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AN ITERATIVE ALGORITHM FOR APPROXIMATING SOLUTIONS OF VARIATIONAL INEQUALITY AND FIXED POINT PROBLEMS IN BANACH SPACES PROBLEMS IN BANACH SPACES

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ABSTRACT. In this paper, an iterative algorithm for approximating a common element of the set of fixed points of a relatively nonexpansive map and the set of solutions of variational inequality problem involving a monotone and Lipschitz continuous map is constructed. Strong convergence of the given iterative sequence in a uniformly smooth and 2-uniformly convex real Banach space is shown.

Keywords: Subgradient extragradient algorithm, monotone map, relatively nonexpansive map, Lipschitz map.

1. INTRODUCTION

Let C be a nonempty subset of a real Banach space E with dual E^*. A mapping $A : C \to E^*$ is called

- *monotone* if for each $x, y \in C$, following inequality holds:

$$\langle Ax - Ay, x - y\rangle \geq 0.$$

- β*-strongly monotone* if there exists $\beta > 0$ such that for each $x, y \in C$ the following inequality holds:

$$\langle Ax - Ay, x - y\rangle \geq \beta||x - y||^2.$$

- α*- inverse strongly monotone* if there exists $\alpha > 0$ such that for each $x, y \in C$ the following inequality holds:

$$\langle Ax - Ay, x - y\rangle \geq \alpha||Ax - Ay||^2.$$

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Also, a mapping $T : C \to E$ is called *L-Lipschitzian* if there exists $L > 0$ such that

$$||Tx - Ty|| \leq L||x - y||, \ \forall x, y \in C. \tag{1}$$

If in inequality (1) $L = 1$, then the mapping T is called *nonexpansive.* A point $x \in C$ is called a *fixed point* of T if $Tx = x$. The set of fixed points of T is denoted by $F(T)$.

Variational inequality problem (VIP) is the problem of finding a point $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0 \ \ \forall \ v \in C, \tag{2}$$

where $A : C \to E^*$ is a nonlinear operator. The set of solutions of the variational inequality problem is denoted by $VI(C, A)$. Variational inequality has been shown to be an important mathematical model in the study of many real problems connected with convex minimization problem, fixed point problem, zero of nonlinear operator and so on (see, e.g, [10, 11, 15, 24, 25]).

In the past years, there are two general approaches to study variational inequality problem, where the operator A is a monotone operator under appropriate conditions: regularized methods and projection methods. In this paper, we focus on projection methods. The simplest projection method for VIPs is the *gradient method* in which only one projection onto a feasible set is performed. However, the convergence of this method requires a slightly strong assumption that operators are strongly monotone or inverse strongly monotone. A well-known algorithm which has allowed for the dropping of the strong assumption above, is the *extragradient method* proposed by Korpelevich [14] for solving saddle point problems. The extragradient method was extended to VIPs in both Euclidean spaces and more general Hilbert spaces. The extragradient method is designed as follows; let $A : H \to H$ be monotone and L-Lipschitz continuous, we have the following algorithm

$$\begin{cases} x_0 \in C, \\ y_n = P_C(x_n - \lambda A(x_n)), \\ x_{n+1} = P_C(x_n - \lambda A(y_n)), \ n \geq 0, \end{cases} \tag{3}$$

where P_C is the *metric projection* onto C, λ is a suitable parameter in $(0, \frac{1}{L})$ and C is a nonempty, closed and convex subset of H. If the solution set $VI(C, A)$ is nonempty then the sequence $\{x_n\}$ generated by (3) converges weakly to an element in $VI(C, A)$.

From the viewpoint of implementation, the extragradient method becomes very useful when the convex set C has a simple structure such that the projection onto it can be evaluated easily. However, if C is arbitrary closed and convex set, one has to calculate in each iteration two projections onto C. This can affect the efficiency of the method. Therefore, Censor *et al.* [5] proposed the following algorithm, called the *subgradient extragradient method*, for variational inequality problem in a real Hilbert space H. Precisely, they obtained the following result.

Let $A : H \to H$ be a monotone and L-Lipschitz mapping on C and λ be a real number such that $\lambda \in (0, \frac{1}{L})$. Suppose that $VI(C, A) \neq \emptyset$. Then the sequence $\{x_n\}$ in H generated by the following algorithm

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \lambda A(x_n)), \\ x_{n+1} = P_{T_n}(x_n - \lambda A(y_n)), \ n \geq 0, \end{cases} \tag{4}$$

with $T_n := \{v \in H : \langle x_n - \lambda A(x_n) - y_n, v - y_n \rangle \leq 0\}$, converges weakly to an element in $VI(C, A)$.
The problem of finding a common solution of a VIP and a fixed point problem has also been studied widely in recent years. The inspiration for studying such a common solution problem is in its possible application to mathematical models whose constraints can be expressed as fixed-point problems and/or variational inequalities, for instance in practical problems such as signal processing, network resource allocation, image recovery, see, for instance [9, 16].

