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# APPROXIMATION OF SOLUTION OF GENERALIZED EQUILIBRIUM PROBLEMS AND COMMON FIXED POINT OF A FINITE FAMILY OF STRICTLY PSEUDOCONTRACTIVE MAPPINGS

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ABSTRACT. This study focuses on the problem of approximating solution of generalized equilibrium problems and common fixed point of finite family of strict pseudocontractive mappings. The result obtained is applied in approximation of solution of generalized mixed equilibrium problems and common fixed point of finite family of strict pseudocontractive mappings. Our theorems improve, complement and unify some existing results that were recently announced by several authors. Corollary obtained and our method of proof are of independent interest.

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## 1. INTRODUCTION

The research embodied in this paper focuses on finding approximate solution of generalized equilibrium problems which is also a common fixed point of finite family of strictly pseudocontractive mappings. Equilibrium problem arises in many areas of applications such as physics, economics, finance, transportation, network and structural analysis, elasticity and optimization (see [23]). The Equilibrium problem was first introduced by Blum and Ottelli [2]

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for solving some important problems such as image recovery, inverse problems, transportation problems, fixed point problems and optimization problems which can be formulated as variational inequality problem. They proved that it includes classical variational inequality and fixed point problems as special cases. Several authors have constructed some iterative schemes for finding a common element of the set of the solutions of equilibrium problems and the set of fixed points of some classes of mappings (see also [7, 16, 17, 23] and the references therein). We now turn to review of happenings in the recent research trend.

Ceng *et al.* [3] proposed an iterative scheme for approximation of solution of eqilibrium problem and fixed point of k-strictly pseudo-contractive mapping $T$ as follows:

$$\begin{cases} f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall\, y \in K, \\ x_{n+1} = \alpha_n u_n + (1-\alpha_n)Tu_n, \ n \geq 1, \end{cases} \tag{1}$$

where $\{\alpha_n\}_{n\geq 1} \subset (0,1)$ and $\{r_n\}_{n\geq 1} \subset (0,+\infty)$ are two sequences of real numbers, satisfying certain conditions. They proved that $\{x_n\}_{n\geq 1}$ and $\{u_n\}_{n\geq 1}$ converge weakly to some $p \in \Omega := F(T) \cap EP(f)$; and further argued that the convergence of the sequences $\{x_n\}_{n\geq 1}$ and $\{u_n\}_{n\geq 1}$ in (1) is strong convergence if and only if

$$\liminf_{n\geq 1} d(x_n, \Omega) = 0,$$

where $d(x_n, \Omega)$ denotes the metric distance from the point $x_n$ to $\Omega$. *It is necessary to observe that only weak convergence is obtained using algorithm (1), and the strong convergence obtained hinged on the assumption that* $\liminf_{n\geq 1} d(x_n, \Omega) = 0$, *which is of course a very strong condition.*

In order to get strong convergence Jaiboon and Kumam [10] constructed CQ algorithm that enable them do so. Precisely, they constructed the following iterative scheme

$$\begin{cases} x_0 \in H, \ C_1 = C, f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \\ \forall\, y \in K, \\ y_{n+1} = \alpha_n u_n + (1-\alpha_n)Tu_n, \ n \geq 1, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \end{cases} \tag{2}$$

where $\{\alpha_n\}_{n\geq 1} \subset (0,1)$ and $\{r_n\}_{n\geq 1} \subset (0,+\infty)$ are sequences of real numbers, satisfying certain conditions: They proved that the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $p = P_{F(T)\cap EP(f)}x_0$. *It is important to note that the scheme* (2) *involves the projection operator, which is not readily available in application. Moreover, the iterative parameter* $\alpha_n, n \in \mathbb{N}$ *in scheme* (2) *does not accommodate the canonical choice* $\alpha_n = \frac{1}{n}, n \in \mathbb{N}$.

In 2012, He [7] introduced an iterative scheme that accommodated the cannonical choice of iterative parameter, which also does not involve projection operator. In fact, He [7] proved the following theorem:

**Theorem 0.1.** *(He [7]) Let $K$ be a nonempty, closed and convex subset of a real Hilbert space $H$ and $f : K \times K \to \mathbb{R}$ be a bifunction satisfying $A_1 - A_4$. Let $T_i : K \to K$ be finite family of $k_i$-strictly pseudocontractive mappings $i = 1, 2, \cdots, N$ such that $\Omega = \bigcap_{i=1}^{N} F(T_i) \bigcap GEP(f) \neq \emptyset$. $0 \leq k_i < 1$. Suppose that $v$ and $x_1$ are arbitrary elements in $K$, for some nonnegative real numbers $\lambda_i$, $i = 1, 2, \cdots, N$, $\sum_{i=1}^{N} \lambda_i = 1$. Let $\{x_n\}$ and $\{u_n\}$ be the sequences generated by*

$$\begin{cases} f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall\, y \in K, \\ x_{n+1} = \alpha_n v + (1-\alpha_n) y_n, \\ y_n = (1-\beta_n)x_n + \beta_n z_n, \\ z_n = (1-\sigma)u_n + \sigma \sum_{i=1}^{N} \lambda_i T_i u_n, \end{cases} \tag{3}$$

*where $\sigma \in (0, 1-k)$, $k := \sup\{k_i : 1 \leq i \leq N\}$, $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ satisfy the following conditions:*

*(i)* $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$ $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

*(ii)* $\{\beta_n\}_{n=1}^{\infty} \subset (0,1)$, $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

*(iii)* $\{r_n\}_{n=1}^{\infty} \subset (0,+\infty)$, $\liminf_{n\to\infty} r_n > 0$, $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

*Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $x^* = P_\Omega(u)$.*

We note that the scheme (3) approximates a solution of *classical equilibrium problem* and a common fixed point of finite family of strict pseudocontractions.

