)\| + (1-\alpha_n)\|y_n - p\| + \alpha_n\|f(p) - p\| \\
&\leq (1 - \alpha_n(1-b))\|x_n - p\| + \alpha_n\|f(p) - p\| \\
&\leq \max\{\|x_n - p\|, \frac{\|f(p) - p\|}{1-b}\}.
\end{aligned}$$

So, by induction,

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{\|f(p) - p\|}{1-b}\}, \quad n \geq 1.$$

Hence $\{x_n\}$ is bounded, also $\{y_n\}$, $\{z_n\}$ and $\{f(x_n)\}$ are all bounded.

Using (18) and and the convexity of $x \to \|x\|^2$, it follows that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \|\alpha_n f(x_n) + (1-\alpha_n) y_n - p\|^2 \\
&\leq \alpha_n \|f(x_n) - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 \\
&\leq \alpha_n \|f(x_n) - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 \\
&\quad -(1-\alpha_n)\theta_n(1-k-d_2\theta_n)\|Tx_n - x_n\|^2.
\end{aligned}$$

Since $\{x_n\}$ is bounded, there exists a constant $B > 0$ such that

$$(1-\alpha_n)\theta_n(1-k-d_2\theta_n)\|Tx_n - x_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n B. \quad (21)$$

We now prove that $\{x_n\}$ *converges strongly to* x^*. The proof will be in two steps.

Case 1. Assume that there is $n_0 \in N$ such that $\{\|x_n - x^*\|\}$ is decreasing from n_0. Since $\{\|x_n - x^*\|\}$ is is bounded, there it is convergent. Clearly, we have

$$\lim_{n\to\infty} \left(\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \right) = 0. \quad (22)$$

From (21), it follows that

$$\lim_{n\to\infty} (1-\alpha_n)\theta_n(1-k-d_2\theta_n)\|Tx_n - x_n\|^2 = 0. \quad (23)$$

Since $\theta_n \in [c, d] \subset \left(0,\ \min\{1, \frac{1-k}{d_2})\right)$, we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \quad (24)$$

Now, observing that,

$$\begin{aligned}
\|z_n - x_n\| &= \|(1-\theta_n)x_n + \theta_n T x_n - x_n\| \\
&= \|(1-\theta_n)x_n + \theta_n T x_n - \theta_n x_n - (1-\theta_n)x_n\| \\
&\leq \|Tx_n - x_n\|,
\end{aligned}$$

from, (24), we have

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \quad (25)$$

We show that $\limsup_{n\to+\infty} \langle x^* - f(x^*), J_\varphi(x^* - x_n)\rangle \leq 0$. Since E is reflexive and $\{x_n\}_{n\geq 0}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that x_{n_j} converges weakly to x^{**} in C and

$$\limsup_{n\to+\infty} \langle x^* - f(x^*), J_\varphi(x^* - x_n)\rangle = \lim_{j\to+\infty} \langle x^* - f(x^*), J_\varphi(x^* - x_{n_j})\rangle.$$

From (24) and the fact that $I - T$ is demiclosed, it follows that $x^{**} \in Fix(T)$. From Lemma 7, the fact that $Q_C(I - \beta_i A_i)$ is nonexpansive and (20), we have

$$\begin{aligned}\|y_n - p\|^2 &= \|\lambda_0 z_n + \sum_{i=1}^{m} \lambda_i Q_C(I - \beta_i A_i)z_n - p\|^2 \\ &\leq \lambda_0\|z_n - p\|^2 + \sum_{i=1}^{m} \lambda_i \|Q_C(I - \beta_i A_i)z_n - p\|^2 - \\ &\quad \lambda_0\lambda_i g(\|Q_C(I - \beta_i A_i)z_n - z_n\|) \\ &\leq \|x_n - p\|^2 - \lambda_0\lambda_i g(\|Q_C(I - \beta_i A_i)z_n - z_n\|).\end{aligned}$$