In 2006, motivated by the idea of Korpelevich's extragradient method [14], Nadezhkina and Takahashi [20], introduced the following iterative process for finding the common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problem for a monotone, Lipschitz continuous mapping. The proposed iterative process was based on the extragradient method. They obtained a weak convergence theorem for two sequences generated by this process. More precisely, given a nonempty, closed and convex set $C \subset H$, a nonexpansive mapping $S : C \to C$ and a monotone mapping $A : C \to H$ which is k-Lipschitz continuous, they introduced the following iterative algorithm in order to find an element of $F(S) \cap VI(C, A)$.

$$\begin{cases} x_0 = x \in C \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda_n A y_n), \end{cases} \tag{5}$$

for all $n \geq 0$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$, $\{\lambda_n\}$ is a sequence in $(0, \frac{1}{k})$ and P_C is the metric projection of H onto C. They showed that if $F(S) \cap VI(C, A) \neq \emptyset$, then the sequences $\{x_n\}, \{y_n\}$ converge weakly to an element $z \in F(S) \cap VI(C, A)$, where $z = \lim_{n\to\infty} P_{F(S)\cap VI(C,A)} x_n$.

Malitsky and Semenov in [28], using a hybrid method which does not require the extrapolation step in the Korpelevich's method, obtained strong convergence result for variational inequality problem involving Lipschitz monotone map in Hilbert spaces. They proved that the sequence generated from the hybrid scheme converges strongly to a solution of the variational inequality problem. Similarly, Nadezhkina and Takahashi [21] in 2006 introduced an iterative scheme by a hybrid method and proved strong convergence of the sequence generated by their algorithm to a point in $F(S) \cap VI(C, A)$ and it is as follows:

$$\begin{cases} x_0 \in C, \\ y_n = P_C(x_n - \mu_n A x_n), \\ z_n = \alpha_n x_n + (1 + \alpha_n) S P_C(x_n - \mu_n A y_n), \\ C_n = \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{6}$$

for all $n \geq 0$, where $0 \leq \alpha_n \leq c < 1$ and $\{\mu_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{k})$. Then the sequence $\{x_n\}$ converges strongly to an element $z \in F(S) \cap VI(C, A)$.

From the foregoing discussion, it is obvious that in Hilbert spaces much have been done in solving problem (2). However, as has been rightly observed by Hazewinkle in the preface of Iona's book [8], "... many and probably most mathematical objects and models do not naturally live in Hilbert space". This fact, perhaps, makes many mathematicians to search and research for methods and techniques for obtaining results that known to hold in Euclidean and more general Hilbert, in more general Banach spaces. In this regard, Cai et al. [29] proposed a method they called *Halpern Subgradient Extragradient Method* for variational inequality problems in 2-uniformly convex and uniformly smooth Banach spaces. They proved that the sequence obtained from their method converges strongly to a solution of variational inequality problems with Lipschitz and monotone maps. In addressing the problem of estimating a Lipschitz constant, a problem associated with implementing these projection methods, they presented a line-search approach using which they were able to obtain another strong convergence theorem for variational inequality problems involving uniformly continuous maps in reflexive Banach spaces. In similar developments, Iusem and Nasri [30] proved convergence theorem for a variant of Korpelevich's method in Banach spaces which include those that are both uniformly smooth and uniformly convex. Their algorithm involves Bregman distance and the convergence is weak to a solution of variational inequality problems for uniformly continuous maps. Closely related to this result of Iusem and Nasri, is the work of Zheng [31] in which strong convergence theorem is given in Banach spaces which include uniformly smooth and uniformly convex Banach spaces. Their algorithms also use Bregman distances.

In [27] Shehu proposed a scheme and proved that the corresponding sequence converges strongly to a common fixed point of a countable family of relatively nonexpansive mappings which also solves a variational inequality problem, in fact which also solves a more general problem, an equilibrium problem. His result holds in Banach spaces which are both uniformly convex and uniformly smooth.

Inspired by above works, it is our purpose in this paper to propose an algorithm using the subgradient extragradient method and prove that the sequence generated by the algorithm constructed converges strongly to a common element of the set of fixed points of a relatively nonexpansive mapping and the set of solutions of variational inequality problem involving a monotone and Lipschitz continuous map in a *uniformly smooth* and *2-uniformly convex* real Banach space. As a consequence of this result, we obtain a strong convergence theorem for approximating a common fixed point for a countable family of relatively nonexpansive maps and a variational inequality problem for a countable family of monotone and Lipschitz-continuous maps. Furthermore, we provide a numerical example which, even though it is in $\mathbb{R}$, illustrates the conclusion of our results. Our results improve and generalize many known results in the literature. For example, our results hold in a more general Banach space than Hilbert, which gives an improvement over, say, the results of Nadezhkina and Takahashi [21] which hold in Hilbert spaces. Furthermore, while our algorithm still requires two projections before getting the next iterate, one of the projections is onto a half-space which is much more tractable than just arbitrary closed convex set. In addition, the update in the algorithm does not require projecting onto an intersection of closed convex sets, which is generally more difficult to compute than projecting onto a single closed convex set.

2. PRELIMINARY

In the sequel, E denotes a real normed space with dual space E^* and J denotes the normalized duality on it. For a sequence $\{x_n\}$ in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$. We shall need the following definitions and results.

