Journal of the Nigerian Mathematical Society Vol. 38, Issue 3, pp. 363-389, 2019

Special Issue: The International Conference on Non-linear Analysis (organized to mark the 70th birthday anniversary of Prof. Charles Ejike Chidume)

SPLIT COMMON FIXED POINT PROBLEM FOR CLASS OF ASYMPTOTICALLY HEMICONTRACTIVE MAPPINGS

E.E. CHIMA[1] AND M.O. OSILIKE[2]

ABSTRACT. Let H_1 and H_2 be two real Hilbert spaces. $T : H_1 \to H_1$ and $S : H_2 \to H_2$ two asymptotically hemicontractive maps. Let $A : H_1 \to H_2$ be a bounded linear operator. The split common fixed point problem (SCFP) for T and S, which is to find a fixed point $x^* \in F(T)$ such that $Ax^* \in F(S)$ is studied. We proved that the set of fixed points of a class of asymptotically hemicontractive maps is closed and convex. We then obtain strong convergence results for the SCFP involving two asymtotically hemicontractive maps using new averaging iterative scheme.

Keywords and phrases: Asymptotically hemicontractive mappings, split common fixed point problem, strong convergence, Hilbert spaces, iterative algorithm.
2010 Mathematical Subject Classification: 47H09,47J25,65J15.

1. INTRODUCTION

The split common fixed point problem (SCFP) is a generalization of the convex feasibility problem (CFP) and split feasibility problem (SFP). SFP in finite- dimensional spaces was first introduced by Censor and Elfving [1] for modelling inverse problems which came from phase retrievals and in medical image reconstruction [2].
Let H_1 and H_2 be two real Hilbert spaces, K and Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. The split feasibility problem is formulated as finding a point $x^* \in H_1$ with the property

Received by the editors December 24, 2018; Revised April 09, 2019 ; Accepted: April 23, 2019
www.nigerianmathematicalsociety.org; Journal available online at https://ojs.ictp.it/jnms/
[1]Corresponding author
[2]Supported by Adeboye's Professorial Chair

$$x^* \in K \text{ such that } Ax^* \in Q. \tag{1.1}$$

Let $T : H_1 \to H_1$ and $S : H_2 \to H_2$ be two given mappings. Let $F(T) = \{x \in H_1 : Tx = x\} \neq \emptyset$, $F(S) = \{x \in H_2 : Sx = x\} \neq \emptyset$ and $A : H_1 \to H_2$ a bounded linear operator. Then the split common fixed point problem (SCFP) for T and S is to find a point $x^* \in F(T)$ such that $Ax^* \in F(S)$.

Li-Jun Zhu etal [3] studied and proved weak convergence results for split common fixed point problem of quasi-pseudocontractions (hemicontractions) in Hilbert spaces. Their work extend the work done on directed operators and demicontractive operators (see for example [6],[15],[16],[17],[18]).

Recently, Y.Yao, Y.Liou and J.Yao [4] studied and proved strong convergence results for the split common fixed point problem of two quasi-pseudo-contractive operators,(hemicontractive operators).

They used the following relaxed algorithm;

Let $x_1 \in H_1$ be arbitrary and let

$$\left\{\begin{array}{ll} v_n = x_n + \delta A^*[(1-\zeta_n)I & \\ \qquad + \zeta_n S((1-\eta_n)I + \eta_n S) - I]Ax_n & n \geq 1 \\ u_n = \alpha_n f(x_n) + (I - \alpha_n B)v_n & n \geq 1 \\ x_{n+1} = (1-\beta_n)u_n + \beta_n T((1-\gamma_n)u_n + \gamma_n T u_n) & n \in N. \end{array}\right\} \tag{1.2}$$

where $\{\zeta_n\}$, $\{\beta_n\}$,$\{\gamma_n\}$,$\{\eta_n\}$ and $\{\alpha_n\}$ are sequences in $(0,1)$ and δ is constant in $(0, \frac{1}{\|A\|^2})$. H_1 and H_2 are two real Hilbert spaces. $T : H_1 \to H_1$ is a uniformly L_1-Lipschitzian quasi-pseudo-contractive operator with $L_1 > 1$, $S : H_2 \to H_2$ is a uniformly L_2-Lipschitzian quasi-pseudo-contractive operator with $L_2 > 1$, $f : H_1 \to H_1$ is a ρ-contraction (i.e., mapping satisfying $\|f(x) - f(y)\| \leq \rho\|x - y\|$, $\forall x, y \in H_1$ for some $\rho \in [0,1)$), $B : H_1 \to H_1$ is a strong positive linear bounded operator with coefficient $\eta > \rho$ and $A : H_1 \to H_2$ is a bounded linear operator with its adjoint A^*.

Motivated by the results of Yao, Liou and Yao [4] , it is our purpose in this paper to introduce an averaging iterative process analogous to the algorithm in (1.2) and then prove strong convergence results for the split common fixed point problem for a class of asymptotically hemicontractive mappings in Hilbert spaces.

2. PRELIMINARIES

In what follows, we shall need the following:

Definition 2.1. Let E be a real Banach space and C a nonempty closed convex subset of E. A mapping $T : C \to C$ is said to be

L-Lipschitzian if there exist $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \ \forall x, y \in C. \tag{2.1}$$

If $L \in [0, 1)$, T is said to be a contraction. T is nonexpansive if $L = 1$.

T is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that for all $x, y \in C$ and integers $n \geq 1$, we have

$$\|T^n x - T^n y\| \leq L\|x - y\|. \tag{2.2}$$

E is said to have the Opial property if for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \ \forall y \in E \text{ with } y \neq x. \tag{2.3}$$

It is well known that every Hilbert space satisfies the Opial condition.

Definition 2.2. Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. Recall that a mapping $T : C \to C$ is called strictly pseudocontractive if there exist $k \in [0, 1)$ such that

$$\|Tx-Ty\|^2 \leq \|x-y\|^2+k\|(I-T)x-(I-T)y\|^2, \ \forall \, x, y \in C. \tag{2.4}$$

T is said to be pseudocontractive mapping if (2.4) is satisfied for $k = 1$, that is

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \ \forall \, x, y \in C.$$

T is called a directed operator if for all $x \in C$, $F(T) \neq \emptyset$ and $x^* \in F(T)$, then

$$\langle Tx - x^*, Tx - x\rangle \leq 0.$$

T is said to be demicontractive mapping if $F(T) \neq \emptyset$ and there exists $k \in [0, 1)$ such that for all $x \in C$ and $x^* \in F(T)$,

$$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 + k\|Tx - x\|^2. \tag{2.5}$$

If $F(T) \neq \emptyset$ and $\forall (x, x^*) \in C \times F(T)$, (2.5) is satisfied for $k = 1$, that is

$$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 + \|Tx - x\|^2,$$

then T is said to be hemicontractive (quasi-pseudocontractive).

Definition 2.3. Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. A mapping $T : C \to C$ is said to be asymptotically k-strictly pseudocontractive if there exists $k \in [0, 1)$ and a sequence $\{k_n\} \subseteq [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$, and

$$\|T^n x - T^n y\|^2 \leq k_n\|x - y\|^2 + k\|(x - T^n x) - (y - T^n y)\|^2. \tag{2.6}$$

T is called an asymptotically pseudocontractive mapping if (2.6) is satisfied for $k = 1$, that is

$$\|T^n x - T^n y\|^2 \leq k_n \|x - y\|^2 + \|(x - T^n x) - (y - T^n y)\|^2.$$

T is called an asymptotically demicontractive mapping [6] if $F(T) \neq \emptyset$ and there exist a sequence $\{k_n\} \subseteq [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ and $k \in [0, 1)$ such that

$$\|T^n x - x^*\|^2 \leq k_n \|x - x^*\|^2 + k\|T^n x - x\|^2, \tag{2.7}$$

for all $n \in N$, $x \in C$ and $\forall x^* \in F(T)$.
T is called an asymptotically hemicontractive mapping [7] if (2.7) is satisfied for $k = 1$, that is

$$\|T^n x - x^*\|^2 \leq k_n \|x - x^*\|^2 + \|T^n x - x\|^2. \tag{2.8}$$

The class of asymptotically hemicontractive maps properly contains the class of asymptotically pseudocontractive maps for which the fixed point set $F(T) := \{x \in C : Tx = x\} \neq \emptyset$.
It is shown in [8] that the class of asymptotically hemicontractive maps and the class of hemicontractive maps are independent.
Some examples of asymptotically hemicontractive maps include:
Example 2.4. Let $\Re$ denote the reals with the usual norm, $C = [-6, 2]$ and define $T : C \to C$ by

$$Tx = \begin{cases} x, & \text{if } x \in [-6, 1) \\ 2x - x^2, & \text{if } x \in [1, 2] \end{cases}.$$

For all integer $n > 1$, it is shown in [5] that with

$$T^n x = \begin{cases} x, & \text{if } x \in [-6, 1) \\ 2x - x^2, & \text{if } x \in [1, 2] \end{cases},$$

$$|T^n x - T^n y|^2 \leq |x - y|^2 + |x - T^n x - (y - T^n y)|^2,$$

and $|T^n x - T^n y| \leq 6|x - y|, \ \forall\ x, y \in [-6, 2], \ n \geq 1$. And since $F(T) \neq \emptyset$, thus T is asymptotically hemicontractive.

Example 2.5. Let $\Re$ denote the reals with the usual norm, $C = [0, 1]$ and define $T : C \to C$ by

$$Tx = (1 - x^{\frac{2}{3}})^{\frac{3}{2}}.$$

For all integer $n \geq 1$ and for all $x \in [0, 1]$, it is also shown in [5] that with

$$T^n x = \begin{cases} (1 - x^{\frac{2}{3}})^{\frac{3}{2}}, & \text{if } n \text{ is odd,} \\ x, & \text{if } n \text{ is even} \end{cases},$$

$|T^n x - T^n y|^2 \leq |x-y|^2 + |x - T^n x - (y - T^n y)|^2, \ \forall\, x, y \in [0,1], n \geq 1,$
and hence T is asymptotically pseudocontractive. Since $F(T) \neq \emptyset$, T is asymptotically hemicontractive.

Definition 2.6. Let H be a Hilbert space. A mapping $T : H \to H$ is said to be demiclosed at p if for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x^*$ and the sequence $\{T(x_n)\}$ converges strongly to p, then $Tx^* = p$.

Lemma 2.7 [12]. Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a uniformly L-Lipschitzian asymptotically pseudocontractive mapping. Then $(I - T)$ is demiclosed at 0.