In 2014, Huang and Ma [9] constructed a modified Mann iterative algorithm that approximates the solution of the generalized equilibrium problem and the common fixed point of finite family of strict pseudocontractive mappings. They proved the following theorem:

**Theorem 0.2.** *(Huang and Ma [9]) Let $K$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T : K \to H$ be a $\lambda$-inverse-strongly monotone mapping and Let $f$ be a bifunction from $K \times K$ to $\mathbb{R}$ which satisfies $A_1$ -$A_4$. Let $S : K \to K$ be a $k$-strict pseudocontractive mapping. Assume that $\Omega := EP(f,T)\bigcap F(S)$ is nonempty. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$, and $\{\delta_n\}_{n=1}^{\infty}$ be sequences in $(0,1)$. Let $\{r_n\}_{n=1}^{\infty}$ be a sequence in $(0,2\lambda)$, and let $\{e_n\}_{n=1}^{\infty}$, be a bounded sequence in K. Let $\{x_n\}_{n=1}^{\infty}$, be a sequence generated by*

$$\begin{cases} x_1 \in K, \\ f(u_n,u) + \langle Tx_n, u-u_n\rangle + \frac{1}{r_n}\langle u-u_n, u_n-x_n\rangle \geq 0 \quad \forall\, u \in K, \\ x_{n+1} = \alpha_n x_n + \beta_n(\delta_n u_n + (1-\delta_n)Su_n) + \gamma_n e_n \quad n \geq 1, \end{cases} \tag{4}$$

*Assume that the sequences $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$, $\{\delta_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ satisfy the following conditions: $0 < a \leq \alpha_n \leq a' < 1$, $0 \leq k \leq \delta_n \leq b < 1$, $0 < c \leq r_n \leq d < 2\lambda$ and $\sum_{n=1}^{\infty}\gamma_n < \infty$. Then, the sequence $\{x_n\}$ converges weakly to some point $x^* \in \Omega$.*

*Observe that the result of Huang and Ma [9] concluded weak convergence. Furthermore, only a single $k$-strict pseudocontractive mapping was considered in the work of Huang and Ma.*

Motivated and inspired by works of He [7], Huang and Ma [9], and that of other authors mentioned above, it is our purpose in this paper to proposed a new iterative algorithm which converges *strongly* to solution of generalized equilibrium problem which is also a common fixed point of finite family of strict pseudocontractions. Our theorem improves and compliments the results of He [7], Huang and Ma [9], and that of a host of other authors.

## 2. PRELIMINARY

In what follows, we assume that H is a real Hilbert space with inner product $\langle .,. \rangle$ and $\|.\|$ as the norm associated with the inner product. K is a nonempty, closed and convex subset of H, $P_K : H \to K$ the metric projection of $H$ onto $K$. $T : K \to K$ is a self mapping,

$F(T)$ denotes the set of fixed points of the operator $T$, $x_n \to x^*$ and $x_n \rightharpoonup x^*$ denote strong and weak convergence of a sequence $\{x_n\}$ to $x^*$, respectively. It is well known that given any nonempety, closed and convex subset $K$ of $H$, then for arbitrary vector $x \in H$, $z = P_K x$ if and only if

$$\langle x - z, y - z \rangle \leq 0, \forall\, y \in K. \tag{5}$$

It is also well known that the following identities hold: $\forall t \in [0,1]$and $\forall\, x, y \in H$,

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2. \tag{6}$$

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y \rangle, \forall x, y \in H. \tag{7}$$

Moreover, it is known that given any vector $y \in H$, there exists $f_y \in H^* = H$ ($H^*$ the dual of $H$) such that $\forall\, x \in H$,

$$f_y(x) = \langle x, y \rangle. \tag{8}$$

**Remark 1.** *It is easy to see using equation (8) that if $x_n \rightharpoonup x^*$, then, $\langle x_n, y \rangle \to \langle x^*, y \rangle$.*

It is necessary to recall the following important definitions.

**Definition 1.** *Let $T : D(T) \subseteq H \to H$ be a mapping (with domain $D(T)$ in $H$), then $T$ is said to be*

*(i) L-Lipschitz if there exists $L \geq 0$ such that $\forall\, x, y \in D(T)$,*

$$\|Tx - Ty\| \leq L\|x - y\|.$$

*If $L \in [0,1)$, then $T$ is called contraction and the mapping $T$ is called nonexpansive if $L \in [0,1]$;*

*(ii) k-strictly pseudocontractive mapping if there exists a constant $k \in [0,1)$ such that $\forall\, x, y \in D(T)$,*

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - Tx - (y - Ty)\|^2;$$

*(iii) k-quasi-strictly pseudocontraction if $F(T)$ is not empty and there exists $k \in [0,1)$ such that $\forall\, x, \in D(T)$, $\forall\, x^*, \in F(T)$,*

$$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 + k\|x - Tx\|^2.$$

*(iv) firmly nonexpansive if*

$$\forall\, x, y \in D(T), \|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y \rangle;$$