Hence,

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq \|\alpha_n f(x_n) + (1-\alpha_n)y_n - p\|^2 \\ &\leq \alpha_n\|f(x_n) - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 \\ &\leq \alpha_n\|f(x_n) - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 \\ &\quad -(1-\alpha_n)\lambda_0\lambda_i g(\|Q_C(I - \beta_i A_i)z_n - z_n\|).\end{aligned}$$

Since $\{x_n\}$ is bounded, then there exists a constant $D > 0$ such that

$$(1-\alpha_n)\lambda_0\lambda_i g(\|Q_C(I-\beta_i A_i)z_n - z_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n D. \quad (26)$$

Thus, we have

$$\lim_{n\to\infty} g(\|Q_C(I - \beta_i A_i)z_n - z_n\|) = 0. \quad (27)$$

Using the properties of g, we have

$$\lim_{n\to\infty} \|Q_C(I - \beta_i A_i)z_n - z_n\| = 0. \quad (28)$$

From (28) and Lemma 2, we obtain $x^{**} \in \bigcap_{i=1}^{m} Fix(Q_C(I - \beta_i A_i))$. Using Lemma 9, we have $x^{**} \in \bigcap_{i=1}^{m} VI(C, A_i)$. Therefore, $x^{**} \in \Gamma$. On the other hand, using the fact that x^* solves (14) and the weak continuity of the duality mapping J_φ, we have

$$\begin{aligned}\limsup_{n\to+\infty}\langle x^* - f(x^*), J_\varphi(x^* - x_n)\rangle &= \lim_{j\to+\infty}\langle x^* - f(x^*), J_\varphi(x^* - x_{n_j})\rangle \\ &= \langle x^* - f(x^*), J_\varphi(x^* - x^{**})\rangle \leq 0.\end{aligned}$$

Finally, we show that $x_n \to x^*$. In fact, since $\Phi(t) = \int_0^t \varphi(\sigma)d\sigma,\ \forall t \geq 0$, and φ is a gauge function, then $\Phi(kt) \leq k\Phi(t)$, for $1 \geq k \geq 0$.

From (10) and Lemma 5, we obtain the following estimates

$$\begin{aligned}
\Phi(\|x_{n+1}-x^*\|) &= \Phi(\|\alpha_n f(x_n)+(1-\alpha_n)y_n-x^*\|)\\
&\le \Phi(\|\alpha_n(f(x_n)-f(x^*))+(1-\alpha_n)(y_n-x^*)\|)\\
&\quad +\alpha_n\langle x^*-f(x^*), J_\varphi(x^*-x_{n+1})\rangle\\
&\le \Phi(\alpha_n\|f(x_n)-f(x^*)\|+\|(1-\alpha_n)(y_n-x^*)\|)\\
&\quad +\alpha_n\langle x^*-f(x^*), J_\varphi(x^*-x_{n+1})\rangle\\
&\le \Phi(\alpha_n b\|x_n-x^*\|+(1-\alpha_n)\|y_n-x^*\|)\\
&\quad +\alpha_n\langle x^*-f(x^*), J_\varphi(x^*-x_{n+1})\rangle\\
&\le \Phi((1-(1-b)\alpha_n)\|x_n-x^*\|)\\
&\quad +\alpha_n\langle x^*-f(x^*), J_\varphi(x^*-x_{n+1})\rangle\\
&\le (1-(1-b)\alpha_n)\Phi(\|x_n-x^*\|)\\
&\quad +\alpha_n\langle x^*-f(x^*), J_\varphi(x^*-x_{n+1})\rangle.
\end{aligned}$$

From Lemma 6, it follows that $x_n \to x^*$.