For all $x, y \in H$, we have the following:

$$(i)\ \|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2,\ t \in [0,1]. \tag{2.9}$$

$$(ii)\ \|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle. \tag{2.10}$$

$$(iii)\ \|x-y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2. \tag{2.11}$$

If $\{x_n\}_{n=1}^{\infty}$ is a sequence in H which converges weakly to z, then

$$(iv)\ \limsup_{n\to\infty} ||x_n - y||^2 = \limsup_{n\to\infty} ||x_n - z||^2 + ||z - y||^2,\ \forall y \in H. \tag{2.12}$$

Lemma 2.8 [9]. Let $\{\Gamma_n\}$ be a sequence of real numbers. Assume $\{\Gamma_n\}$ does not decrease at infinity, that is, there exists at least a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ such that $\Gamma_{n_k} \leq \Gamma_{n_k+1}$ for all $k \geq 0$. For every $n \geq n_0$, define an integer sequence $\{\tau(n)\}$ as

$$\tau(n) = \max\{i \leq n : \Gamma_{n_i} \leq \Gamma_{n_i+1}\}.$$

Then $\tau(n) \to \infty$ *as* $n \to \infty$, and for all $n \geq n_0$,

$$\max\{\Gamma_{\tau(n)}, \Gamma_n\} \leq \Gamma_{\tau(n)+1}.$$

Lemma 2.9 [10]. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of non-negative real numbers satisfying

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\beta_n + \gamma_n,\ n \geq 0, \tag{2.13}$$

where $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$, and $\{\gamma_n\}_{n=1}^{\infty}$ satisfy the conditions:

(i) $\{\alpha_n\}_{n=1}^{\infty} \subset [0,1],\ \sum \alpha_n = \infty$ or equivalently $\prod_{n=1}^{\infty}(1-\alpha_n) = 0$;
(ii) $\limsup_{n\to\infty} \beta_n \leq 0$;
(iii) $\gamma_n \geq 0, (n \geq 0),\ \sum_{n=1}^{\infty} \gamma_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Definition 2.10. Let H be a real Hilbert space. A is said to be strongly positive operator, if there exist a constant $\gamma > 0$ such that

$$\langle Ax, x\rangle \geq \gamma\|x\|^2,\ \forall x \in H.$$

Lemma 2.11 [19]. Assume A is a strongly positive linear operator on a Hilbert space H with coefficient $\gamma > 0$ and $0 < \lambda \leq \|A\|^{-1}$. Then $\|I - \lambda A\| \leq 1 - \lambda\gamma$.

3. MAIN RESULTS

We begin with following important properties of asymptotically hemicontractive maps which will play crucial role in the proof of our convergence results.

Proposition 3.1. Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a uniformly L-Lipschitzian asymptotically hemicontractive mapping. Then, $F(T) = \{x \in C : Tx = x\}$ is closed and convex.

Proof: (i) Let $\{p_n\}_{n=1}^{\infty} \subseteq F(T)$ be such that $p_n \to p$. We prove that $p \in F(T)$.

$$\begin{aligned} ||p - Tp|| &\leq ||p - p_n|| + ||p_n - Tp|| \\ &= ||p - p_n|| + ||Tp_n - Tp|| \\ &\leq (1 + L)||p_n - p|| \to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence $p \in F(T)$, and $F(T)$ is closed.

Let $p_1, p_2 \in F(T)$ and let $\lambda \in [0, 1]$ be arbitrary. Set $p := \lambda p_1 + (1 - \lambda)p_2$. We prove that $p \in F(T)$. Observe that $||p - p_1|| = (1 - \lambda)||p_1 - p_2||$ and $||p - p_2|| = \lambda||p_1 - p_2||$. Set

$$G_n x := T^n\Big((1 - \beta)x + \beta T^n x\Big),$$

where $\beta \in \Big(0, \frac{2}{(1+\lambda)+\sqrt{(1+\lambda)^2+4L^2}}\Big)$, and $\lambda := \sup_{n\geq 1} k_n$. Then $G_n p_1 = p_1$, and $G_n p_2 = p_2$. Observe that

$$\begin{aligned} ||p - G_n p||^2 &= ||\lambda(p_1 - G_n p) + (1 - \lambda)(p_2 - G_n p)||^2 \\ &= \lambda||p_1 - G_n p||^2 + (1 - \lambda)||p_2 - G_n p||^2 \\ &\quad -\lambda(1 - \lambda)||p_1 - p_2||^2. \end{aligned} \tag{3.1}$$

Observe also that

$$\begin{aligned}
||G_np-p_1||^2 &= ||T^n\Big((1-\beta)p+\beta T^np\Big)-p_1||^2\\
&= ||T^n\Big((1-\beta)p+\beta T^np\Big)-T^np_1||^2\\
&\leq k_n||(1-\beta)(p-p_1)+\beta(T^np-T^np_1)||^2\\
&\quad +\|((1-\beta)p+\beta T^np)-T^n\Big((1-\beta)p+\beta T^np\Big)\|^2
\end{aligned}$$

$$\begin{aligned}
&= k_n(1-\beta)\|p-p_1\|^2+k_n\beta\|T^np-p_1\|^2-k_n(1-\beta)\beta\|p-T^np\|^2\\
&\quad +\|((1-\beta)p+\beta T^np)-G_np\|^2\\
&\leq k_n(1-\beta)\|p-p_1\|^2+k_n\beta\big[k_n\|p-p_1\|^2+\|p-T^np\|\big]\\
&\quad -k_n\beta(1-\beta)\|p-T^np\|^2+(1-\beta)\|p-G_np\|^2\\
&\quad +\beta\|T^np-T^n\Big((1-\beta)p+\beta T^np\Big)\|^2-\beta(1-\beta)\|p-T^np\|^2\\
&= k_n[1+\beta(k_n-1)]\|p-p_1\|^2+k_n\beta\|p-T^np\|\\
&\quad -k_n\beta(1-\beta)\|p-T^np\|^2+(1-\beta)\|p-G_np\|^2\\
&\quad +\beta^3L^2\|p-T^np\|^2-\beta(1-\beta)\|p-T^np\|^2\\
&= k_n[1+\beta(k_n-1)]\|p-p_1\|^2-\beta[1-\beta(k_n+1)-\beta^2L^2]\|p-T^np\|^2\\
&\quad +(1-\beta)\|p-G_np\|^2\\
&\leq k_n[1+\beta(k_n-1)]\|p-p_1\|^2-\beta[1-\beta(k_n+1)-\beta^2L^2]\|p-T^np\|^2\\
&\quad +(1-\beta)\|p-G_np\|^2\\
&= k_n[1+\beta(k_n-1)]\|p-p_1\|^2-\beta[1-\beta(k_n+1)-\beta^2L^2]\|p-T^np\|^2\\
&\quad +(1-\beta)\|p-G_np\|^2 \qquad (3.2)
\end{aligned}$$

Similarly,

$$\begin{aligned}
||G_np-p_2||^2 \leq\ & k_n[1+\beta(k_n-1)]\|p-p_2\|^2+(1-\beta)||p-G_np||^2\\
&-\beta[1-\beta(k_n+1)-\beta^2L^2]\|p-T^np\|^2. \qquad (3.3)
\end{aligned}$$

Thus, using (3.2) and (3.3) in (3.1), we have that

$$\begin{aligned}
||p-G_np||^2 &\leq (1-\lambda)^2\lambda k_n[1+\beta(k_n-1)]\|p_1-p_2\|^2\\
&\quad +(1-\lambda)\lambda^2k_n[1+\beta(k_n-1)]\|p_1-p_2\|^2\\
&\quad -\beta[1-\beta(k_n+1)-\beta^2L^2]||p-T^np||^2\\
&\quad +(1-\beta)||p-G_np||^2-\lambda(1-\lambda)\|p_1-p_2\|^2\\
&= \lambda(1-\lambda)(1+\beta k_n)(k_n-1)\|p_1-p_2\|^2\\
&\quad -\beta[1-\beta(k_n+1)-\beta^2L^2]||p-T^np||^2\\
&\quad +(1-\beta)||p-G_np||^2
\end{aligned}$$

and it follows that

$\beta||p-G_np||^2\leq\lambda(1-\lambda)(1+\beta k_n)(k_n-1)\|p_1-p_2\|^2\to 0$ as $n\to\infty$.

Hence $G_np \to p$ as $n \to \infty$.
Observe that

$$\begin{aligned} ||p - T^np|| &\leq ||p - G_np|| + ||G_np - T^np|| \\ &\leq ||p - G_np|| + L\beta||p - T^np||. \end{aligned}$$

Thus

$$(1 - L\beta)||p - T^np|| \leq ||p - G_np|| \to 0 \text{ as } n \to \infty,$$

and hence $T^np \to p$ as $n \to \infty$. Since T is continuous we have

$$T^{n+1}p \to Tp \text{ as } n \to \infty.$$

Thus $Tp = p$.
We now introduce our algorithm and then prove our strong convergence results for solving the split common fixed point problem for a class of asymptotically hemicontractive mappings in real Hilbert spaces.
Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ a bounded linear operator with its adjoint operator A^*, $T : H_1 \to H_1$ a uniformly L_1- Lipchitzian asymptotically hemicontractive mapping, $S : H_2 \to H_2$ a uniformly L_2- Lipchitzian asymptotically hemicontractive mapping, F(T) and F(S) are sets of fixed points of T and S respectively, $f : H_1 \to H_1$ a ρ-contractive operator and $B : H_1 \to H_1$ a strong positive linear bounded operator with $\nu > 2\rho$. The objective here is to solve the following two-set split common fixed point problem: $p \in F(T)$ and $Ap \in F(S)(i.e., p \in \Gamma = \{p \in H_1 : p \in F(T) \text{ } and \text{ } Ap \in F(S)\}$. For this purpose we modify algorithm in (1.2) to suit our class of mappings.
Algorithm 3.2: For $x_1 \in H_1$, define a sequence $\{x_n\}$ as follows:

$$\left\{ \begin{array}{ll} v_n = x_n + \delta A^*[(1-\zeta_n)I & \\ \qquad + \zeta_n S^n((1-\eta_n)I + \eta_n S^n) - I]Ax_n & n \geq 1 \\ u_n = \alpha_n f(x_n) + (I - \alpha_n B)v_n & n \geq 1 \\ x_{n+1} = (1-\beta_n)u_n + \beta_n T^n((1-\gamma_n)u_n + \gamma_n T^n u_n) & n \in N. \end{array} \right\} \quad (3.4)$$

where $\{\zeta_n\}$, $\{\beta_n\}$,$\{\gamma_n\}$,$\{\eta_n\}$ and $\{\alpha_n\}$ are sequences in $(0,1)$ and δ is a constant in $(0, \frac{1}{\|A\|^2})$.