*(v) monotone if $\forall\, x, y \in D(T), \langle Tx - Ty, x - y \rangle \geq 0$;*

*(vi)* *$\alpha$-inverse strongly Monotone if there exists $\alpha > 0$ such that $\forall\ x, y \in D(T)$,*

$$\langle Tx - Ty, x - y\rangle \geq \alpha \|Tx - Ty\|^2.$$

**Remark 2.** *It is easy to see that every nonexpansive mapping is $0$-strictly pseudocontractive. Hence, the class of $k$-strictly pseudocontractive mappings contains the class of nonexpansive mappings. It is worthy to note that the converse is false. The following example portrays this fact:*

**Example 1.** *Let $T : H \to H$ be defined for any $x \in H$ by*

$$T(x) = -2x.$$

*It is clear that $T$ is Lipschitz but not nonexpansive.*

It could easily be shown that T is strictly Pseudocontractive, thus the class of $k$-strictly pseudocontractive mappings properly contains the class of nonexpansive mappings. It is easy to see that every $k$-strictly pseudocontraction with nonempty fixed point set is $k$-quasi-strictly pseudocontraction.

**Definition 2.** *The generalized mixed equilibrium problems (abbrreviated GMEP) for operators $f,\ \Phi,\ B$ is a problem of finding $u^* \in K$ such that*

$$f(u^*, y) + \Phi(y) - \Phi(u^*) + \langle Bu^*, y - u^*\rangle \geq\ 0, \forall\ y \in K, \qquad (9)$$

*where $f$ is a real valued bifunction with domain $K \times K$, $\Phi$ is a proper extended real valued function with domain $K$, that is, $\Phi : K \to \Re \cup \{+\infty\}$ and $B$ a nonlinear monotone operator.The solution set of (9) is denoted by*

$$GMEP(f, \Phi, B)$$
$$:= \{u \in K : f(u, y) + \Phi(y) - \Phi(u) + \langle Bu, y - u\rangle \geq\ 0, \forall\ y \in K\}.$$

*If $\Phi \equiv 0 \equiv B$ in (9), then, inequality (9) reduces to the* ***Classical Equilibrium Problem*** *, that is the problem of finding $u^* \in K$ such that*

$$f(u^*, y) \geq\ 0,\ \forall\ y \in K; \qquad (10)$$

*and in this case, $GMEP(f, 0, 0)$ is denoted by*

$$EP(f) := \{u \in K : f(u, y) \geq\ 0, \forall\ y \in K\}.$$

*If $\Phi \equiv 0 \equiv f$ in (9), then (9) reduces to the* ***Classical Variational Inequality Problem****, that is, the problem of finding $u^* \in K$ such that*

$$\langle Bu^*, y - u^* \rangle \geq 0, \forall\, y \in K; \tag{11}$$

*and in this case, $GMEP(0, 0, B)$ is denoted by*

$$V.I(B, K) = \{u \in K : \langle Bu, y - u \rangle \geq 0\ , \forall\, y \in K\}.$$

*If $B \equiv 0 \equiv f$ in (9), then (9) reduces to the following minimization problem: find $u^* \in K$ s.t*

$$\Phi(y) \geq \Phi(u^*), \forall\, y \in K; \tag{12}$$

*in this case, $GMEP(0, \Phi, 0)$ is denoted by $Argmin(\Phi)$, where*

$$Argmin(\Phi)\ = \{u \in K : \Phi(y) \geq \Phi(u), \forall\, y \in K\}.$$

*If $B \equiv 0$ in (9), then (9) reduces to the* ***Mixed Equilibrium Problem****, that is, the problem of finding $u^* \in K$ such that*

$$f(u^*, y) + \Phi(y) - \Phi(u^*) \geq 0, \forall\, y \in K; \tag{13}$$

*and in this case,l $GMEP(f, \Phi, 0)$ is denoted by*

$$MEP(f, \Phi)\ = \{u \in K : f(u, y) + \Phi(y) - \Phi(u) \geq 0, \forall\, y \in K\}.$$

*If $\Phi \equiv 0$ in (9), then (9) reduces to the* ***Generalized Equilibrium Problem****, that is, the problem of finding $u^* \in K$ such that*

$$f(u^*, y) + \langle Bu^*, y - u^* \rangle \geq 0, \forall\, y \in K; \tag{14}$$

*and in this case, $GEP(f, 0, B)$ is denoted by*

$$GEP(f, B) = \{u \in K : f(u, y) + \langle Bu, y - u \rangle \geq 0, \forall\, y \in K\}.$$

*If $f \equiv 0$ in (9), then (9) reduces to the* ***Generalized Variarational Inequality Problems*** *i.e the problem of finding $u^* \in K$ such that*

$$\Phi(u^*) - \Phi(y) + \langle Bu^*, y - u^* \rangle \geq 0, \forall\, y \in K. \tag{15}$$

*In this case, $GEP(0, \Phi, B)$ is denoted by*

$$GVI(\Phi, B, K) = \{u \in K : \Phi(u) - \Phi(y) + \langle Bu, y - u \rangle \geq 0, \forall\, y \in K.\}$$

*From the forgoing analysis we observe that (9) solves three diferent types of problems simultaneously, that is, it solves problem of optimization, variational inequality and equilibrium problems.*

In solving equilbrium problem, we impose the following conditions on our bifunction $f$, namely:

$A_1$ $\quad f(x,x)=0, \forall\, x\in K;$

$A_2$ f is monotone in the sense that

$$f(x,y)+f(y,x)\leq 0, \forall\, y,x\in K;$$

$A_3$ $f$ is hemi-continuous

$$i.e \qquad \limsup_{t\to 0^+} f(tz+(1-t)x,y)\leq f(x,y), \forall\, y,x\in K;$$

$A_4$ The function $f(x,.)$ is convex and lower semicontinous $\forall\, x\in K$.