Case 2. Assume that the sequence $\{\|x_n-x^*\|\}$ is not monotonically decreasing. Set $B_n = \|x_n - x^*\|$. Let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined from for $n \ge n_0$, for some n_0 large enough, by $\tau(n) = \max\{k \in \mathbb{N} : k \le n,\ B_k \le B_{k+1}\}$. Obviously, $\{\tau(n)\}$ is a non-decreasing and $\tau(n) \to \infty$ as $n \to \infty$ and $B_{\tau(n)} \le B_{\tau(n)+1}$ for $n \ge n_0$. From (21), we obtain

$$(1-\alpha_{\tau(n)})\theta_{\tau(n)}(1-k-d_2\theta_{\tau(n)})\|Tx_{\tau(n)}-x_{\tau(n)}\|^2 \le \alpha_{\tau(n)}B.$$

Furthermore, we have

$$\lim_{n\to\infty}\theta_{\tau(n)}(1-k-d_2\theta_{\tau(n)})\|Tx_{\tau(n)}-x_{\tau(n)}\| = 0.$$

Since $1-k-d_2\theta_{\tau(n)} > 0$, it follows that

$$\lim_{n\to\infty}\|x_{\tau(n)}-Tx_{\tau(n)}\| = 0. \tag{29}$$

Using the same arguments as in case 1, we can show that $x_{\tau(n)}$ and $y_{\tau(n)}$ are bounded in C and $\limsup\limits_{\tau(n)\to+\infty}\langle x^*-f(x^*), J_\varphi(x^*-x_{\tau(n)})\rangle \le 0$.

For all $n \ge n_0$, we have

$$\begin{aligned}
0 &\le \Phi(\|x_{\tau(n)+1}-x^*\|)-\Phi(\|x_{\tau(n)}-x^*\|)\\
&\le \alpha_{\tau(n)}[-(1-b)\Phi(\|x_{\tau(n)}-x^*\|)+\langle x^*-f(x^*), J_\varphi(x^*-x_{\tau(n)+1})\rangle],
\end{aligned}$$

which implies that

$$\Phi(\|x_{\tau(n)}-x^*\|) \le \frac{1}{1-b}\langle x^*-f(x^*), J_\varphi(x^*-x_{\tau(n)+1})\rangle.$$

Then, we have

$$\lim_{n\to\infty} \Phi(\|x_{\tau(n)} - x^*\|) = 0.$$

Therefore,

$$\lim_{n\to\infty} B_{\tau(n)} = \lim_{n\to\infty} B_{\tau(n)+1} = 0.$$

Thus, by Lemma 8, we conclude that

$$0 \le B_n \le \max\{B_{\tau(n)},\ B_{\tau(n)+1}\} = B_{\tau(n)+1}.$$

Hence, $\lim_{n\to\infty} B_n = 0$, that is $\{x_n\}$ converges strongly to x^*. This completes the proof. □

Corollary 7. *Assume that $E = l_p, 2 \le p < \infty$. Let C be a nonempty, closed and convex subset of E. Let Q_C be a sunny nonexpansive retraction from E onto C and $f : C \to C$ be a b-contraction. For $i = 1, 2, ..., m$, let $A_i : C \to E$ be a α_i-inverse strongly accretive and let $T : C \to C$ be a k-demicontractive mapping such that $\left(\bigcap_{i=1}^{m} VI(C, A_i)\right) \bigcap Fix(T) \neq \emptyset$. Assume that $I - T$ is demiclosed at origin and $\beta_i \in \left[a, \frac{\alpha_i}{K^2}\right]$ for some $a > 0$ where K is the 2-uniformly smoothness constant of E. Let $\{x_n\}$ be a sequence defined as follows:*

$$\begin{cases} x_0 \in C, \ \text{choosen arbitrarily}, \\ z_n = \theta_n x_n + (1-\theta_n)Tx_n, \\ y_n = \lambda_0 z_n + \displaystyle\sum_{i=1}^{m} \lambda_i Q_C(I - \beta_i A_i)z_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)y_n, \end{cases} \tag{30}$$

where $\sum_{i=0}^{m} \lambda_i = 1$, $\theta_n \in [c, d] \subset \left(0, \min\{1, \frac{1-k}{p-1}\}\right)$ and $\{\alpha_n\} \subset (0, 1)$. Assume that the above control sequences satisfy the following conditions:

(a) $\lim_{n\to\infty} \alpha_n = 0$;

(b) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then, the sequence $\{x_n\}$ generated by (30) converges strongly to $x^* \in \left(\bigcap_{i=1}^{m} VI(C, A_i)\right) \bigcap Fix(T)$, the unique solution of the following *variational inequality:*

$$\langle x^* - f(x^*), J_\varphi(x^* - p)\rangle \le 0, \quad \forall p \in \Big(\bigcap_{i=1}^{m} VI(C, A_i)\Big) \bigcap Fix(T). \tag{31}$$

Proof. since l_p-spaces , $2 \le p < \infty$ are 2-uniformly smooth and p-uniformly convex Banach spaces with weakly continuous duality mapping, then the proof follows from Theorem 6 and Corollary 5. □

In the special case, where $T \equiv I$, the indentity map, then Theorem 6 is reduced to the following:

Theorem 8. *Let E be a 2-uniformly smooth and uniformly convex real Banach space having a weakly continuous duality map J_φ and let C be a nonempty, closed and convex subset of E. Let Q_C be a sunny nonexpansive retraction from E onto C and let $f : C \to C$ be a b-contraction. For $i = 1, 2, ..., m$, let $A_i : C \to E$ be α_i-inverse strongly accretive such that $\Gamma := \bigcap_{i=1}^{m} VI(C, A_i) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined as follows:*

$$\begin{cases} x_0 \in C, \text{ choosen arbitrarily,} \\ y_n = \lambda_0 x_n + \sum_{i=1}^{m} \lambda_i Q_C(I - \beta_i A_i)x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) y_n, \end{cases} \tag{32}$$

where $\sum_{i=0}^{m} \lambda_i = 1$, $\beta_i \in \left[a, \frac{\alpha_i}{K^2}\right]$ and $\{\alpha_n\} \subset (0, 1)$. Assume that the above control sequences satisfy the following conditions:
(a) $\lim_{n\to\infty} \alpha_n = 0$;
(b) $\sum_{n=0}^{\infty} \alpha_n = \infty$.
Then, the sequence $\{x_n\}$ generated by algorithm (32) *converges strongly to an element of $\bigcap_{i=1}^{m} VI(C, A_i)$.*

In the special case, where $A_i \equiv 0$, for $i = 1, 2, ..., m$, then Theorem 6 is reduced to the following:

Theorem 9. *Let E be a 2-uniformly smooth and uniformly convex real Banach space having a weakly continuous duality map J_φ and C be a nonempty, closed and convex subset of E. Let Q_C be a sunny nonexpansive retraction from E onto C and let $f : C \to C$ be a b-contraction. Let $T : C \to C$ be a k-demicontractive mapping such that $Fix(T) \neq \emptyset$ and $I - T$ is demiclosed at origin. Let $\{x_n\}$ be a sequence defined as follows:*

$$\begin{cases} x_0 \in C, \ \textit{choosen arbitrarily}, \\ \\ z_n = \theta_n x_n + (1-\theta_n)Tx_n, \\ \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)z_n, \end{cases} \tag{33}$$

where $\sum_{i=0}^{m} \lambda_i = 1$, $\theta_n \in [c,d] \subset \Big(0, \ \min\{1, \frac{1-k}{d_2}\}\Big)$ *and* $\{\alpha_n\} \subset (0,1)$. *Assume that the above control sequences satisfy the following conditions:*
(a) $\lim_{n\to\infty} \alpha_n = 0$;
(b) $\sum_{n=0}^{\infty} \alpha_n = \infty$.
Then, the sequence $\{x_n\}$ generated by algorithm (33) *converges strongly to an element of $Fix(T)$.*

4. APPLICATION TO CONVEX MINIMIZATION PROBLEMS

In this section, we study the problem of finding a common solution of a finite family of convex minimization problems coupled with fixed point problem in a real Hilbert spaces. Precisely, find an x^* with the property:

$$x^* \in \Big(\bigcap_{i=1}^{m} \operatorname{argmin}_{x\in C}\, g_i(x)\Big) \bigcap Fix(T), \tag{34}$$

where for, for each i, $1 \leq i \leq m$, g_i is a continuously Fréchet differentiable and convex functional on C.