Theorem 3.3: Let H_1 and H_2 be two real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator, and $f : H_1 \to H_1$ be a ρ-contractive operator. Let $B : H_1 \to H_1$ is a strong positive linear bounded operator with coefficient $\nu > 2\rho$ and $0 < \alpha_n \leq \|B\|^{-1}$. Let $T : H_1 \to H_1$ be uniformly L_1-Lipschitzian asymptotically hemicontractive mapping with sequence $\{k_n^{(1)}\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty}(k_n^{(1)} - 1) < \infty$, $S : H_2 \to H_2$ be uniformly L_2-Lipschitzian asymptotically hemicontractive mapping with sequence $\{k_n^{(2)}\}_{n=1}^{\infty}$such that

$\sum_{n=1}^{\infty}(k_n^{(2)}-1)<\infty$ and F(T) and F(S) are sets of fixed points of T and S respectively. Let $k_n=\max\{k_n^{(1)},k_n^{(2)}\}$, assume that $(I-T)$ and $(I-S)$ are demiclosed at zero and $\Gamma\neq\emptyset$. Suppose that the following conditions are satisfied:
(B_1) : $\sum_{n=1}^{\infty}\alpha_n<\infty$;
(B_2) : $0<a_1<\beta_n<c_1<\gamma_n<b_1<\frac{2}{(1+\lambda)+\sqrt{(1+\lambda)^2+4L_1^2}}$;where $\lambda:=\sup_{n\geq 1}k_n$.
(B_3) : $0<a_2<\zeta_n<c_2<\eta_n<b_2<\frac{2}{(1+\lambda)+\sqrt{(1+\lambda)^2+4L_2^2}}$.
Then the sequence $\{x_n\}$ generated by (3.4) converges strongly to $p=P_\Gamma(f+I-B)p$, where $\Gamma=\{p\in H_1 : p\in F(T)\ and\ Ap\in F(S)\}$.
Proof: Let $p=P_\Gamma(f+I-B)p$. Then we have $p\in F(T)$ and $Ap\in F(S)$.
From (2.9), we have

$$\begin{aligned}
&\|[(1-\zeta_n)I+\zeta_nS^n((1-\eta_n)I+\eta_nS^n)]Ax_n-Ap\|^2\\
=\ &\|(1-\zeta_n)(Ax_n-Ap)+\zeta_n(S^n((1-\eta_n)I+\eta_nS^n)Ax_n-Ap)\|^2\\
=\ &(1-\zeta_n)\|Ax_n-Ap\|^2+\zeta_n\|S^n((1-\eta_n)I+\eta_nS^n)Ax_n-Ap\|^2\\
&\quad-\zeta_n(1-\zeta_n)\|S^n((1-\eta_n)I+\eta_nS^n)Ax_n-Ax_n\|^2. \qquad (3.5)\\
&\|S^n\big(((1-\eta_n)I+\eta_nS^n)Ax_n\big)-Ap\|^2\\
\leq\ &k_n\|((1-\eta_n)I+\eta_nS^n)Ax_n-Ap\|^2\\
&+\|((1-\eta_n)I+\eta_nS^n)Ax_n-S^n\big((1-\eta_n)I+\eta_nS^n\big)Ax_n\|^2.
\end{aligned}$$

Observe that from (2.8)

$$\begin{aligned}
=\ &k_n\|(1-\eta_n)(Ax_n-Ap)+\eta_n(S^nAx_n-Ap)\|^2\\
&+\|((1-\eta_n)I+\eta_nS^n)Ax_n-S^n((1-\eta_n)I+\eta_nS^n)Ax_n\|^2\\
=\ &k_n(1-\eta_n)\|Ax_n-Ap\|^2+k_n\eta_n\|S^nAx_n-Ap\|^2\\
&-k_n\eta_n(1-\eta_n)\|Ax_n-S^nAx_n\|^2\\
&+\|((1-\eta_n)I+\eta_nS^n)Ax_n-S^n((1-\eta_n)I+\eta_nS^n)Ax_n\|^2\\
\leq\ &k_n(1-\eta_n)\|Ax_n-Ap\|^2\\
&+k_n\eta_n\big[k_n\|Ax_n-Ap\|^2+\|S^nAx_n-Ax_n\|^2\big]\\
&-k_n\eta_n(1-\eta_n)\|Ax_n-S^nAx_n\|^2\\
&+\|((1-\eta_n)(Ax_n-S^n((1-\eta_n)Ax_n+S^n\eta_nAx_n))\\
&+\eta_n(S^nAx_n-S^n((1-\eta_n)Ax_n+\eta_nS^nAx_n))\|^2
\end{aligned}$$

$$\begin{aligned}
&= k_n(1-\eta_n)\|Ax_n - Ap\|^2 + k_n^2\eta_n\|Ax_n - Ap\|^2 \\
&\quad +k_n\eta_n\|S^nAx_n - Ax_n\|^2 \\
&\quad -k_n\eta_n(1-\eta_n)\|Ax_n - S^nAx_n\|^2 \\
&\quad +(1-\eta_n)\|Ax_n - S^n((1-\eta_n)Ax_n + \eta_nS^nAx_n)\|^2 \\
&\quad +\eta_n\|S^nAx_n - S^n((1-\eta_n)Ax_n + \eta_nS^nAx_n)\|^2 \\
&\quad -\eta_n(1-\eta_n)\|S^nAx_n - Ax_n\|^2 \\
&\leq k_n[1+\eta_n(k_n-1)]\|Ax_n - Ap\|^2 \\
&\quad +k_n\eta_n\|S^nAx_n - Ax_n\|^2 \\
&\quad -k_n\eta_n(1-\eta_n)\|Ax_n - S^nAx_n\|^2 \\
&\quad +(1-\eta_n)\|Ax_n - S^n((1-\eta_n)Ax_n + \eta_nS^nAx_n)\|^2 \\
&\quad +\eta_n^3L^2\|Ax_n - S^nAx_n\|^2 \\
&\quad -\eta_n(1-\eta_n)\|S^nAx_n - Ax_n\|^2 \\
&= k_n[1+\eta_n(k_n-1)]\|Ax_n - Ap\|^2 \\
&\quad -\eta_n[1-\eta_n(1+k_n) - \eta_n^2L^2]\|S^nAx_n - Ax_n\|^2 \\
&\quad +(1-\eta_n)\|Ax_n - S^n((1-\eta_n)Ax_n + \eta_nS^nAx_n)\|^2. \qquad (3.6)
\end{aligned}$$

Since $\eta_n < \frac{2}{(1+\lambda)+\sqrt{(1+\lambda)^2+4L^2}}$, where $\lambda := \sup_{n\geq 1} k_n$, we have that $1-\eta_n(1+k_n)-\eta_n^2L^2 > 0$.
It follows that

$$\begin{aligned}
&\|S^n((1-\eta_n)I+\eta_n)Ax_n - Ap\|^2 \\
&\leq k_n[1+\eta_n(k_n-1)]\|Ax_n - Ap\|^2 \\
&\quad +(1-\eta_n)\|Ax_n - S^n((1-\eta_n)Ax_n + \eta_nS^nAx_n)\|^2. \qquad (3.7)
\end{aligned}$$

Substituting (3.7) in (3.5), we have

$$\begin{aligned}
&\|[(1-\zeta_n)I + \zeta_nS^n((1-\eta_n)I+\eta_n)]Ax_n - Ap\|^2 \\
&\leq (1-\zeta_n)\|Ax_n - Ap\|^2 \\
&\quad +\zeta_n\Big(k_n[1+\eta_n(k_n-1)]\|Ax_n - Ap\|^2 \\
&\quad +(1-\eta_n)\|Ax_n - S^n((1-\eta_n)Ax_n + \eta_nS^nAx_n)\|^2\Big) \\
&\quad -\zeta_n(1-\zeta_n)\|S^n((1-\eta_n)I+\eta_nS^n)Ax_n - Ax_n\|^2 \\
&= \big[1+\zeta_n(k_n-1)(1+\eta_nk_n)\big]\|Ax_n - Ap\|^2 \\
&\quad +\zeta_n(\zeta_n-\eta_n)\|S^n((1-\eta_n)I+\eta_nS^n)Ax_n - Ax_n\|^2. \qquad (3.8)
\end{aligned}$$

With $\zeta_n < \eta_n$, it implies that

$$\begin{aligned} & \|[(1-\zeta_n)I + \zeta_n S^n((1-\eta_n)I + S^n\eta_n)]Ax_n - Ap\|^2 \\ \leq & \left[1 + \zeta_n(k_n - 1)(1 + \eta_n k_n)\right]\|Ax_n - Ap\|^2. \end{aligned} \tag{3.9}$$

Following as in the derivation of (3.7), we deduce that

$$\begin{aligned} & \|T^n((1-\gamma_n)u_n + \gamma_n T^n u_n) - p\|^2 \\ \leq & k_n[1 + \gamma_n(k_n - 1)]\|u_n - p\|^2 \\ & +(1-\gamma_n)\|u_n - T^n((1-\gamma_n)u_n + \gamma_n T^n u_n\|^2. \end{aligned} \tag{3.10}$$

This together with (2.8), (3.4) and $\beta_n < \gamma_n$ imply

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|(1-\beta_n)u_n + \beta_n T^n((1-\gamma_n)u_n + \gamma_n T^n u_n) - p\|^2 \\ &= (1-\beta_n)\|u_n - p\|^2 \\ &\quad +\beta_n\|T^n((1-\gamma_n)u_n + \gamma_n T^n u_n) - p\|^2 \\ &\quad -\beta_n(1-\beta_n)\|u_n - T^n((1-\gamma_n)u_n + \gamma_n T^n u_n)\|^2 \\ &\leq (1-\beta_n)\|u_n - p\|^2 \\ &\quad +\beta_n\Big[k_n(1 + \gamma_n(k_n - 1))\|u_n - p\|^2 \\ &\quad +(1-\gamma_n)\|u_n - T^n((1-\gamma_n)u_n + \gamma_n T^n u_n)\|^2\Big] \\ &\quad -\beta_n(1-\beta_n)\|u_n - T^n((1-\gamma_n)u_n + \gamma_n T^n u_n)\|^2 \\ &= [1 + \beta_n(k_n - 1)(1 + \gamma_n k_n)]\|u_n - p\|^2 \\ &\quad -\beta_n(\gamma_n - \beta_n)\|u_n - T^n((1-\gamma_n)u_n + \gamma_n T^n u_n)\|^2 \qquad (3.11) \\ &\leq [1 + \beta_n(k_n - 1)(1 + \gamma_n k_n)]\|u_n - p\|^2. \qquad (3.12) \end{aligned}$$