In what follows, the following Lemmas shall play important roles:

**Lemma 1.** [9] *Let $K$ be a closed and convex subset of $H$ and let $f:K\times K\to\mathbb{R}$ be a bifunction satisfying conditions $A_1$ to $A_4$. Let $r>0$ and $x\in H$, then there exists $z\in K$ s.t*

$$f(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geq 0\ \forall\, y\in K.$$

**Lemma 2.** [9] *Let $K$ be a nonempty closed and convex subset of a real Hilbert space $H$ and let $f:K\times K\to\mathbb{R}$ be a bifunction satisfying $A_1-A_4$. For $r>0$ and $x\in H$, define a mapping $T_r$ as follows:*

$$T_r(x)=\{z\in K: f(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geq 0\ \forall\, y\in K\}.$$

*Then, the following holds:*

*(i) $T_r$ is single valued*
*(ii) $T_r$ is firmly nonexpansive ie $\forall x,y\in H$, $\|T_rx-T_ry\|^2\leq\langle x-y,T_rx-T_ry\rangle$*
*(iii) $F(T_r)=EP(f)$*
*(iv) $EP(f)$ is closed and convex.*

**Lemma 3.** [22] *Suppose that $\{a_n\}_{n\geq 1}$ is a sequence of nonnegative numbers such that*

$$a_{n+1}=(1-\alpha_n)a_n+\sigma_n\ \forall n\geq 1,$$

*where* $\{\alpha_n\}_{n=1}^{\infty}$ *is a sequence in* $(0,1)$ *and* $\{\sigma_n\}_{n\geq}$ *is a sequence of nonnegative numbers such that* $\sum_{n=1}^{\infty}\alpha_n = \infty$ *and* $\sigma_n = o(\alpha_n)$ *( that is,* $\lim_{n\to\infty}\frac{\sigma_n}{\alpha_n} = 0$*) or* $\sum_{n=1}^{\infty}\sigma_n < \infty$*, then* $\lim_{n\to\infty} a_n = 0$*.*

**Lemma 4.** [14] *Let* $K$ *be a nonempty closed convex subset of a Hilbert space* $H$*, and let* $T : K \to K$ *be a self mapping.*
*(i) If* $T$ *is k-strict pseudocontraction, then the mapping* $T$ *is* $\frac{1+k}{1-k}-$ *Lipschitz; in addition, the mapping* $(I-T)$ *is demiclosed at 0; that is, if* $\{x_n\}$ *is a sequence in* $K$ *with* $x_n \rightharpoonup x$ *and* $x_n - Tx_n \to 0$ *, then* $x \in F(T)$*.*

*(ii) If* $T$ *is k-quasi-strict pseudocontraction, then the fixed point set,* $F(T)$*, of* $T$ *is closed and convex.*

**Lemma 5.** [13] *Let* $\{\gamma_n\}$ *be a sequence of real numbers such that there exists a subsequence* $\{\gamma_{n_j}\}$ *such that* $\gamma_{n_j} < \gamma_{n_j+1}$*,* $\forall j \in \mathbb{N}$*. Consider the sequence of integers* $\{\tau(n)\}$ *defined by*

$$\tau(n) = \max_k \{k \leq n : \gamma_k < \gamma_{k+1}\}.$$

*Then,*

*(i)* $\{\tau(n)\}$ *is a non decreasing sequence for all* $n \geq n_0$
*(ii)* $\lim_{n\to\infty}\tau(n) = \infty$;
*(iii)* $\gamma_{\tau(n)} < \gamma_{\tau(n)+1} \quad \forall\, n \geq n_0$
*(iv)* $\gamma_n < \gamma_{\tau(n)+1} \quad \forall\, n \geq n_0$

## 3. MAIN RESULT

In this section, we will introduce our iterative scheme, present our main result and its detailed proof. We shall also demonstrate how our main result can be used to approximate the common solution of generalised mixed equilibrium problems and common fixed point of a finite family of strict pseudocontractions.

Let $K$ be a nonempty subset of a real Hilbert space $H$, let $f : K \times K \to \mathbb{R}$ and $A : K \to H$ be given functions. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0,1)$. The schemes $\{x_n\}_{n\geq 1}$ is defined from arbitrary elements $x_1, u \in K$ by

$$\begin{cases} f(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0,\ \forall\, y \in K,\\ z_n = (1-\beta_n)u_n + \beta_n T_{i(n)}u_n,\\ x_{n+1} = \alpha_n u + (1-\alpha_n)z_n, \end{cases} \tag{16}$$

were $\{r_n\}_{n\geq 1}$ is a sequence of positive real numbers, and $\{u_n\}_{n\geq 1}$ is a sequence guaranteed by Lemma 2, provided the mappings $f$ and $A$ satify appropriate conditions.