Lemma 11. *(Baillon and Haddad* [5]*) Let H be a real Hilbert space and g be a continuously Fréchet differentiable and convex functional on H where ∇g denotes its gradient. If ∇g is $\frac{1}{\alpha}$-Lipschitz continuous, then ∇g is α-inverse strongly monotone.*

Remark 3. A necessary condition for a point $x^* \in K$ to be a solution of the minimization problem (34) is that :

$$x^* \in \Big(\bigcap_{i=1}^{m} VI(\nabla g_i, K)\Big) \bigcap Fix(T).$$

Hence, one has the following result.

Theorem 10. *Let H be a real Hilbert space. Let C be a nonempty, closed and convex subset of H and $f : C \to C$ be a b-contraction. For $i = 1, 2, ..., m$, let $g_i : C \to \mathbb{R}$ be a continuously Fréchet differentiable and convex functional on C with a $\frac{1}{\alpha_i}$-Lipschitz continuous ∇g_i. Let $T : C \to C$ is a k-demicontractive mapping such that $\Gamma := \Big(\bigcap_{i=1}^{m} VI(C, \nabla g_i)\Big) \bigcap Fix(T) \neq \emptyset$ and $I - T$ is demiclosed at origin. Let $\{x_n\}$ be a sequence defined as follows:*

$$\begin{cases} x_0 \in C, \ \textit{choosen arbitrarily,} \\ \\ z_n = \theta_n x_n + (1-\theta_n) T x_n, \\ \\ y_n = \lambda_0 z_n + \displaystyle\sum_{i=1}^{m} \lambda_i P_C (I - \beta_i \nabla g_i) z_n, \\ \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) y_n, \end{cases} \tag{35}$$

where $\sum_{i=0}^{m} \lambda_i = 1$, $\theta_n \in [c, d] \subset (0,\, 1)$, $\{\alpha_n\} \subset (0, 1)$, $\beta_i \in [a, 2\alpha_i]$ for some $a > 0$. Assume that the above control sequences satisfy the following conditions:
(a) $\lim_{n \to \infty} \alpha_n = 0$;
(b) $\sum_{n=0}^{\infty} \alpha_n = \infty$.
Then, the sequence $\{x_n\}$ generated by (35) converges strongly to a solution of problem (34).

Proof. We set $H = E$ and $\nabla g_i = A_i$, into Theorem 6. Then, the proof follows Theorem 6 and Remark 3. $\square$

Remark 4. Real sequence that satisfy conditions (a), and (b) is given by: $\alpha_n = \frac{1}{\sqrt{n}}$.

REFERENCES

[1] S. O. Kim, *Thermal stability of a reactive non-Newtonian flow in a sphere*, Int. Commu. Heat and Mass Transfer **60** (1) 70-81, 2009.

[2] Ya. Alber, *Metric and generalized Projection Operators in Banach space: properties and applications in Theory and Applications of Nonlinear Operators of Accretive and Monotone Type,*(A. G Kartsatos, Ed.), Marcel Dekker, New York (1996), pp. 15-50.

[3] K. Aoyama, H. Iiduka, W. Takahashi, Weak convergence of an iterative sequence for accretive operators in Banach spaces, Fixed Point Theory Appl. 2006 (2006) doi:10.1155/FPTA/2006/35390. Article ID 35390, 13 pages.

[4] M. Aslam Noor, Some development in general variational inequalities, Appl. Math. Comput. 152 (2004) 199-277.

[5] J. B. Baillon and G. Haddad, Quelques proprits des oprateurs angle-borns et n-cycliquement monotones, Israel J. Math. 26 (1977) 137 - 150.

[6] F. E. Browder, *Convergenge theorem for sequence of nonlinear operator in Banach spaces,* Z., 100 (1967) 201-225.