Observe that using Lemma 2.11,

$$\begin{aligned} \|u_n - p\| &= \|\alpha_n(f(x_n) - Bp) + (I - \alpha_n B)(v_n - p)\| \\ &\leq \alpha_n\|f(x_n) - Bp\| + \|I - \alpha_n B\|\|v_n - p\| \\ &\leq \alpha_n\|f(x_n) - f(p)\| + \alpha_n\|f(p) - Bp\| \\ &+ (1 - \alpha_n\nu)\|v_n - p\| \\ &\leq \alpha_n\rho\|x_n - p\| + \alpha_n\|f(p) - Bp\| \\ &+ (1 - \alpha_n\nu)\|v_n - p\|. \end{aligned} \tag{3.13}$$

By standard argument, we obtain that

$$\begin{aligned} \|v_n - p\|^2 &= \|x_n - p + \delta A^*[(1-\zeta_n)I \\ &\quad +\zeta_n S^n((1-\eta_n)I + \eta_n S^n) - I]Ax_n\|^2 \\ &= \|x_n - p\|^2 + \delta^2\|A^*[(1-\zeta_n)I \\ &\quad +\zeta_n S^n((1-\eta_n)I + \eta_n S^n) - I]Ax_n\|^2 \\ &\quad +2\delta\langle x_n - p, A^*[(1-\zeta_n)I \\ &\quad +\zeta_n S^n((1-\eta_n)I + \eta_n S^n) - I]Ax_n\rangle. \end{aligned} \tag{3.14}$$

Since A is a linear operator, with adjoint A^*, we have

$$\begin{aligned} &\langle x_n - p, A^*[(1-\zeta_n)I + \zeta_n S^n((1-\eta_n)I + \eta_n S^n) - I]Ax_n\rangle \\ = &\langle A(x_n - p), [(1-\zeta_n)I + \zeta_n S^n((1-\eta_n)I + \eta_n S^n) - I]Ax_n\rangle \\ = &\langle [(1-\zeta_n)I + \zeta_n S^n((1-\eta_n)I + \eta_n S^n)]Ax_n - Ap, [(1-\zeta_n)I \\ &+\zeta_n S^n((1-\eta_n)I + \eta_n S^n) - I]Ax_n\rangle \\ &-\|[(1-\zeta_n)I + \zeta_n S^n((1-\eta_n)I + \eta_n S^n) - I]Ax_n\|^2. \end{aligned} \tag{3.15}$$

Also, using (2.11), we obtain

$$\begin{aligned} &\langle [(1-\zeta_n)I + \zeta_n S^n((1-\eta_n)I + \eta_n S^n)]Ax_n - Ap, [(1-\zeta_n)I \\ &+\zeta_n S^n((1-\eta_n)I + \eta_n S^n) - I]Ax_n\rangle \\ &= \frac{1}{2}\Big(\|[(1-\zeta_n)I + \zeta_n S^n((1-\eta_n)I + \eta_n S^n)]Ax_n - Ap\|^2 \\ &+\|[(1-\zeta_n)I + \zeta_n S^n((1-\eta_n)I + \eta_n S^n) - I]Ax_n\|^2 \\ &-\|Ax_n - Ap\|^2\Big). \end{aligned} \tag{3.16}$$

From (3.9), (3.15) and (3.16), we get

$$\begin{aligned} &\langle x_n - p, A^*[(1-\zeta_n)I + \zeta_n S^n((1-\eta_n)I + \eta_n S^n) - I]Ax_n\rangle \\ = &\frac{1}{2}\Big(\|[(1-\zeta_n)I + \zeta_n S^n((1-\eta_n)I + \eta_n S^n)]Ax_n - Ap\|^2 \\ &+\|[(1-\zeta_n)I + \zeta_n S^n((1-\eta_n)I + \eta_n S^n) - I]Ax_n\|^2 - \|Ax_n - Ap\|^2\Big) \\ &-\|[(1-\zeta_n)I + \zeta_n S^n((1-\eta_n)I + \eta_n S^n) - I]Ax_n\|^2 \\ \leq &\frac{1}{2}\Big([1 + \zeta_n(k_n - 1)(1 + \eta_n k_n)]\|Ax_n - Ap\|^2 \\ &+\|[(1-\zeta_n)I + \zeta_n S^n((1-\eta_n)I + \eta_n S^n) - I]Ax_n\|^2 \\ &-\|Ax_n - Ap\|^2\Big) \\ &-\|[(1-\zeta_n)I + \zeta_n S^n((1-\eta_n)I + \eta_n S^n) - I]Ax_n\|^2 \\ = &\frac{1}{2}[\zeta_n(k_n - 1)(1 + \eta_n k_n)]\|Ax_n - Ap\|^2 \\ &-\frac{1}{2}\|[(1-\zeta_n)I + \zeta_n S^n((1-\eta_n)I + \eta_n S^n) - I]Ax_n\|^2. \end{aligned} \tag{3.17}$$

Then

$$\begin{aligned}\|v_n - p\|^2 &= \|x_n - p + \delta A^*[(1-\zeta_n)I \\ &\quad +\zeta_n S^n((1-\eta_n)I + \eta_n S^n) - I]Ax_n\|^2 \\ &= \|x_n - p\|^2 + \delta^2\|A^*[(1-\zeta_n)I \\ &\quad +\zeta_n S^n((1-\eta_n)I + \eta_n S^n) - I]Ax_n\|^2 \\ &\quad +2\delta\langle x_n - p, A^*[(1-\zeta_n)I \\ &\quad +\zeta_n S^n((1-\eta_n)I + \eta_n S^n) - I]Ax_n\rangle\end{aligned}$$

$$\begin{aligned}&\leq \|x_n - p\|^2 \\ &\quad +\delta^2\|A\|^2\|[(1-\zeta_n)I + \zeta_n S^n((1-\eta_n)I + \eta_n S^n) - I]Ax_n\|^2 \\ &\quad +2\delta\Big(\frac{1}{2}([\zeta_n(k_n-1)(1+\eta_n k_n)]\|Ax_n - Ap\|^2) \\ &\quad -\frac{1}{2}\|[(1-\zeta_n)I + \zeta_n S^n((1-\eta_n)I + \eta_n S^n) - I]Ax_n\|^2\Big) \\ &= \|x_n - p\|^2 + \delta^2\|A\|^2\|[(1-\zeta_n)I + \zeta_n S^n((1-\eta_n)I + \eta_n S^n) - I]Ax_n\|^2 \\ &\quad +[\delta\|A\|^2\zeta_n(k_n-1)(1+\eta_n k_n)]\|x_n - p\|^2 \\ &\quad -\delta\|[(1-\zeta_n)I + \zeta_n S^n((1-\eta_n)I + \eta_n S^n) - I]Ax_n\|^2 \\ &= \|x_n - p\|^2 \\ &\quad +(\delta^2\|A\|^2 - \delta)\|[(1-\zeta_n)I + \zeta_n S^n((1-\eta_n)I + \eta_n S^n) - I]Ax_n\|^2 \\ &\quad +[\delta\|A\|^2\zeta_n(k_n-1)(1+\eta_n k_n)]\|x_n - p\|^2 \\ &= [1 + \delta\|A\|^2\zeta_n(k_n-1)(1+\eta_n k_n)]\|x_n - p\|^2 \\ &\quad +(\delta^2\|A\|^2 - \delta)\|[(1-\zeta_n)I + \zeta_n S^n((1-\eta_n)I + \eta_n S^n) - I]Ax_n\|^2 \\ &\leq [1 + \delta\|A\|^2\zeta_n(k_n-1)(1+\eta_n k_n)]\|x_n - p\|^2. \qquad (3.18)\end{aligned}$$

So, it follows that

$$\begin{aligned}\|v_n - p\| &\leq \sqrt{[1 + \delta\|A\|^2\zeta_n(k_n-1)(1+\eta_n k_n)]\|x_n - p\|^2} \\ &\leq [1 + \delta\|A\|^2\zeta_n(k_n-1)(1+\eta_n k_n)]\|x_n - p\|. \qquad (3.19)\end{aligned}$$

Now substituting (3.19) in (3.13), we have that

$$\begin{aligned}\|u_n - p\| &\leq \alpha_n\rho\|x_n - p\| + \alpha_n\|f(p) - Bp\| + (1-\alpha_n\nu)\|v_n - p\| \\ &\leq \alpha_n\rho\|x_n - p\| + \alpha_n\|f(p) - Bp\| \\ &\quad +(1-\alpha_n\nu)\big([1 + \delta\|A\|^2\zeta_n(k_n-1)(1+\eta_n k_n)]\|x_n - p\|\big) \\ &= \big[\alpha_n\rho \\ &\quad +(1-\alpha_n\nu)(1 + \delta\|A\|^2\zeta_n(k_n-1)(1+\eta_n k_n))\big]\|x_n - p\| \\ &\quad +\alpha_n\|f(p) - Bp\|. \qquad (3.20)\end{aligned}$$

Again, from (3.12), we deduce that

$$\begin{aligned}\|x_{n+1} - p\| &\leq \sqrt{[1 + \beta_n(k_n-1)(1+\gamma_n k_n)]\|u_n - p\|^2} \\ &\leq [1 + \beta_n(k_n-1)(1+\gamma_n k_n)]\|u_n - p\|. \qquad (3.21)\end{aligned}$$