To appreciate how $\{z_n\}_{n\geq 1}$ in (16) runs, we compute some terms of the sequence . Observe that

$$\begin{aligned}
z_1 &= (1-\beta_1)u_1 + \beta_1 T_1 u_1,\\
z_2 &= (1-\beta_2)u_2 + \beta_2 T_2 u_2,\\
&\cdots,\\
z_N &= (1-\beta_N)u_N + \beta_N T_N u_N,\\
z_{N+1} &= (1-\beta_{N+1})u_{N+1} + \beta_{N+1} T_1 u_{N+1},\\
z_{N+2} &= (1-\beta_{N+2})u_{N+2} + \beta_{N+2} T_2 u_{N+2},\\
&\cdots,\\
z_{2N} &= (1-\beta_{2N})u_{2N} + \beta_2 N T_N u_{2N},\\
z_{2N+1} &= (1-\beta_{2N+1})u_{2N+1} + \beta_{2N+1} T_1 u_{2N+1},\\
&\cdots,\\
z_{kN+i} &= (1-\beta_{kN+i})u_{kN+i} + \beta_{kN+i} T_i u_{kN+i},\\
&\cdots,\ \text{where } u_n = T_{r_n}(I - r_n A)x_n, k \geq 0 \text{ and } i = 1, 2, \cdots, N.
\end{aligned}$$

**Remark 3.** *Thus, the sequence $\{z_n\}_{n\geq 1}$ can be written in compact form as*

$$z_n = (1-\beta_n)u_n + \beta_n T_{i(n)}u_n, \qquad i(n) \in \{1, 2, \cdots, N\}.$$

We are now ready to present our main result.

**Theorem 0.3.** *Let $K$ be a nonempty, closed and convex subset of a real Hilbert space $H$ and $f : K \times K \to \mathbb{R}$ be a bifunction satisfying $A_1 - A_4$ and $A$ be $\alpha$-inverse strongly monotone mapping of $K$ into $H$. Let $T_i : K \to K$ be a finite family of $k_i$-strictly pseudocontractive mappings $i = 1, 2, \cdots, N$ , $0 \leq k_i < 1$ and $k := \sup\{k_i : 1 \leq i \leq N\}$. Let $\{x_n\}, \{u_n\}$ be sequences defined from arbitrary elements $x_1, u \in K$ by (16). Suppose that $\Omega = \bigcap_{i=1}^N F(T_i) \bigcap GEP(f, A) \neq \emptyset$, and that (i) $\lim \alpha_n = 0,\ \sum_{n=1}^{\infty} \alpha_n = \infty$;*

*(ii)* $0 < \delta \leq \beta_n \leq 1 - k - \delta < 1$; *(iii)* $0 < b \leq r_n \leq c \leq 2\alpha$, *then* $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $x^* = P_\Omega(u)$.

*Proof.* Observe that for any $x, y \in K$, for any $n \in \mathbb{N}$, and using the fact that $0 < b \leq r_n \leq c \leq 2\alpha$, we obtain that

$$\begin{aligned}
\|(I - r_nA)x - (I - r_nA)y\|^2 &= \|x - y - r_n(Ax - Ay)\|^2 \\
&= \|x - y\|^2 - 2r_n \langle x - y, Ax - Ay\rangle \\
&\quad + {r_n}^2\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2 - 2\alpha r_n\|Ax - Ay\|^2 \\
&\quad + {r_n}^2\|Ax - Ay\|^2 \\
&= \|x - y\|^2 + r_n(r_n - 2\alpha)\|Ax - Ay\|^2 \\
&\leq \|x - x\|^2.
\end{aligned}$$

Thus, $I - r_nA$ is nonexpansive. Moreover, it is clearly from Lemmas 1, 2 and Remark 3 that (16) is well defined. Now, let $x^* = P_\Omega(u)$; we note that $P_\Omega$ is well defined since $\Omega$ is intersection of nonempty closed and convex sets (see (iv) of Lemma 2 and (ii) of Lemma 4). We now show that

$$\|z_n - x^*\| \leq \|x_n - x^*\|, \ \forall\, n \geq 1.$$

Using the fact that $T_i$ are $k_i$- strictly pseudocontractive mapping, we have

$$\begin{aligned}
\|z_n - x^*\|^2 &= \|(1 - \beta_n)u_n + \beta_nT_{i(n)}u_n - x^*\|^2 \\
&= \|(1 - \beta_n)(u_n - x^*) + \beta_n(T_{i(n)}u_n - x^*)\|^2 \\
&= (1 - \beta_n)\|u_n - x^*\|^2 + \beta_n\|T_{i(n)}u_n - x^*\|^2 \\
&\quad - \beta_n(1 - \beta_n)\|T_{i(n)}u_n - u_n\|^2 \\
&\leq (1 - \beta_n)\|u_n - x^*\|^2 \\
&\quad + \beta_n[\|u_n - x^*\|^2 + k\|T_{i(n)}u_n - u_n\|^2] \\
&\quad - \beta_n(1 - \beta_n)\|T_{i(n)}u_n - u_n\|^2 \\
&= \|u_n - x^*\|^2 - \beta_n(1 - k - \beta_n)\|T_{i(n)}u_n - u_n\|^2.
\end{aligned}$$