The *normalized duality mapping* on E, $J : E \to 2^{E^*}$, is defined for each $x \in E$ by

$$Jx := \{x^* \in E^* : \langle x, x^* \rangle = ||x||||x^*||, ||x|| = ||x^*||\},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between elements of E and E^*. Let $U = \{x \in E : \|x\| = 1\}$. A Banach space E is said to be *strictly convex* if for any $x, y \in U$ and $x \neq y$ implies $\left\|\frac{x+y}{2}\right\| < 1$. It is also said to be *uniformly convex* if for each $\epsilon \in (0, 2]$, there exist $\delta > 0$ such that for any $x, y \in U$, $\|x-y\| \geq \epsilon$ implies $\left\|\frac{x+y}{2}\right\| \leq 1-\delta$. It is known that a uniformly convex Banach space is reflexive and strictly convex. A Banach space E is said to be *smooth* if the limit $\lim\limits_{t\to 0} \dfrac{\|x+ty\| - \|x\|}{t}$ exists for all $x, y \in U$. It is also *uniformly smooth* if the limit is attained uniformly for $x, y \in U$. It is well-known that the Hilbert and the Lebesgue L^q $(1 < q \leq 2)$ spaces are 2-uniformly convex and uniformly smooth.

The following properties of the duality mapping J can be found in [2]:

(i) If E is smooth, then J is single-valued.
(ii) If E is strictly convex, then J is one-to-one.

(iii) If E is uniformly smooth, the J is uniformly norm-to-norm continuous on each bounded subset of E.

(iv) If E is a smooth, strictly convex and reflexive Banach space, them J is single-valued, one-to-one and onto.

Let E be a smooth real Banach space. We define the following *Lyapunov functional* due to Alber [1] by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \ \forall x, y \in E. \tag{7}$$

If $E = H$, a real Hilbert space, equation (7) reduces to $\phi(x,y) = \|x-y\|^2 \quad \forall\, x, y \in H$. It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \le \phi(x,y) \le (\|x\| + \|y\|)^2 \quad \forall\, x, y \in E. \tag{8}$$

In addition, the function ϕ has the following property:

$$\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, Jz - Jy\rangle \ \forall\ x, y, z \in E. \tag{9}$$

Define a map $V : E \times E^* \to \mathbb{R}$ by

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2. \tag{10}$$

Then, it is easy to see that

$$V(x, x^*) = \phi(x, J^{-1}(x^*)) \ \forall x \in E,\ x^* \in E^*. \tag{11}$$

Lemma 1 (Alber, [1])**.** *Let E be a reflexive strictly convex and smooth Banach space with E^* as its dual. Then,*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \le V(x, x^* + y^*) \tag{12}$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma 2 (Kamimura and Takahashi, [10])**.** *Let E be a real smooth and uniformly convex Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \to 0$ as $n \to \infty$, then $\|x_n - y_n\| \to 0$ as $n \to \infty$.*

Lemma 3 (Matsushita and Takahashi, [19])**.** *Let E be a strictly convex and smooth Banach space, then $\phi(x,y) = 0$ if and only if $x = y$.*

Let E be a reflexive, strictly convex and smooth real Banach space and C be a nonempty, closed and convex subset of E. For each $x \in E$, there exists a unique element $x_0 \in C$ (denoted by $\Pi_C(x)$) such that

$$\phi(x_0, x) = \min_{y \in C} \phi(y, x).$$

The map $\Pi_C : E \to C$, defined by $\Pi_C(x) = x_0$, is called the generalized projection map from E onto C. If $E = H$, a real Hilbert space, then Π_C coincides with the metric projection P_C (see e.g., Alber [1], for more details).

Lemma 4 (Alber [1], Kamimura and Takahashi [10]). *Let C be a nonempty closed and convex subset of a smooth real Banach space E and $x \in E$. Then $x_0 = \Pi_C x$ if and only if $\langle x_0 - y, Jx - Jx_0\rangle \geq 0, \quad \forall y \in C$.*

Lemma 5 (Alber [1], Kamimura and Takahashi [10]). *Let E be a reflexive, strictly convex and smooth real Banach space and C be a nonempty closed and convex subset of E. Then for any $x \in E$,*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall\ y \in C. \tag{13}$$

Lemma 6 (Nakajo, [12]). *Let E be a 2-uniformly convex and smooth real Banach space. Then, for every $x, y \in E$, $\phi(x, y) \geq c_1||x - y||^2$, where $c_1 > 0$ is the 2-uniform convexity constant of E.*

Let $T : H \to H$ be a mapping and $Fix(T)$ denote the fixed point set of T. A point p in C is said to be an *asymptotic fixed point* of T (see [17, 18, 19, 22]) if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$. A mapping T from C into itself is called *relatively nonexpansive* if

(C1) $F(T) \neq \emptyset$;
(C2) $\phi(p, Tx) \leq \phi(p, x)$ for $x \in C$ and
(C3) $F(T) = \hat{F}(T)$.

If E is a real Hilbert space, then the class of relatively nonexpansive mappings contains the class of nonexpansive mappings with $F(T) \neq \emptyset$ (see, e.g., [26]).