Substituting (3.20) in (3.21), we have

$$\begin{aligned}
\|x_{n+1}-p\| &\leq [1+\beta_n(k_n-1)(1+\gamma_n k_n)] \\
&\quad \times\Big(\big[\alpha_n\rho \\
&\quad +(1-\alpha_n\nu)(1+\delta\|A\|^2\zeta_n(k_n-1)(1+\eta_n k_n))\big]\|x_n-p\| \\
&\quad +\alpha_n\|f(p)-Bp\|\Big) \\
&\leq (1+\sigma_n)(1+\xi_n)\big([1-\alpha_n(\nu-\rho)]\|x_n-p\| \\
&\quad +\alpha_n\|f(p)-Bp\|\big), \qquad (3.22)
\end{aligned}$$

where $\sigma_n=\beta_n(k_n-1)(1+\gamma_n k_n)$ and $\xi_n=\delta\|A\|^2\zeta_n(k_n-1)(1+\eta_n k_n)$. Hence,

$$\begin{aligned}
\|x_{n+1}-p\| &\leq (1+\sigma_n)(1+\xi_n)\big([1-\alpha_n(\nu-\rho)]\|x_n-p\| \\
&\quad +\alpha_n\|f(p)-Bp\|\big) \\
&= (1+\omega_n)\big([1-\alpha_n(\nu-\rho)]\|x_n-p\| \\
&\quad +\alpha_n\|f(p)-Bp\|\big) \\
&\leq (1+\omega_n)\Big([1-\alpha_n(\nu-\rho)]\max\big\{\|x_n-p\|,\frac{\|f(p)-Bp\|}{\nu-\rho}\big\} \\
&\quad +\alpha_n\|f(p)-Bp\|\Big)
\end{aligned}$$

$$\begin{aligned}
&= (1+\omega_n)\Big([1-\alpha_n(\nu-\rho)]\max\big\{\|x_n-p\|,\frac{\|f(p)-Bp\|}{\nu-\rho}\big\} \\
&\quad +\alpha_n(\nu-\rho)\frac{\|f(p)-Bp\|}{\nu-\rho}\Big) \\
&\leq (1+\omega_n)\Big([1-\alpha_n(\nu-\rho)]\max\big\{\|x_n-p\|,\frac{\|f(p)-Bp\|}{\nu-\rho}\big\} \\
&\quad +\alpha_n(\nu-\rho)\max\big\{\|x_n-p\|,\frac{\|f(p)-Bp\|}{\nu-\rho}\big\}\Big) \\
&= (1+\omega_n)\max\big\{\|x_n-p\|,\frac{\|f(p)-Bp\|}{\nu-\rho}\big\} \\
&\ \ \vdots \\
&\leq \prod_{j=1}^{n}(1+w_j)\max\big\{\|x_1-p\|,\frac{\|f(p)-Bp\|}{\nu-\rho}\big\}. \qquad (3.23)
\end{aligned}$$

Since $\omega_n=\sigma_n+\xi_n+\sigma_n\xi_n$ and $\Sigma\omega_n<\infty$, it follows that $\{x_n\}$ is bounded, so also v_n and u_n.
To complete the proof, we now consider the following two cases;

Case 1. Suppose there exists some integer $n_0 > 0$ such that $\{\|x_n - p\|\}_{n>n_0}$ is a monotone decreasing sequence. Then $\lim_{n\to\infty}\{\|x_n - p\|\}$ exists and it follow from (3.12) that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq (1+\beta_n(k_n-1)(1+\gamma_n k_n))\|u_n-p\|^2\\
&= (1+\sigma_n)\|u_n-p\|^2\\
&\leq (1+\sigma_n)\Big\{\alpha_n\rho\|x_n-p\|+\alpha_n\|f(p)-Bp\|\\
&\quad +(1-\alpha_n\nu)\|v_n-p\|\Big\}^2\\
&= (1+\sigma_n)\Big\{\alpha_n^2(\rho\|x_n-p\|+\|f(p)-Bp\|)^2\\
&\quad +2\alpha_n(1-\alpha_n\nu)(\rho\|x_n-p\|+\|f(p)-Bp\|)\\
&\quad \times\|v_n-p\|+(1-\alpha_n\nu)^2\|v_n-p\|^2\Big\}\\
&= (1+\sigma_n)\Big\{\alpha_n\big(\rho\|x_n-p\|+\|f(p)-Bp\|\big)\\
&\quad \times\big[\alpha_n\big(\rho\|x_n-p\|+\|f(p)-Bp\|\big)\\
&\quad +2\alpha_n(1-\alpha_n\nu)\|v_n-p\|\big]\\
&\quad +(1-\alpha_n\nu)^2\|v_n-p\|^2\Big\}
\end{aligned}$$

$$\begin{aligned}
&\leq (1+\sigma_n)\Big\{\alpha_n\{(\rho\|x_n-p\|+\|f(p)-Bp\|)\\
&\quad \times(\rho\|x_n-p\|+\|f(p)-Bp\|)+2(1-\alpha_n\nu)\\
&\quad [(1+\delta\|A\|^2\zeta_n(k_n-1)(1+\eta_n k_n)\|x_n-p\|)]\}\\
&\quad +(1-\alpha_n\nu)^2\|v_n-p\|^2\Big\}\\
&= (1+\sigma_n)\Big\{\alpha_n\{(\rho\|x_n-p\|+\|f(p)-Bp\|)\\
&\quad \times(\rho\|x_n-p\|+\|f(p)-Bp\|)\\
&\quad +2(1-\alpha_n\nu)(1+\xi_n)\|x_n-p\|\}\\
&\quad +(1-\alpha_n\nu)^2\|v_n-p\|^2\Big\},\ \text{(where}\\
&\quad \xi_n=\delta\|A\|^2\zeta_n(k_n-1)(1+\eta_n k_n))\\
&\leq (1+\sigma_n)\big(\alpha_n M+(1-\alpha_n\nu)^2\|v_n-p\|^2\big), \qquad (3.24)
\end{aligned}$$

where $M > 0$ is a constant such that

$$\begin{aligned}
\sup\{&(\rho\|x_n-p\|+\|f(p)-Bp\|)(\rho\|x_n-p\|+\|f(p)-Bp\|)\\
&+2(1-\alpha_n\nu)(1+\xi_n)\|x_n-p\|\}\leq M.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq (1+\sigma_n)\big(\alpha_n M+(1-\alpha_n\nu)^2\|v_n-p\|^2\big)\\
&\leq (1+\sigma_n)\Big(\alpha_n M+(1-\alpha_n\nu)^2\Big\{(1+\xi_n)\|x_n-p\|^2\\
&\quad +(\delta^2\|A\|^2-\delta)\|[(1-\zeta_n)I\\
&\quad +\zeta_n S^n((1-\eta_n)+\eta_n S^n)-I]Ax_n\|^2\Big\}\Big)\\
&= (1+\sigma_n)\Big(\alpha_n M+(1-\alpha_n\nu)^2(1+\xi_n)\|x_n-p\|^2\\
&\quad +(1-\alpha_n\nu)^2(\delta^2\|A\|^2-\delta)\|[(1-\zeta_n)I\\
&\quad +\zeta_n S^n((1-\eta_n)+\eta_n S^n)-I]Ax_n\|^2\Big)\\
&= \alpha_n M(1+\sigma_n)+(1+\sigma_n)(1-\alpha_n\nu)^2(1+\xi_n)\|x_n-p\|^2\\
&\quad +(1+\sigma_n)(1-\alpha_n\nu)^2(\delta^2\|A\|^2-\delta)\\
&\quad \times\|[(1-\zeta_n)I+\zeta_n S^n((1-\eta_n)+\eta_n S^n)-I]Ax_n\|^2.
\end{aligned}$$

Thus

$$\begin{aligned}
&(1+\sigma_n)(1-\alpha_n\nu)^2(\delta-\delta^2\|A\|^2)\|[(1-\zeta_n)I\\
&\qquad +\zeta_n S^n((1-\eta_n)+\eta_n S^n)-I]Ax_n\|^2\\
&\leq \alpha_n M(1+\sigma_n)+(1+\sigma_n)(1-\alpha_n\nu)^2(1+\xi_n)\|x_n-p\|^2\\
&-\|x_{n+1}-p\|^2.
\end{aligned}$$

Since $\lim_{n\to\infty}\|x_n-p\|$ exist, $\alpha_n\to 0$, $\sigma_n\to 0$, and $\xi_n\to 0$, we have that

$$\lim_{n\to\infty}\|[(1-\zeta_n)I+\zeta_n S^n((1-\eta_n)I+\eta_n S^n)-I]Ax_n\|=0. \quad (3.25)$$

Hence

$$\lim_{n\to\infty}\|Ax_n-S^n((1-\eta_n)I+\eta_n S^n)Ax_n\|=0. \quad (3.26)$$

We now have

$$\begin{aligned}
\|Ax_n-S^nAx_n\| &\leq \|Ax_n-S^n((1-\eta_n)I+\eta_n S^n)Ax_n\|\\
&\quad +\|S^n((1-\eta_n)I+\eta_n S^n)Ax_n-S^nAx_n\|\\
&\leq \|Ax_n-S^n((1-\eta_n)I+\eta_n S^n)Ax_n\|\\
&\quad +L_2\eta_n\|Ax_n-S^nAx_n\|.
\end{aligned}$$

Therefore
$\|Ax_n-S^nAx_n\|\leq\frac{1}{1-L_2\eta_n}\|Ax_n-S^n((1-\eta_n)I+\eta_n S^n)Ax_n\|$.
Thus

$$\lim_{n\to\infty}\|Ax_n-S^nAx_n\|=0. \quad (3.27)$$

Observe that

$$\begin{aligned}\|u_n - x_n\| &= \|\delta A^*[(1-\zeta_n)I\zeta_n S^n((1-\eta_n)I + \eta_n S^n) - I]Ax_n \\ &\quad +\alpha_n\big(f(x_n) - Bx_n - \delta BA^*[(1-\zeta_n)I \\ &\quad +\zeta_n S^n((1-\eta_n)I + \eta_n S^n) - I]Ax_n\big)\| \\ &\leq \delta\|A\|\|[(1-\zeta_n)I + \zeta_n S^n((1-\eta_n)I + \eta_n S^n) - I]Ax_n\| \\ &\quad +\alpha_n\|Bx_n + \delta BA^*[(1-\zeta_n)I \\ &\quad +\zeta_n S^n((1-\eta_n)I + \eta_n S^n) - I]Ax_n - f(x_n)\|.\end{aligned}$$

This together with (3.25) gives that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.28}$$

From (3.11) and (3.24) we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq [1 + \beta_n(k_n - 1)(1 + \gamma_n k_n)]\|u_n - p\|^2 \\ &\quad -\beta_n(\gamma_n - \beta_n)\|u_n - T^n((1-\gamma_n)u_n + \gamma_n T^n u_n)\|^2 \\ &\leq (1+\sigma_n)[\alpha_n M + (1-\alpha_n\nu)^2(1+\xi_n)\|x_n - p\|^2] \\ &\quad -\beta_n(\gamma_n - \beta_n)\|u_n - T^n((1-\gamma_n)u_n + \gamma_n T^n u_n)\|^2.\end{aligned}$$

This implies that

$$\begin{aligned}&\beta_n(\gamma_n - \beta_n)\|u_n - T^n((1-\gamma_n)u_n + \gamma_n T^n u_n)\|^2 \\ \leq\ &(1+\sigma_n)[\alpha_n M + (1-\alpha_n\nu)^2(1+\xi_n)]\|x_n - p\|^2 \\ &-\|x_{n+1} - p\|^2.\end{aligned}$$

Hence

$$\lim_{n\to\infty} \|u_n - T^n((1-\gamma_n)u_n + \gamma_n T^n u_n)\| = 0. \tag{3.29}$$

Note that

$$\begin{aligned}\|u_n - T^n u_n\| &\leq \|u_n - T^n((1-\gamma_n)u_n + \gamma_n T^n u_n)\| \\ &\quad +\|T^n((1-\gamma_n)u_n + \gamma_n T^n u_n) - T^n u_n\| \\ &\leq \|u_n - T^n((1-\gamma_n)u_n + \gamma_n T^n u_n)\| \\ &\quad +L_1\gamma_n\|u_n - T^n u_n\|.\end{aligned}$$

Hence,
$\|u_n - T^n u_n\| \leq \frac{1}{1-L_1\gamma_n}\|u_n - T^n((1-\gamma_n)u_n + \gamma_n T^n u_n)\|$
This together with (3.29) gives that

$$\lim_{n\to\infty} \|u_n - T^n u_n\| = 0. \tag{3.30}$$

Next, we prove that,
$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$ and $\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0$.