From condition $(ii)$, we have that

$$0 < \delta \leq \beta_n \leq 1 - k - \delta < 1. \tag{17}$$

Multiplying inequality (17) by $-1$ and adding $1 - k$ to the resulting inequality, we obtain that

$$\delta \leq 1 - k - \beta_n,$$

which implies that

$$-\beta_n(1-k-\beta_n) \le -\delta^2.$$

Thus,

$$\|z_n - x^*\|^2 \le \|u_n - x^*\|^2 - \delta^2 \|T_{i(n)}u_n - u_n\|^2. \tag{18}$$

Next, Using Lemma 2 (ii) and definition of $u_n$, we have that $u_n = T_{r_n}(I - r_nA)x_n$. Furthermore, using the fact that $x^* \in GEP(f, A)$, we also have that $x^* = T_{r_n}(I - r_nA)x^*$. Thus, by nonexpansiveness of $T_{r_n}$ and condition $(iii)$ we have that

$$\begin{aligned}
\|u_n - x^*\|^2 &= \|T_{r_n}(I - r_nA)x_n - T_{r_n}(I - r_nA)x^*\|^2 \\
&\le \|(I - r_nA)x_n - (I - r_nA)x^*\|^2 \\
&= \|x_n - x^* - r_n(Ax_n - Ax^*)\|^2 \\
&= \|x_n - x^*\|^2 - 2r_n \langle x_n - x^*, Ax_n - Ax^*\rangle \\
&\quad + {r_n}^2 \|Ax_n - Ax^*\|^2 \\
&\le \|x_n - x^*\|^2 - 2\alpha r_n \|Ax_n - Ax^*\|^2 \\
&\quad + {r_n}^2 \|Ax_n - Ax^*\|^2 \\
&= \|x_n - x^*\|^2 + r_n(r_n - 2\alpha)\|Ax_n - Ax^*\|^2 \\
&\le \|x_n - x^*\|^2. \qquad (19)
\end{aligned}$$

Using inequalities (18) and (19), we obtain that

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|\alpha_n(u - x^*) + (1-\alpha_n)(z_n - x^*)\| \\
&\le \alpha_n\|u - x^*\| + (1-\alpha_n)\|z_n - x^*\| \\
&\le \alpha_n\|u - x^*\| + (1-\alpha_n)\|x_n - x^*\|. \qquad (20)
\end{aligned}$$

It is easy to deduce from (20) that

$$\|x_n - x^*\| \le \max\{\|u - x^*\|, \|x_1 - x^*\|\}.$$

Hence, $\{x_n\}_{n\ge1}$ is bounded. Consequently, $\{z_n\}, \{u_n\}$ are bounded.

We now consider the following two cases:

**Case 1:** $\|x_{n+1} - x^*\| \le \|x_n - x^*\|,\ \forall\, n \ge n_0,\ \textit{for some } n_0 \ge 1.$ Therefore, $\{||x_n - x^*||\}$ is a monotone nonincreasing sequence, hence its limit exists.

Next we show that

$$\lim_{n\to\infty} ||z_n - u_n|| = \lim ||x_n - z_n|| = 0.$$

It follows from inequalities (18) and (19), that

$$\begin{aligned} ||x_{n+1}-x^*||^2 &= ||\alpha_n(u-x^*)+(1-\alpha_n)(z_n-x^*)||^2 \qquad (21)\\ &\leq \alpha_n\|u-x^*\|^2+(1-\alpha_n)\|z_n-x^*\|^2 \\ &\leq \alpha_n\|u-x^*\|^2 \\ &\quad +(1-\alpha_n)\left(\|u_n-x^*\|^2-\delta^2\|T_{i(n)}u_n-u_n\|^2\right) \\ &\leq \alpha_n\|u-x^*\|^2 \\ &\quad +(1-\alpha_n)\left(\|x_n-x^*\|^2-\delta^2\|T_{i(n)}u_n-u_n\|^2\right) \\ &\leq \alpha_n||u-x^*||^2+\|x_n-x^*\| \\ &\quad -(1-\alpha_n)\delta^2\|T_{i(n)}u_n-u_n\|^2, \end{aligned}$$

which implies that

$$(1-\alpha_n)\delta^2||T_{i(n)}u_n-u_n||^2 \leq \alpha_n||u-x^*||^2+||x_n-x^*||^2-||x_{n+1}-x^*||^2. \qquad (22)$$

Thus,

$$\lim_{n\to\infty}\|T_{i(n)}u_n-u_n\|=0. \qquad (23)$$

Since $\|z_n-u_n\|=\beta_n\|T_{i(n)}u_n-u_n\|$ and $\{\beta_n\}$ is bounded, we have that

$$\lim_{n\to\infty}\|z_n-u_n\|=0. \qquad (24)$$

Also, for some $M_0>0$,

$$\begin{aligned} ||x_{n+1}-z_n|| &= \alpha_n||u-z_n|| \\ &\leq \alpha_n M_0. \end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\|x_{n+1}-z_n\|=0. \qquad (25)$$

Next, we show that $\lim ||Ax_n-Ax^*||=0$.
Using convexity of $\|.\|^2$, inequalities (18) and (19)

$$\begin{aligned} ||x_{n+1}-x^*||^2 &\leq \alpha_n\|u-x^*\|^2+(1-\alpha_n)\|z_n-x^*\|^2 \\ &\leq \alpha_n||u-x^*||^2+(1-\alpha_n)\|u_n-x^*\|^2 \\ &\leq \alpha_n||u-x^*||^2+(1-\alpha_n)\|x_n-x^*\|^2 \\ &\quad +r_n(r_n-2\alpha)||Ax_n-Ax^*||^2, \end{aligned}$$

which implies that

$$\begin{aligned} -r_n(r_n-2\alpha)||Ax_n-Ax^*||^2 &\leq \alpha_n||u-x^*||^2 \\ &+(1-\alpha_n)||x_n-x^*||^2 \\ &-||x_{n+1}-x^*||^2. \end{aligned}$$