Lemma 7 (Matsushita and Takahashi, [19]). *Let E be a strictly convex and smooth real Banach space and C be a closed convex subset of E. Let T be a relatively nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.*

We denote by $N_C(v)$ the normal cone for C at a point $v \in C$, that is,

$$N_C(v) = \{x^* \in E^* : \langle v - y, x^*\rangle \geq 0\ \forall\ y \in C\}.$$

The following result is well-known:

Lemma 8 (Rockafellar,[23]). *Let C be a nonempty closed convex subset of a real Banach space E and A be a monotone and hemi-continuous map from C into E^* with $C = D(A)$. Let T be a map defined by:*

$$Tv = \begin{cases} Av + N_C(v), & v \in C, \\ \emptyset, & v \notin C. \end{cases} \tag{14}$$

Then, T is maximal monotone and $T^{-1}(0) = VI(C, A)$.

Lemma 9 (Kohsaka and Takahashi [13]). *Let C be a closed convex subset of a uniformly smooth and 2-uniformly convex Banach space E and $(S_i)_{i=1}^{\infty}$ be a family of relatively nonexpansive maps such that $\bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset$. Let $(\eta_i)_{i=1}^{\infty} \subset (0,1)$ and $(\mu_i)_{i=1}^{\infty} \subset (0,1)$ be sequences such that $\sum_{i=1}^{\infty} \eta_i = 1$. Consider the map $T : C \to E$ defined by*

$$Tx = J^{-1}\left(\sum_{i=1}^{\infty} \eta_i(\mu_i Jx + (1-\mu_i)JS_i x)\right) \quad \text{for each } x \in C. \tag{15}$$

Then T is relatively nonexpansive and $F(T) = \bigcap_{i=1}^{\infty} F(S_i)$.

3. MAIN RESULT

In this section, we introduce and study subgradient extragradient-like algorithm and prove strong convergence theorem of the sequence generated by the algorithm. In what follows, except stated otherwise, let E be a 2- uniformly convex and uniformly smooth real Banach space with dual space E^* with c_1 as the 2-uniformly convexity constant. Also C is a nonempty closed convex subset of E, $U : E \to E$ is relatively nonexpansive and $A : C \to E^*$ is monotone and Lipschitz on C with Lipschitz constant $L > 0$ and we denote $\Omega := F(U) \cap VI(C, A) \neq \emptyset$. We have the following algorithm.

$$\begin{cases} x_0 \in C_0 = C, \\ y_n = \Pi_C J^{-1}(Jx_n - \mu A x_n), \\ z_n = \Pi_{T_n} J^{-1}(Jx_n - \mu A y_n), \\ w_n = J^{-1}\big((1-\alpha)Jx_n + \alpha JUz_n\big) \\ C_{n+1} = \big\{z \in C_n : \phi(z, w_n) \le \phi(z, x_n) - \alpha c\,(\phi(z_n, y_n) + \phi(y_n, x_n))\,\big\} \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{16}$$

for all $n \geq 0$, where $T_n = \{x \in E : \langle Jx_n - \mu A x_n - Jy_n, x - y_n \rangle \le 0\}$, $\alpha \in (0,1)$, μ and c are positive constants.

Remark 1. *The sequence $\{x_n\}$ generated by the algorithm is well-defined if T_n is nonempty for all n (T_n is a halfspace) and C_n is closed, convex and nonempty for all $n \geq 0$. Concerning T_n, we show $C \subseteq T_n$. By definition, $y_n = \Pi_C J^{-1}(Jx_n - \mu Ax_n)$. This implies that $\langle Jy_n - Jx_n + \mu Ax_n, x - y_n\rangle \geq 0$ for all $x \in C$ by Lemma 4. Thus $\langle Jy_n - Jx_n + \mu Ax_n, y - y_n\rangle \geq 0$ and so $y \in T_n$. For the sets C_n:*

• *C_n is convex for all $n \geq 0$; We proceed by induction. Clearly, for $n = 0$, $C_n = C$ is convex. Suppose C_n is convex for some $n \geq 0$. From the definition of C_{n+1} and using the definition of ϕ, it is seen that $C_{n+1} := \big\{z \in C_n : 2\langle z, Jx_n - Jw_n\rangle \leq -\alpha c\left(\phi(z_n, y_n) + \phi(y_n, x_n)\right) + \|x_n\|^2 - \|w_n\|^2\big\}$. Let $x, y \in C_{n+1}$ and $\lambda \in [0,1]$.*

$$\begin{aligned}
2\langle \lambda x + (1-\lambda)y, Jx_n - Jw_n\rangle &= 2\langle \lambda x, Jx_n - Jw_n\rangle + 2\langle (1-\lambda)y, Jx_n - Jw_n\rangle \\
&= 2\lambda\langle x, Jx_n - Jw_n\rangle + 2(1-\lambda)\langle y, Jx_n - Jw_n\rangle \\
&\leq \lambda\left[-\alpha c\left(\phi(z_n, y_n) + \phi(y_n, x_n)\right) + \|x_n\|^2 - \|w_n\|^2\right] \\
&\quad + (1-\lambda)\left[-\alpha c\left(\phi(z_n, y_n) + \phi(y_n, x_n)\right) + \|x_n\|^2 - \|w_n\|^2\right] \\
&= -\alpha c\left(\phi(z_n, y_n) + \phi(y_n, x_n)\right) + \|x_n\|^2 - \|w_n\|^2.
\end{aligned}$$

Hence, $\lambda x + (1-\lambda)y \in C_{n+1}$. It follows that C_n is convex for all $n \geq 0$.