Observe that

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \|(1-\beta_n)u_n \\
&\quad +\beta_n T^n((1-\gamma_n)u_n+\gamma_n T^n u_n)-x_n\| \\
&= \|(1-\beta_n)u_n+\beta_n T^n((1-\gamma_n)u_n \\
&\quad +\gamma_n T^n u_n)-x_n+u_n-u_n\| \\
&= \|(1-\beta_n)u_n-u_n+\beta_n T^n((1-\gamma_n)u_n \\
&\quad +\gamma_n T^n u_n)+u_n-x_n\| \\
&\leq \beta_n\|T^n((1-\gamma_n)u_n+\gamma_n T^n u_n)-u_n\| \\
&\quad +\|x_n-u_n\| \to 0,\ \text{as } n\to\infty.
\end{aligned}$$

It follows that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.31}$$

Again

$$\begin{aligned}
\|u_{n+1}-u_n\| &= \|\alpha_{n+1}f(x_{n+1})+(I-\alpha_{n+1}B)v_{n+1} \\
&\qquad -(\alpha_n f(x_n)+(1-\alpha_n B)v_n)\| \\
&= \|\alpha_{n+1}f(x_{n+1})+v_{n+1}-\alpha_{n+1}Bv_{n+1} \\
&\qquad -\alpha_n f(x_n)-v_n+\alpha_n Bv_n\| \\
&\leq \|\alpha_{n+1}(f(x_{n+1})-Bv_{n+1})\|+\|\alpha_n\big(Bv_n-f(x_n)\big)\| \\
&\quad +\|v_{n+1}-v_n\| \\
&= \alpha_{n+1}\|f(x_{n+1})-Bv_{n+1}\|+\alpha_n\|Bv_n-f(x_n)\| \\
&\quad +\|x_{n+1}+\delta A^*[(1-\zeta_{n+1})I \\
&\quad +\zeta_{n+1}S^{n+1}\big((1-\eta_{n+1})I+\eta_{n+1}S^{n+1}\big)-I]Ax_{n+1} \\
&\quad -\Big(x_n+\delta A^*\Big[(1-\zeta_n)I \\
&\quad +\zeta_n S^n\big((1-\eta_n)I+\eta_n S^n\big)-I\Big]Ax_n\Big)\| \\
&\leq \alpha_{n+1}\|f(x_{n+1})-Bv_{n+1}\|+\alpha_n\|Bv_n-f(x_n)\| \\
&\quad +\|x_{n+1}-x_n\| \\
&\quad +\delta\|A\|\|[(1-\zeta_{n+1})I \\
&\quad +\zeta_{n+1}S^{n+1}\big((1-\eta_{n+1})I+\eta_{n+1}S^{n+1}\big)-I]Ax_{n+1}\| \\
&\quad +\delta\|A\|\|[(1-\zeta_n)I \\
&\quad +\zeta_n S^n\big((1-\eta_n)I+\eta_n S^n\big)-I]Ax_n\| \to 0,\ \text{as } n\to\infty.
\end{aligned}$$

This implies that

$$\lim_{n\to\infty}\|u_{n+1}-u_n\|=0. \tag{3.32}$$

Also, we prove that

$\|Ax_n-SAx_n\|\to 0,\ as\ n\to\infty$ and $\|u_n-Tu_n\|\to 0,\ \text{as } n\to\infty.$

From (3.27), we set

$$z_n := \|Ax_n - S^n Ax_n\| \to 0, \text{ as } n \to \infty. \tag{3.33}$$

Since S is uniformly L_2-Lipschitzian, it follows from (3.27) and (3.33) that

$$\begin{aligned}
\|Ax_n - SAx_n\| &\leq \|Ax_n - S^n Ax_n\| + \|S^n Ax_n - SAx_n\| \\
&\leq z_n + L_2\|S^{n-1}Ax_n - Ax_n\| \\
&\leq z_n + L_2\{\|S^{n-1}Ax_n - S^{n-1}Ax_{n-1}\| \\
&\quad +\|S^{n-1}Ax_{n-1} - Ax_n\|\}
\end{aligned}$$

$$\begin{aligned}
&\leq z_n + L_2^2\|Ax_n - Ax_{n-1}\| \\
&\quad +L_2\|S^{n-1}Ax_{n-1} - Ax_{n-1} + Ax_{n-1} - Ax_n\| \\
&\leq z_n + L_2(1+L_2)\|Ax_n - Ax_{n-1}\| \\
&\quad +L_2 z_{n-1} \to 0 \text{ as } n \to \infty.
\end{aligned}$$

It follows that

$$\lim_{n\to\infty} \|Ax_n - SAx_n\| = 0. \tag{3.34}$$

Similarly from (3.30), set

$$t_n := \|u_n - T^n u_n\| \to 0, \ (as\ n \to \infty). \tag{3.35}$$

Since T is uniformly L_1-Lipschitzian, it follows from (3.30) and (3.35) that

$$\begin{aligned}
\|u_n - Tu_n\| &\leq \|u_n - T^n u_n\| + \|T^n u_n - Tu_n\| \\
&\leq t_n + L_1\|T^{n-1}u_n - u_n\| \\
&\leq t_n + L_1\{\|T^{n-1}u_n - T^{n-1}u_{n-1}\| + \|T^{n-1}u_{n-1} - u_n\|\} \\
&\leq t_n + L_1^2\|u_n - u_{n-1}\| \\
&\quad +L_1\|T^{n-1}u_{n-1} - u_{n-1} + u_{n-1} - u_n\| \\
&\leq t_n + L_1(1+L_1)\|u_n - u_{n-1}\| \\
&\quad +L_1 t_{n-1} \to 0 \text{ as } n \to \infty.
\end{aligned}$$

This implies that

$$\lim_{n\to\infty} \|u_n - Tu_n\| = 0. \tag{3.36}$$

Moreover, we show that
$\limsup_{n\to\infty}\langle f(p) - Bp, u_n - p\rangle \leq 0.$
Choose a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$\lim_{n\to\infty} \sup\langle f(p) - Bp, u_n - p\rangle = \lim_{k\to\infty}\langle f(p) - Bp, u_{n_k} - p\rangle. \tag{3.37}$$

Since the sequence $\{u_{n_k}\}$ is bounded, we can choose a subsequence $\{u_{n_{k_i}}\}$ of $\{u_{n_k}\}$ such that $\{u_{n_{k_i}}\} \rightharpoonup y$. Without loss of generality, we assume that $\{u_{n_k}\} \rightharpoonup y$.
Thus, we derive from the above conclusions that

$$x_{n_k} \rightharpoonup y \text{ and } Ax_{n_k} \rightharpoonup Ay \text{ as } k \to \infty. \tag{3.38}$$

By the assumption that $I - T$ and $I - S$ are demi-closed at 0, we have that $Ay \in F(S)$ and $y \in F(T)$, which means that $y \in \Gamma$.
Hence,

$$\begin{aligned}
\limsup_{n\to\infty} \langle f(p) - Bp, u_n - p\rangle &= \lim_{k\to\infty} \langle f(p) - Bp, u_{n_k} - p\rangle \\
&= \lim_{k\to\infty} \langle f(p) - Bp, y - p\rangle \\
&\leq 0. \qquad (3.39)
\end{aligned}$$

From (3.12)

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq [1 + \beta_n(k_n - 1)(1 + \gamma_n k_n)]\|u_n - p\|^2 \\
&= (1 + \sigma_n)\|u_n - p\|^2.
\end{aligned}$$

Using (2.10), we have

$$\begin{aligned}
\|u_n - p\|^2 &= \|\alpha_n f(x_n) + (I - \alpha_n B)v_n - p\|^2 \\
&= \|(I - \alpha_n B)(v_n - p) + \alpha_n(f(x_n) - Bp)\|^2 \\
&\leq (1 - \alpha_n \nu)\|v_n - p\|^2 + 2\alpha_n\langle f(x_n) - Bp, u_n - p\rangle \\
&\leq (1 - \alpha_n \nu)(1 + \delta\|A\|^2\zeta_n(k_n - 1)(1 + \eta_n k_n))\|x_n - p\|^2 \\
&\quad +2\alpha_n\langle f(x_n) - Bp, u_n - p\rangle \\
&= (1 - \alpha_n \nu)(1 + \xi_n)\|x_n - p\|^2 + 2\alpha_n\langle f(x_n) - f(p), u_n - p\rangle \\
&\quad +2\alpha_n\langle f(p) - Bp, u_n - p\rangle \\
&\leq (1 - \alpha_n \nu)(1 + \xi_n)\|x_n - p\|^2 + 2\alpha_n\rho\|x_n - p\|\|u_n - p\| \\
&\quad +2\alpha_n\langle f(p) - Bp, u_n - p\rangle \\
&\leq (1 - \alpha_n \nu)(1 + \xi_n)\|x_n - p\|^2 + \alpha_n\rho\|x_n - p\|^2 \\
&\quad +\alpha_n\rho\|u_n - p\|^2 + 2\alpha_n\langle f(p) - Bp, u_n - p\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|u_n - p\|^2 &\leq \frac{[(1 - \alpha_n \nu)(1 + \xi_n) + \alpha_n\rho]\|x_n - p\|^2}{1 - \alpha_n\rho} \\
&\quad + \frac{2\alpha_n}{1 - \alpha_n\rho}\langle f(p) - Bp, u_n - p\rangle. \qquad (3.40)
\end{aligned}$$