Using condition $(iii)$, we obtain that

$$\begin{aligned} b(2\alpha-c)||Ax_n-Ax^*||^2 &\leq \alpha_n||u-x^*||^2 \\ &+(1-\alpha_n)||x_n-x^*||^2-||x_{n+1}-x^*||^2. \end{aligned}$$

Hence,

$$\lim_{n\to\infty}||Ax_n-Ax^*||^2=0. \tag{26}$$

Using the fact that $u_n = T_{r_n}(I-r_nA)x_n$, $x^* = T_{r_n}(I-r_nA)x^*$, $x^*\in GEP(f,A)$, firmly nonexpansiveness of $T_{r_n}$, equation (7) and the fact that $I-r_nA$ is nonexpansive, we obtain that

$$\begin{aligned} \|u_n-x^*\|^2 &= \|T_{r_n}(I-r_nA)x_n-T_{r_n}(I-r_nA)x^*\|^2 \\ &\leq \langle (x_n-r_nAx_n)-(x^*-r_nAx^*), u_n-x^*\rangle \\ &= \frac{1}{2}\left(\|(x_n-r_nAx_n)-(x^*-r_nAx^*)\|^2+\|u_n-x^*\|^2\right) \\ &- \frac{1}{2}\left(\|\left((x_n-r_nAx_n)-(x^*-r_nAx^*)\right)-(u_n-x^*)\|^2\right) \\ &\leq \frac{1}{2}\left(\|x_n-x^*\|^2+\|u_n-x^*\|^2\right) \\ &\quad -\frac{1}{2}\|(x_n-u_n)-r_n(Ax_n-Ax^*)\|^2, \end{aligned}$$

which implies that

$$\begin{aligned} \|u_n-x^*\|^2 &\leq \|x_n-x^*\|^2-\|(x_n-u_n)-r_n(Ax_n-Ax^*)\|^2 \\ &= \|x_n-x^*\|^2-\|x_n-u_n\|^2-{r_n}^2\|Ax_n-Ax^*\|^2 \\ &\quad +2r_n\langle x_n-u_n, Ax_n-Ax^*\rangle \\ &\leq \|x_n-x^*\|^2-\|x_n-u_n\|^2 \\ &\quad +2r_n\|x_n-u_n\|.\|Ax_n-Ax^*\|. \end{aligned} \tag{27}$$

Substituting inequality (27) in inequality (22),using the facts that $\{x_n\},\{u_n\}$ are bounded and equation (26), we obtain that

$$\begin{aligned} \|x_{n+1}-x^*\|^2 &\leq \alpha_n\|u-x^*\|^2+\|x_n-x^*\|^2 \\ &\quad -\|x_n-u_n\|^2+2r_n\|x_n-u_n\|\|Ax_n-Ax^*\|. \end{aligned}$$

Thus,

$$\begin{aligned}\|x_n - u_n\|^2 &\leq \alpha_n\|u - x^*\|^2 + \|x_n - x^*\|^2 \\ &\quad -\|x_{n+1} - x^*\|^2 + 2r_n\|x_n - u_n\|.\|Ax_n - Ax^*\|,\end{aligned}$$

consequently,

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{28}$$

Since we have from (24) and (28) that

$$\lim_{n\to\infty} \|u_n - z_n\| = \lim_{n\to\infty} \|u_n - x_n\| = 0$$

, it follows that

$$\begin{aligned}\|x_n - z_n\| &= \|x_n - u_n + u_n - z_n\| \\ &\leq \|x_n - u_n\| + \|u_n - z_n\| \to 0 \; as \; n \to \infty.\end{aligned}$$

Thus,

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{29}$$

Next, we show that

$$\lim_{n\to\infty} \|u_{n+j} - u_n\| = 0$$

and

$$\lim_{n\to\infty} \|x_{n+j} - x_n\| = 0, \; \forall \; j \in \{1, 2, \cdots, N\}.$$

We proceed as follows:

$$\begin{aligned}\|u_{n+1} - u_n\| &\leq \|u_{n+1} - x_{n+1}\| + \|x_{n+1} - u_n\| \\ &\leq \|u_{n+1} - x_{n+1}\| + \|x_{n+1} - z_n\| \\ &\quad +\|z_n - u_n\| \to 0 \; as \; n \to \infty.\end{aligned}$$

Consequently,

$$\begin{aligned}\|u_{n+j} - u_n\| &\leq \|u_{n+j} - u_{n+j-1}\| \\ &\quad +\|u_{n+j-1} - u_{n+j-2}\| + \|u_{n+j-2} - u_{n+j-3}\| \\ &+ \cdots + \|u_{n+1} - u_n\| \to 0 \; as \; n \to \infty.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \|u_{n+j} - u_n\| = 0. \; \forall \; j \in \{1, 2, \cdots, N\}. \tag{30}$$

Similarly, using (28) and (30), we obtain that

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq \|x_{n+1} - u_{n+1}\| + \|u_{n+1} - u_n\| \\ &\quad +\|u_n - x_n\| \to 0 \; as \; n \to \infty.\end{aligned} \tag{31}$$