• *C_n is closed for all $n \geq 0$; We proceed by induction again. For $n = 0$, $C_n = C$ is closed. Suppose C_n is closed for some $n \geq 0$. Let $(v_m)_m$ be a sequence in C_{n+1} such that $v_m \to \bar{v} \in E$ as $m \to \infty$. Now $v_m \in C_{n+1}$ implies $\phi(v_m, w_n) \leq \phi(v_m, x_n) - \alpha c\left(\phi(z_n, y_n) + \phi(y_n, x_n)\right)$. By continuity of But ϕ with respect to the first component we have $\phi(\bar{v}, w_n) \leq \phi(\bar{v}, x_n) - \alpha c\left(\phi(z_n, y_n) + \phi(y_n, x_n)\right)$ by letting $m \to \infty$. Hence $\bar{v} \in C_{n+1}$. Therefore, C_n is closed $\forall\ n \geq 0$.*

• *$C_n \neq \emptyset$ for each $n \geq 0$; We show that $\Omega \subset C_n$ for each $n \geq 0$. For $n = 0$, we have that $\Omega \subset C_0$. Suppose $\Omega \subset C_n$ for some $n \geq 0$. We show that $\Omega \subset C_{n+1}$. Let $p \in \Omega$. We first show that*

$$\phi(p, z_n) \leq \phi(p, x_n) - \left(1 - \frac{\mu L}{c_1}\right)\big[\phi(z_n, y_n) + \phi(y_n, x_n)\big]. \tag{17}$$

Indeed, since $z_n = \Pi_{T_n} J^{-1}(Jx_n - \mu Ay_n)$, using Lemma 5 and the definition of ϕ, we have

$$\begin{aligned}
\phi(p, z_n) &= \phi(p, \Pi_{T_n} J^{-1}(Jx_n - \mu Ay_n)) \\
&\leq \phi(p, J^{-1}(Jx_n - \mu Ay_n)) - \phi(z_n, J^{-1}(Jx_n - \mu Ay_n)) \\
&= \phi(p, x_n) - \phi(z_n, x_n) + 2\mu\langle p - z_n, Ay_n\rangle \\
&\leq \phi(p, x_n) - \phi(z_n, x_n) + 2\mu\langle y_n - z_n, Ay_n\rangle. \qquad (18)
\end{aligned}$$

Thus, from definition of ϕ, *(9), Lemma 4, Lipschitz continuity of* A *and Lemma 6, we have*

$$\begin{aligned}
\phi(z_n,x_n)-2\mu\langle y_n-z_n,Ay_n\rangle &= \phi(z_n,y_n)+\phi(y_n,x_n)+2\langle y_n-z_n,Jx_n-Jy_n\rangle\\
&- 2\mu\langle y_n-z_n,Ay_n\rangle\\
&= \phi(z_n,y_n)+\phi(y_n,x_n)-2\langle z_n-y_n,Jx_n-\mu Ay_n-Jy_n\rangle\\
&\geq \phi(z_n,y_n)+\phi(y_n,x_n)-2\mu\langle z_n-y_n,Ax_n-Ay_n\rangle\\
&\geq \phi(z_n,y_n)+\phi(y_n,x_n)-2\mu||z_n-y_n||||Ax_n-Ay_n||\\
&\geq \phi(z_n,y_n)+\phi(y_n,x_n)-2L\mu||z_n-y_n||||x_n-y_n||\\
&\geq \phi(z_n,y_n)+\phi(y_n,x_n)-L\mu(||z_n-y_n||^2+||x_n-y_n||^2)\\
&\geq \phi(z_n,y_n)+\phi(y_n,x_n)-\frac{\mu L}{c_1}(\phi(z_n,y_n)+\phi(y_n,x_n))\\
&= c(\phi(z_n,y_n)+\phi(y_n,x_n)), \qquad (19)
\end{aligned}$$

where $c=1-\frac{\mu L}{c_1}$. *From inequalities (18) and (19), we have*

$$\phi(p,z_n)\leq\phi(p,x_n)-c(\phi(z_n,y_n)+\phi(y_n,x_n)). \qquad (20)$$

Now to show that $p\in\Omega$, *by the fact that* U *is relatively nonexpansive and inequality* (17), *we have that*

$$\begin{aligned}
\phi(p,w_n) &= \phi(p,J^{-1}((1-\alpha)Jx_n+\alpha JUz_n))\\
&= V(p,(1-\alpha)Jx_n+\alpha JUz_n)\\
&\leq (1-\alpha)\phi(p,x_n)+\alpha\phi(p,Uz_n)\\
&\leq (1-\alpha)\phi(p,x_n)+\alpha\Big[\phi(p,x_n)-c\big(\phi(z_n,y_n)+\phi(y_n,x_n)\big)\Big]\\
&= \phi(p,x_n)-\alpha c(\phi(z_n,y_n)+\phi(y_n,x_n)).
\end{aligned}$$

This implies that $p\in C_{n+1}$. *Hence* $\Omega\subset C_n$, $\forall n\geq 0$.