Thus

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le (1+\sigma_n)\|u_n-p\|^2 \\
&\le (1+\sigma_n)\Big\{\frac{[(1-\alpha_n\nu)(1+\xi_n)+\alpha_n\rho]\|x_n-p\|^2}{1-\alpha_n\rho} \\
&\quad +\frac{2\alpha_n}{1-\alpha_n\rho}\langle f(p)-Bp, u_n-p\rangle\Big\}
\end{aligned}$$

$$\begin{aligned}
&= (1+\sigma_n)\Big\{\{\frac{1-\alpha_n(\nu-\rho)+\xi_n(1-\alpha_n\nu)}{1-\alpha_n\rho}\}\|x_n-p\|^2 \\
&\quad +\frac{2\alpha_n}{1-\alpha_n\rho}\langle f(p)-Bp, u_n-p\rangle\Big\} \\
&= \{1-\frac{\alpha_n(\nu-2\rho)}{1-\alpha_n\rho}\}\|x_n-p\|^2 \\
&\quad +(1+\sigma_n)\frac{2\alpha_n}{1-\alpha_n\rho}\langle f(p)-Bp, u_n-p\rangle
\end{aligned}$$

$$\begin{aligned}
&+\{\frac{\sigma_n[1-\alpha_n(\nu-\rho)+\xi_n(1-\alpha_n\nu)]+\xi_n(1-\alpha_n\nu)}{1-\alpha_n\rho}\}\|x_n-p\|^2 \\
\le\ & \{1-\frac{\alpha_n(\nu-2\rho)}{1-\alpha_n\rho}\}\|x_n-p\|^2 \\
&+(1+\sigma_n)\frac{2\alpha_n}{1-\alpha_n\rho}\langle f(p)-Bp, u_n-p\rangle \\
&+\{\frac{\sigma_n[1-\alpha_n(\nu-\rho)+\xi_n(1-\alpha_n\nu)]+\xi_n(1-\alpha_n\nu)}{1-\alpha_n\rho}\}D \\
=\ & \{1-\frac{\alpha_n(\nu-2\rho)}{1-\alpha_n\rho}\}\|x_n-p\|^2 \\
&+(1+\sigma_n)\frac{2\alpha_n}{1-\alpha_n\rho}\langle f(p)-Bp, u_n-p\rangle+\theta_n, \qquad (3.41)
\end{aligned}$$

where $\theta_n = \{\frac{\sigma_n[1-\alpha_n(\nu-\rho)+\xi_n(1-\alpha_n\nu)]+\xi_n(1-\alpha_n\nu)}{1-\alpha_n\rho}\}D$, and $D > 0$ is a constant such that $\|x_n-p\|^2 \le D,\ \forall n \ge 1$.

Using (3.39) in (3.41) and applying Lemma 2.9, we have that $x_n \to p$.

Case 2: Suppose that $\{\|x_n-p\|\}_{n=1}^{\infty}$ is not a monotone decreasing sequence. Then for any n_0, there exists an integer $m \ge n_0$ such that $\|x_m-p\| \le \|x_{m+1}-p\|$. Let n_0 be such that $\|x_{n_0}-p\| \le \|x_{n_0+1}-p\|$. Set $\varphi_n = \|x_n-p\|$. Then we have

$\varphi_{n_0} \le \varphi_{n_0+1}$.

Let $\tau : N \to N$ be a mapping defined for all $n \ge n_0$ for some

sufficiently large n_0 by
$\tau(n) := \max\{k \in N : n_0 \leq k \leq n,\ \varphi_k \leq \varphi_{k+1}\}$
Then τ is a non-decreasing sequence such that $\tau(n) \to \infty\ as\ n \to \infty$ and $\varphi_{\tau(n)} \leq \varphi_{\tau(n)+1}$ for all $n \geq n_0$.
Following a similar argument as in case 1, we obtain that
$\lim_{n\to\infty} \|Ax_{\tau(n)} - SAx_{\tau(n)}\| = 0$ and $\lim_{n\to\infty} \|u_{\tau(n)} - Tu_{\tau(n)}\| = 0$.
This implies that
$\varphi_w(u_{\tau(n)}) \subset \Gamma$.
Hence, we have

$$\lim_{n\to\infty} \sup\langle f(p) - Bp, u_{\tau(n)} - p\rangle \leq 0. \tag{3.42}$$

Again, since $\varphi_{\tau(n)} \leq \varphi_{\tau(n)+1}$, we have from (3.41)

$$\begin{aligned}
\varphi^2_{\tau(n)} &\leq \varphi^2_{\tau(n)+1} \\
&\leq \Big[1 - \frac{(\nu - 2\rho)\alpha_{\tau(n)}}{1 - \alpha_{\tau(n)}\rho}\Big]\varphi^2_{\tau(n)} \\
&\quad +(1 + \sigma_{\tau(n)})\frac{2\alpha_{\tau(n)}}{1 - \alpha_{\tau(n)}\rho}\langle f(p) - Bp, u_{\tau(n)} - p\rangle \\
&\quad +\frac{1}{1 - \alpha_{\tau(n)}\rho}\{\sigma_{\tau(n)}[1 - \alpha_{\tau(n)}(\nu - \rho) + \xi_{\tau(n)}(1 - \alpha_{\tau(n)}\nu)] \\
&\quad +\xi_{\tau(n)}(1 - \alpha_{\tau(n)}\nu)\}D.
\end{aligned} \tag{3.43}$$

It follows that

$$\begin{aligned}
\varphi^2_{\tau(n)} &\leq (1 + \sigma_{\tau(n)})\frac{2}{\nu - 2\rho}\langle f(p) - Bp, u_{\tau(n)} - p\rangle \\
&\quad +\{\sigma_{\tau(n)}[1 - \alpha_{\tau(n)}(\nu - \rho) + \xi_{\tau(n)}(1 - \alpha_{\tau(n)}\nu)] \\
&\quad +\xi_{\tau(n)}\}D.
\end{aligned} \tag{3.44}$$

Using (3.42) in (3.44), and using the fact that $\sum \sigma_{\tau(n)} < \infty$, we have

$$\lim_{n\to\infty} \varphi_{\tau(n)} = 0 \tag{3.45}$$

From (3.43), we obtain that
$\limsup_{n\to\infty} \varphi^2_{\tau(n)+1} \leq \lim_{n\to\infty} \sup \varphi^2_{\tau(n)}$.
This together with (3.45) implies that
$\lim_{n\to\infty} \varphi_{\tau(n)+1} = 0$.
Apply Lemma 2.8 to have
$0 \leq \varphi_n \leq \max\{\varphi_{\tau(n)}, \varphi_{\tau(n)+1}\}$.
Hence, $\lim_{n\to\infty} \varphi_n = 0$, and this implies that $\{x_n\}_{n=1}^{\infty}$ converges strongly to p.

Corollary 3.3: Let H_1 and H_2 be two real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator, and $f : H_1 \to H_1$ is a ρ-contractive operator. Let $B : H_1 \to H_1$ is a strong positive linear bounded operator with coefficient $\nu > 2\rho$ and $0 < \alpha_n \leq \|B\|^{-1}$. Let $T : H_1 \to H_1$ a uniformly L_1-Lipschitzian asymptotically pseudocontractive mapping with sequence $\{k_n^{(1)}\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty}(k_n^{(1)} - 1) < \infty$, and $F(T) \neq \emptyset$; $S : H_2 \to H_2$ a uniformly L_2-Lipschitzian asymptotically pseudocontractive mapping with sequence $\{k_n^{(2)}\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty}(k_n^{(2)} - 1) < \infty$, and $F(S) \neq \emptyset$. If $\Gamma \neq \emptyset$ and suppose that the following conditions are satisfied:
$(B_1):\ \sum_{n=1}^{\infty} \alpha_n < \infty$;
$(B_2):\ 0 < a_1 < \beta_n < c_1 < \gamma_n < b_1 < \frac{2}{(1+h)+\sqrt{(1+h)^2+4L^2}}$;
$(B_3):\ 0 < a_2 < \zeta_n < c_2 < \eta_n < b_2 < \frac{2}{(1+h)+\sqrt{(1+h)^2+4L^2}}$,
$h = \sup_{n\geq 1} k_n$.
Then the sequence $\{x_n\}$ generated by (3.4) converges strongly to $p = P_{\Gamma}(f + I - B)p$.

4. NUMERICAL EXAMPLE

Let $H_1 = l_2$, let $f : l_2 \to l_2$ be define by

$$f(x_1, x_2, x_3, ...) = (0, \frac{x_1}{2}, \frac{x_2}{3}, \frac{x_3}{4}, ...)$$

and $B : l_2 \to l_2$ be define by

$$B(x_1, x_2, x_3, ...) = (4x_1, 4x_2, 4x_3, ...).$$

Let $T : l_2 \to l_2$ be defined by

$$T(x_1, x_2, x_3, ...) = (0, x_1, x_2, ...)$$

. Let $H_2 = R^n$, and for each $x = (x_1, x_2, x_3, ..., x_n) \in R^n$, define $S : R^n \to R^n$ by

$$Sx = \begin{cases} (x_1, x_2, x_3, \ldots, x_n), & \text{if } \prod_{j=1}^{n} x_j < 0, \\ (-x_1, -x_2, -x_3, \ldots, -x_n), & \text{if } \prod_{j=1}^{n} x_j \geq 0. \end{cases}$$

Then $F(T) = \{(0, 0, 0, \ldots)\}$ and

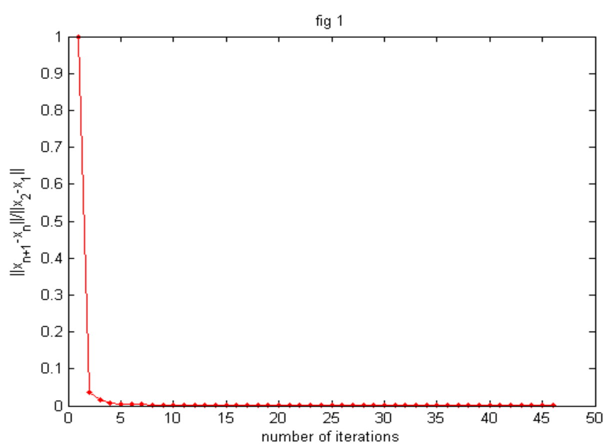
$$F(S) = \{(0, 0, 0, \ldots, 0)\} \cup \{(x_1, x_1, x_3, \ldots, x_n) : \prod_{j=1}^{n} x_j < 0\}.$$