[7] L. C. Ceng and J. C. Yao, An extragradient-like approximation method for variational inequality problems and fixed point problems, Appl. Math. Comput., 190 (2007), 205-215.

[8] Y. Censor, A.N. Iusem, S.A. Zenios, An interior point method with Bregman functions for the variational inequality problem with paramonotone operators, Math. Program. 81 (1998) 373-400.

[9] I. Cioranescu, *Geometry of Banach space, duality mapping and nonlinear problems,* Kluwer, Dordrecht, (1990).

[10] S. Chang, J. K. Kim, X. R. Wang, *Modified block iterative algorithm for solving convex feasibility problems in Banach spaces,* Journal of Inequalities and Applications, vol. (2010), Article ID 869684, 14 pages.

[11] Chidume, C. E., *Geometric Properties of Banach spaces and Nonlinear Iterations*, Springer Verlag Series: Lecture Notes in Mathematics, Vol. 1965, (2009), ISBN 978-1-84882-189.

[12] K. Goebel, and W. A. Kirk, Topics in metric fixed poit theory, Cambridge Studies, in Advanced Mathemathics, 28, University Cambridge Press, Cambridge 1990.

[13] B. Halpern, Fixed points of nonexpansive maps, Bull. Amer. Math. Soc., 3 (1967), 957-961.

[14] H . Iiduka and W. Takahashi, Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings, Nonlinear Anal., 61 (2005), 341-350.

[15] T. Kotzer, N. Cohen and J. Shamir, Images to ration by a novel method of parallel projection onto constraint sets, Optim. Lett., 20 (1995), 1172-1174.

[16] G. Marino and H. K. Xu, Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces, J. Math. Anal. Appl., 329 (2007), 336-346.

[17] J. K. Kim, S. Y. Cho, X. Qin, Some results on generalized equilibrium problems involving strictly pseudocontractive mappings, Acta Math. Sci. Ser. B Engl. Ed., 31 (2011), 2041-2057.

[18] T.C. Lim, H.K. Xu, *Fixed point theorems for assymptoticaly nonexpansive mapping,* Nonliear Anal 22(1994), no. 11, 1345-1355.

[19] W.R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc., 4 (1953) 506-510.

[20] G. Marino, H.K. Xu, *Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces* , J. Math. Math. Appl., 329(2007), 336-346.

[21] P. E. Mainge, *Strong convergence of projected subgradient methods for non-smooth and nonstrictly convex minimization, Set-Valued Analysis,* 16, 899-912 (2008).

[22] A. Moudafi, Viscosity approximation methods for fixed-point problems, J. Math. Anal. Appl. 241 (2000), 46-55.

[23] Z. Opial, *Weak convergence of sequence of succecive approximation of non-expansive mapping,* Bull. Am. Math. Soc. 73 (1967), 591 597.

[24] X. Qin and S. M. Kang, Convergence theorems on an iterative method for variational inequality problems and fixed point problems, Bull. Malays. Math. Sci. Soc., 33 (2010), 155-167.

[25] H.K. Xu, T.H. Kim, Convergence of hybrid steepest-descent methods for variational inequalities, J. Optim. Theory Appl. 119 (2003) 185-201.

[26] J.C. Yao, Variational inequalities with generalized monotone operators, Math. Oper. Res. 19 (1994) 691-705.

[27] H.K. Xu, *Iterative algorithms for nonlinear operators,* J. London Math. Soc. **66** (2002), no. 2, 240-256.

[28] I. Yamada, The hybrid steepest-descent method for variational inequality problems over the intersection of the fixed point sets of nonexpansive mappings, in: D. Butnariu, Y. Censor, S. Reich (Eds.), Inherently Parallel Algorithms in Feasibility and Optimization and their Applications, North-Holland, Amsterdam, Holland, 2001, pp. 473-504.

[29] Y. Zhang, Q. Yuan, Iterative common solutions of fixed point and variational inequality problems, J. Nonlinear Sci. Appl. 9 (2016), 1882-1890.

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