We now prove the main theorem.

Theorem 1. *Let* E *be a 2-uniformly convex, uniformly smooth real Banach space and* C *be a nonempty closed convex subset of* E. *Let* $U:E\to E$ *be a relatively nonexpansive mapping and* $A:C\to E^*$ *be a monotone and L-Lipschitz mapping on* C. *Let* μ *be a real number satisfying* $\mu<\frac{c_1}{L}$. *Suppose that* Ω *is nonempty. Then the sequence* $\{x_n\}$ *generated by Algorithm 16 converges strongly to* $\Pi_\Omega x_0$.

Proof. We divide our proof into the following steps.
Step 1: We show that $\lim_{n\to\infty}\phi(x_n,x_0)$ exists.
Since $x_n=\Pi_{C_n}x_0$ and $\Omega\subset C_n$ $\forall$ $n\geq 0$, then using Lemma 5, we

have that for any $p \in \Omega$

$$\begin{aligned} \phi(x_n, x_0) &\leq \phi(p, x_0) - \phi(p, x_n) \\ &\leq \phi(p, x_0). \end{aligned} \tag{21}$$

It follows that the sequence $\{\phi(x_n, x_0)\}$ is bounded and so by inequality (8) we get that $\{x_n\}$ is bounded. Since $C_{n+1} \subset C_n \ \forall\, n \geq 0$ and $x_n = \Pi_{C_n} x_0$, we obtain that for $x_{n+1} \in C_n$

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0). \tag{22}$$

Therefore, $\{\phi(x_n, x_0)\}$ is monotone nondecreasing and bounded above by $\phi(p, x_0)$. Hence, $\lim_{n\to\infty} \phi(x_n, x_0)$ exists.

Step 2: We show $\{x_n\}$ converges to $\Pi_\Omega x_0$.
Using the fact that $x_n = \Pi_{C_n} x_0$ and $x_{n+1} \in C_n$, we have from Lemma 5 that for $m > n$,

$$\phi(x_m, x_n) \leq \phi(x_m, x_0) - \phi(x_n, x_0). \tag{23}$$

This and **Step 1** imply that

$$\phi(x_m, x_n) \to 0 \text{ as } m, n \to \infty \tag{24}$$

By Lemma 2, we have

$$\|x_n - x_m\| \to 0 \text{ as } n, m \to \infty. \tag{25}$$

Hence $\{x_n\}$ is Cauchy which implies that there exists $x^* \in E$ such that $x_n \to x^*$ as $n \to \infty$. Since $\{x_n\}_{n\geq 1}$ is in C and C is closed, then $x^* \in C$. Again, since $x_{n+1} \in C_{n+1}$, we have

$$\begin{aligned} \phi(x_{n+1}, w_n) &\leq \phi(x_{n+1}, x_n) - \alpha c\big(\phi(z_n, y_n) + \phi(y_n, x_n)\big) \quad (26) \\ &\leq \phi(x_{n+1}, x_n) \to 0 \ \text{ as } n \to \infty \text{ by (24)}. \end{aligned}$$

Therefore using Lemma 2, we have $\|x_{n+1} - w_n\| \to 0$ as $n \to \infty$. We obtain from (25) that $\|x_n - x_{n+1}\| \to 0$ as $n \to \infty$. So,

$$\|x_n - w_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - w_n\| \to 0.$$

We have $\|x_n - w_n\| \to 0$ as $n \to \infty$. Similarly, using the fact that $x_{n+1} \in C_{n+1}$ and using inequality (26), we have that $\phi(z_n, y_n) \to 0$ and $\phi(y_n, x_n) \to 0$ as $n \to \infty$ which imply by Lemma 2 that $\|z_n - y_n\| \to 0$ and $\|y_n - x_n\| \to 0$ as $n \to \infty$. Also,

$$\|x_n - z_n\| \leq \|x_n - y_n\| + \|y_n - z_n\| \to 0.$$

So, $\|x_n - z_n\| \to 0$ as $n \to \infty$. Now, using the fact that J is norm to norm uniformly continuous on bounded sets and since the sequences $\{x_n\}$ and $\{w_n\}$ are bounded, $\alpha \in (0,1)$ and the fact that $\|x_n - w_n\| \to 0$, we have $\|Jx_n - Jw_n\| \to 0$, which implies that

$$\|Jx_n - JUz_n\| = \frac{1}{|\alpha|}\|Jw_n - Jx_n\| \to 0, \quad \text{as } n \to \infty.$$

By the norm to norm uniform continuity of J^{-1} on bounded sets and the fact that $\{z_n\}$ is bounded, we have

$$\|x_n - Uz_n\| \to 0, \text{ as } n \to \infty.$$

Thus,

$$\|z_n - Uz_n\| \leq \|z_n - x_n\| + \|x_n - Uz_n\| \to 0, \quad \text{as } n \to \infty.$$