Define $A : l_2 \to R^n$ by $Ax = (x_1, x_2, x_3, \ldots, x_n)$ for each $x = (x_1, x_2, x_3, \ldots) \in l_2$. Then A is a bounded linear operator with adjoint operator $A^*y = (x_1, x_2, x_3, \ldots, x_n, 0, 0, 0, \ldots)$ for

$y = (x_1, x_2, x_3, \ldots, x_n) \in R^n$. Again, $||A|| = ||A^*|| = 1$.
The following shows the efficacy and computability of our iterative scheme.
Case I. Let $\alpha_n = \frac{1}{(n+1)^2}$, $\beta_n = \frac{n}{4(n+1)(1+\sqrt{2})} = \zeta_n$, $\gamma_n = \frac{n}{2(n+1)(1+\sqrt{2})} = \eta_n$, $\delta = \frac{1}{2}$.Taking the initial value x $(x = x_1 \in l_2 = H_1)$ as $x = (1, 1, 1, 1, 1, 0, 0, 0, \cdots)$, $y = \frac{\|x_{n+1}-x_n\|}{\|x_2-x_1\|}$ and using (3.4) at each iterative step with the stopping criterion of 10^{-4} yields convergence as shown in Table 1 and Figure 1 below

Table 1

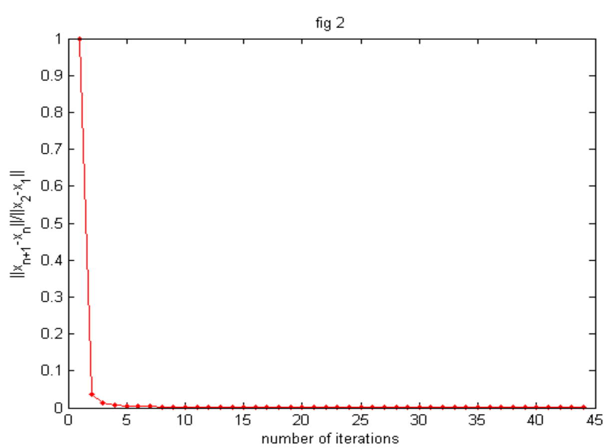
1.0000	0.0367	0.0150	0.0067	0.0058
0.0032	0.0033	0.0019	0.0022	0.0013
0.0015	0.0010	0.0011	0.0007	0.0008
0.0006	0.0006	0.0005	0.0005	0.0004
0.0004	0.0003	0.0003	0.0003	0.0003
0.0002	0.0002	0.0002	0.0002	0.0002
0.0002	0.0002	0.0002	0.0002	0.0001
0.0001	0.0001	0.0001	0.0001	0.0001
0.0001	0.0001	0.0001	0.0001	0.0001
0.0001				



Case II. Let $\alpha_n = \frac{1}{(n+1)^2}$, $\beta_n = \frac{n}{5(n+1)(1+\sqrt{2})}$, $\zeta_n = \frac{n}{6(n+1)(1+\sqrt{2})}$, $\gamma_n = \frac{n}{(n+1)(1+\sqrt{2})}$ $\eta_n = \frac{n}{3(n+1)(1+\sqrt{2})}$, $\delta = \frac{1}{2}$. Taking the initial value x $(x = x_1 \in l_2 = H_1)$ as $x = (1, 1, 1, 1, 1, 0, 0, 0, \cdots)$, $y = \frac{\|x_{n+1}-x_n\|}{\|x_2-x_1\|}$ and using (3.4) at each iterative step with the stopping criterion of 10^{-4} yields convergence as shown in Table 2 and Figure 2 below

Table 2

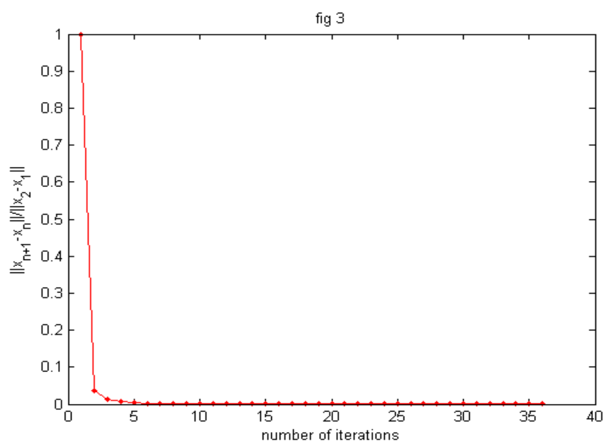
1.0000	0.0366	0.0141	0.0066	0.0053
0.0030	0.0030	0.0018	0.0021	0.0013
0.0015	0.0009	0.0011	0.0007	0.0009
0.0006	0.0007	0.0005	0.0006	0.0004
0.0005	0.0003	0.0004	0.0003	0.0003
0.0002	0.0003	0.0002	0.0002	0.0002
0.0002	0.0002	0.0002	0.0001	0.0001
0.0001	0.0001	0.0001	0.0001	0.0001
0.0001	0.0001	0.0001	0.0001	



Case III. Let $\alpha_n = \frac{1}{(n+1)^2}$, $\beta_n = \frac{n}{5(n+1)(1+\sqrt{2})}$, $\zeta_n = \frac{n}{6(n+1)(1+\sqrt{2})}$, $\gamma_n = \frac{n}{(n+1)(1+\sqrt{2})}$ $\eta_n = \frac{n}{3(n+1)(1+\sqrt{2})}$, $\delta = \frac{1}{2}$. Taking the initial value x $(x = x_1 \in l_2 = H_1)$ as $x = (1, 1, 1, 1, -1, 0, 0, 0, \cdots)$, $y = \frac{\|x_{n+1}-x_n\|}{\|x_2-x_1\|}$ and using (3.4) at each iterative step with the stopping criterion of 10^{-4} yields convergence as shown in Table 3 and Figure 3 below

Table 3

1.0000	0.0363	0.0139	0.0065	0.0051
0.0028	0.0029	0.0017	0.0019	0.0012
0.0014	0.0009	0.0011	0.0007	0.0008
0.0005	0.0006	0.0004	0.0005	0.0003
0.0004	0.0003	0.0003	0.0002	0.0003
0.0002	0.0002	0.0002	0.0002	0.0001
0.0002	0.0001	0.0001	0.0001	0.0001
0.0001				



ACKNOWLEDGEMENTS

Authors are very thankful to reviewers for their valuable comments and suggestions.

REFERENCES

[1] Y. Censor and T. Elfving, *A multiprojection algorithm using Bregman Product space*, Numerical Algorithms , **1**, 221-239, 1994.

[2] C. Byrne, *Iterative oblique projection onto convex sets and the split feasibility problem*, Inverse Problems **18** (2), 2002.

[3] L. Zhu, Y. Liou, J. Yao and Y. Yao, *New algorithms designed for the split common fixed point problem of quasi-pseudocontractive* Journal of Inequalities and Applications, **2014**, 2014:304.

[4] Y. Yao, Y.Liou and J. Yao, *Split Common Fixed Point Problem for two quasi-pseudo-contractive operators and its algorithm construction*, Fixed point theory and Applications, **2015**, 2015:127.

[5] E.E. Chima and M.O. Osilike, *Split Common Fixed Point Problem for a Class of Total Asymptotic Pseudocontractions*, Journal of Appl Math., **2016**, Article ID 3435078.

[6] L. Qihou, *Convergence theorems of the iteration for asymtotically demicontractive and hemicontractive mappings*, Nonlinear Anal, Theory Methods Appl., **261** (11) 1835-1842, 1996.
[7] Q. Liu, *On Naimpally and Sigh's open question*, *Anal.Appl*, **124**, 157-164, 1987.
[8] M.O. Osilike, A. Udoemene, D.I. Igbokwe, B.G. Akuchu, *Demicloseness principle and convergence theorems for k-strictly asymtotically pseudocontractive maps*, J. Math. Anal. Appl.,**326** 1334-1345, 2006.
[9] Mainge, PE, *Approximation methods for common fixed points of nonexpansive mappings in Hilbert spaces* J. Math.Anal.Appl., **325**,469-479, 2007.
[10] H.K. Xu, *Inequalities in Banach Spaces with Applications*, Nonlinear Analysis, **16**(2) 1127-1138, 1991.
[11] H. Zegeye, N. Shahzad and M.A. Alghamdi,*Convergence of Ishikawa's iteration method for pseudocontractive mappings*, Nonlinear Anal. **74**, 7304-7311, 2011.
[12] M.O. Osilike, P.U. Nwokoro, E.E. Chima, *Strong Convergence of new Iterative algorithms for certain classes of asymptotically pseudocontraction*, Fixed point theory and Applications, **2013**, 2013:334.
[13] M.O. Osilike and A.C. Onah, *Strong Convergence of the Ishikawa iteration for α-hemicontractive mappings*, Seria Mathematică-Informatică, **LIII** (1) 151-161, 2015.
[14] M.O. Osilike, F.O. Isiogugu and F.U. Attah, *Strong Convergence of a modified Ishikawa iterative algorithm for Lipschitz pseudocontractive mappings*, J. Appl. Math. and Informatics, **31**, 565-575, 2013.
[15] Y. Censor and A. Segal, *The split common fixed point problem for directed operators*, J. Convex Anal, **16**, 587-600, 2009.
[16] A. Moudafi, *A note on the split common fixed-point problem for quasi-nonexpansive operators*, Nonlinear Analysis, **74** (12) 4083-4087, 2011.
[17] A. Moudafi, *The split common fixed-point problem for demicontractive mappings*, Inverse Probl, **26**, 055007(2010).
[18] J. Zhao and S. He, *Alternating Mann iterative algorithms for the split common fixed- point problem for quasi-nonexpansive mappings*, Fixed point theory and Appl, **2013** (288) 2013.
[19] G. Marino, H-K. Xu,*A general iterative method for nonexpansive mappings in Hilbert spaces*, J.Math. Anal. Appl., **318**, 43-52, 2006.

DEPARTMENT OF MATHEMATICS, BINGHAM UNIVERSITY, KARU, NIGERIA; AND FIXED POINT THEORY AND APPLICATION RESEARCH GROUP (FPTA-RG), DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NIGERIA, NSUKKA,NIGERIA
E-mail addresses: epukeezinneuk@yahoo.com; ezinneuk@gmail.com

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NIGERIA, NSUKKA,NIGERIA
E-mail addresses: micah.osilike@unn.edu.ng; osilike@yahoo.com