Therefore,

$$\|z_n - Uz_n\| \to 0, \text{ as } n \to \infty. \tag{27}$$

Now, we prove that $x^* \in \Omega$. Since $x_n \to x^*$ and $\|x_n - z_n\| \to 0$, we have that $z_n \to x^*$. Using the fact that U is relatively nonexpansive and $\|z_n - Uz_n\| \to 0$, it follows that $x^* \in F(U)$. Next, we show that $x^* \in VI(C, A)$. By Lemma 8 we have that the map $T : E \to 2^{E^*}$ defined by

$$Tv = \begin{cases} Av + N_C(v), & \text{if } v \in C; \\ \emptyset, & \text{if } v \notin C, \end{cases}$$

is maximal monotone, where $N_C(v)$ is the normal cone of C at v. For all $(v, u^*) \in G(T)$, we have the $u^* - A(v) \in N_C(v)$. By definition of $N_C(v)$, we find that

$$\langle v - y, u^* - Av\rangle \geq 0 \qquad \forall\, y \in C.$$

Since $y_n \in C$, we have

$$\langle v - y_n, u^*\rangle \geq \langle v - y_n, Av\rangle. \tag{28}$$

By the definition of y_n and Lemma 4 , we get

$$\langle v - y_n, Ax_n\rangle \geq \left\langle v - y_n, \frac{Jx_n - Jy_n}{\mu}\right\rangle. \tag{29}$$

Therefore, it follows from inequalities (28), (29) and monotonicity of A that

$$\begin{aligned} \langle v - y_n, u^*\rangle &\geq \langle v - y_n, Av\rangle \\ &= \langle v - y_n, Av - Ay_n\rangle + \langle v - y_n, Ay_n - Ax_n\rangle + \langle v - y_n, Ax_n\rangle \\ &\geq \langle v - y_n, Ay_n - Ax_n\rangle + \left\langle v - y_n, \frac{Jx_n - Jy_n}{\mu}\right\rangle. \end{aligned} \tag{30}$$

Since $||x_n - y_n|| \to 0$ as $n \to \infty$ and A is Lipschitz continuous, we have

$$\lim_{n\to\infty} ||Ay_n - Ax_n|| = 0. \tag{31}$$

Taking limit in inequality (30) and using (31) with $y_n \to x^*$, we have $\langle v - x^*, u^* \rangle \geq 0 \ \ \forall\ (v, u^*) \in G(T)$. And since T is maximal monotone, we have $x^* \in T^{-1}0 = VI(C, A)$. Hence, $x^* \in VI(C, A)$. Therefore, $x^* \in F(U) \cap VI(C, A)$.

Finally, we show the $x^* = \Pi_\Omega x_0$. Let $w = \Pi_\Omega x_0$, using the fact that $x^* \in \Omega$, we have

$$\phi(w, x_0) \leq \phi(x^*, x_0). \tag{32}$$

Since $x_n = \Pi_{C_n} x_0$ and $w \in \Omega \subseteq C_n$, we have

$$\phi(x_n, x_0) \leq \phi(w, x_0).$$

But we have that $x_n \to x^*$ as $n \to \infty$. This implies that by continuity of $\phi(\cdot, x_0)$, we have

$$\phi(x^*, x_0) \leq \phi(w, x_0). \tag{33}$$

Hence, with inequalities (32) and (33), we have

$$\phi(x^*, x_0) = \phi(w, x_0). \tag{34}$$

We observe that, from Lemma 5 and Lemma 3 and equation (34), we have

$$0 \leq \quad \phi(x^*, w) \quad \leq \phi(x*, x_0) - \phi(w, x_0) = 0.$$

Thus, $\phi(x^*, w) = 0$ and so $x^* = w = \Pi_\Omega x_0$. □

4. APPLICATION

In this section, we use Theorem 1 to obtain a strong convergence theorem for approximating a common fixed point for a countable family of relatively nonexpansive maps and variational inequality problem for a countable family of monotone and Lipschitz-continuous maps.

Theorem 2. *Let E be a uniformly smooth and 2-uniformly convex real Banach space. let C be a nonempty, closed and convex subset of E such that $J(C)$ is convex. Let $A_i : E \to E^*, i = 1, 2, \ldots, N$ be a countable family of monotone and L_i-Lipschitz continuous maps and $U_i : E \to E, i = 1, 2, 3, \ldots$ be a countable family of relatively nonexpansive maps such that $\bigcap_{i=1}^{\infty} F(U_i) \neq \emptyset$. Suppose $(\eta_j)_{j=1}^{\infty} \subset (0, 1)$ and $(\xi_k)_{k=1}^{\infty} \subset (0, 1)$ be sequences such that $\sum_{j=1}^{\infty} \eta_j = 1$ and*

$U : E \to E$ defined by

$$Ux = J^{-1}\left(\sum_{i=1}^{\infty} \eta_i\big(\xi_i Jx + (1-\xi_i)JU_i x\big)\right) \quad \text{for each } x \in C. \tag{35}$$

We have the following algorithm

$$\begin{cases} x_0 \in C, \\ y_n = \Pi_C J^{-1}(Jx_n - \mu A_i x_n), \\ T_n = \{x \in E : \langle Jx_n - \mu A_i x_n - Jy_n, x - y_n\rangle \leq 0\}, \\ z_n = \Pi_{T_n} J^{-1}(Jx_n - \mu A_i y_n), \\ w_n = J^{-1}\big((1-\alpha)Jx_n + \alpha JUz_n\big) \\ C_{n+1} = \big\{z \in C_n : \phi(z, w_n) \leq \phi(z, x_n)\big\} \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{36}$$

for all $n \geq 0$, where $\alpha \in (0,1)$ and μ be a real number satisfying $\mu < \frac{c_1}{L}$. Then the sequences generated by the algorithm converge strongly to $\Pi_\Omega x_0$.

Proof. From Lemma 9, U is relatively nonexpansive and $F(U) = \bigcap_{i=1}^{\infty} F(U_i)$ and $VI(C,\ A) = \bigcap_{i=1}^{\infty} VI(C,\ A_i)$. The conclusion follows from Theorem 1. □

4. NUMERICAL EXPERIMENT

In this section, we provide a numerical example using our newly proposed method considered in Algorithm 16 of this paper. All codes were written in Python 2.7.12 and run on *hp* Intel Core i5-6500 CPU @ 3.20GHz × 4, 7.7 GiB Ram Desktop.

Example 1. *Let $E = \mathbb{R}$ and $C = [-4, \infty] \subset \mathbb{R}$. For all $x \in \mathbb{R}$, define A, U, P_C by $Ax = 3x$; $Ux = \sin x$ and $P_C x = \begin{cases} x, & \text{if } x \in C, \\ \frac{\beta x}{|x|}, & \text{if } x \notin C \end{cases}$ Let $x_0 = 5$, $\mu = 0.1$, $\alpha = 0.5$, $L = 3$ and $c_1 = 0.5$.*

It is easy to see that A is monotone and Lipschitz with Lipschitz constant 3 and U is nonexpansive. Then the sequences generated by Algorithm 16 converges to zero. The representation is shown below;

Table 1. Numerical results as regards Example 1

n	x_n
0	2.00000000e+00
5	6.32812500e-01
10	1.50169373e-01
15	3.56358960e-02
20	8.45656517e-03
25	2.00678256e-03
30	4.76218907e-04
35	1.13008979e-04
40	2.68175604e-05
45	6.36393280e-06
50	1.51019108e-06

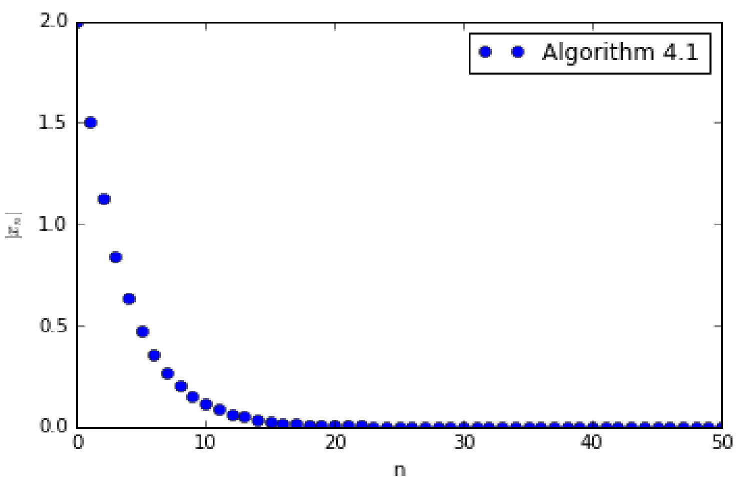

Fig. 1. Figure as regard Table 1

Remark 2. *Theorem 1 improves the results of Nadezhkina and Takahashi* [21] *in the following ways:*

(i) Our result is proved in more general real Banach space than real Hilbert space in a uniformly smooth and 2-uniformly convex real Banach space while the results of Nadezhkina and Takahashi [21] *are proved in Hilbert space.*

(ii) In our algorithm of Theorem 1, it involves parameter that is fixed μ. *This reduces computational cost, while in the algorithm of Nadezhkina and Takahashi* [21], *it involves parameters* μ_n *that are computed for each iteration.*

4. CONCLUDING REMARKS

This present article presents a strong convergence result for variational inequality problems involving monotone and Lipschitz continuous operator and fixed point problems involving a relatively nonexpansive operator in a real Banach space. Variational inequality theory has theoretical and applied importance. Numerous problems in physics, optimization and economics can be reduced to finding a solution of variation inequality. On the other hand, the fixed point problem has also many applications. In this paper, by using modified subgradient extragradient method, we investigate an iterative algorithm for finding a common element of the set of solutions of variational inequalities for monotone and Lipschitz continuous mappings and the set of fixed points of a relatively nonexpansive mapping in a 2-uniformly convex and uniformly smooth real Banach space.

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