Consequently,

$$\begin{aligned}\|x_{n+j}-x_n\| &\leq \|x_{n+j}-x_{n+j-1}\| \\ &\quad +\|x_{n+j-1}-x_{n+j-2}\|+\|x_{n+j-2}-x_{n+j-3}\| \\ &+ \cdots + \|x_{n+1}-x_n\| \to 0 \ as\ n\to\infty.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\|x_{n+j}-x_n\| = 0 \ \forall\ j\in\{1,2,\cdots,N\}.$$

Next, we show that $\lim_{n\to\infty}\|u_n - T_ju_n\| = 0 = \lim_{n\to\infty}\|x_n - T_jx_n\|$, $\forall\ j\in\{1,2,\cdots,N\}$.

Observe that for any $n\in\mathbb{N}$, there exist $k(n)\in\mathbb{N}\cup\{0\}, i(n)\in\{1,2,\cdots,N\}$ such that $n=k(n)N+i(n)$. Now, for any $n\geq 1$, set $T_n = T_{i(n)}$.Then, it follows that

$$T_{n+j} = \begin{cases} T_{(i(n)+j)}, \ i(n)+j\leq N \\ T_{(i(n)+j-N)}, \ otherwise. \end{cases} \tag{32}$$

Hence, using (23) we have that

$$\|u_n - T_nu_n\| \ = \ \|u_n - T_{i(n)}u_n\| \to 0 \ as\ n\to\infty. \tag{33}$$

Consequently, for any $j\in\{1,2,\cdots,N\}$, we have using equation (30), (33) and Lemma 4 that

$$\begin{aligned}\|u_n - T_{n+j}u_n\| &\leq \|u_n - u_{n+j}\| + \|u_{n+j}-T_{n+j}u_{n+j}\| \\ &\quad +\|T_{n+j}u_{n+j}-T_{n+j}u_n\| \to 0 \ as\ n\to\infty.\end{aligned}$$

Thus,

$$\lim_{n\to\infty}\|u_n - T_ju_n\| = 0, \quad \forall\ j\in\{1,2,\cdots,N\}. \tag{34}$$

Moreover, for each $j\in\{1,2,\cdots,N\}$, we observe from equations (28) and (34) and Lemma 4 that

$$\begin{aligned}\|x_n - T_jx_n\| &\leq \|x_n - u_n\| + \|u_n - T_ju_n\| \\ &\quad +\|T_ju_n - T_jx_n\| \to 0 \ as\ n\to\infty. \qquad (35) \\ & \qquad\qquad\qquad\qquad\qquad\qquad (36)\end{aligned}$$

Next, we show that $\limsup_{n\to\infty}\langle(u-x^*), x_{n+1}-x^*\rangle \leq 0$. From properties of limit superior of bounded sequence of real numbers, we have that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u-x^*, x_n - x^*\rangle = \lim_{k\to\infty}\langle u-x^*, x_{n_k}-x^*\rangle. \tag{37}$$

Since $\{x_{n_k}\}_{k\geq 1}$ is bounded, there exists a subsequence $\{x_{n_{k_m}}\}_{m\geq 1}$ of $\{x_{n_k}\}_{k\geq 1}$ such that $x_{n_{k_m}} \rightharpoonup p^* \in K$ $m \to \infty$

**Claim** : $p^* \in \Omega$.

**Proof of Claim:** We first show that $p^* \in F(T_i) \quad \forall\, j \in \{1, 2, \cdots, N\}$. It follows from ((35)) that

$$\lim_{m\to\infty} \|x_{n_{k_m}} - T_j x_{n_{k_m}}\| = 0, \quad \forall\, j \in \{1, 2, \cdots, N\}$$

Hence by Lemma 4, we have that $p^* \in F(T_j) \quad \forall\, j \in \{1, 2, \cdots, N\}$. Since $u_n = T_{r_n}(I - r_n A)x_n$, for any $y \in K$, we have that

$$f(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall\, y \in K$$

From $A_2$, we have that

$$\begin{aligned} f(u_n, y) + f(y, u_n) \leq 0 \quad &\leq \quad f(u_n, y) + \langle Ax_n, y - u_n\rangle \\ &\quad + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle, \ \forall\, y \in K. \end{aligned}$$

This implies that

$$\langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq f(y, u_n), \ \forall\, y \in K. \quad (38)$$

Relacing $n$ with $n_{km}$ in (38), we obtain that

$$\begin{aligned} \left\langle Ax_{n_{km}}, y - u_{n_{km}}\right\rangle \quad & + \frac{1}{r_{n_{km}}}\left\langle y - u_{n_{km}}, u_{n_{km}} - x_{n_{km}}\right\rangle \\ & \geq \quad f(y, u_{n_{km}}), \ \forall\, y \in K. \end{aligned} \quad (39)$$

Now, for $t \in (0, 1)$ and $y \in K$, set $z_t = ty + (1-t)p^*$. Clearly, $z_t \in K$ since $K$ is convex. Therefore, using monotonicity of A and the conditions on $\{r_n\}$, we have that

$$\begin{aligned}
\left\langle z_t - u_{n_{km}}, Az_t\right\rangle &\geq \left\langle z_t - u_{n_{km}}, Az_t\right\rangle - \left\langle Ax_{n_{km}}, z_t - u_{n_{km}}\right\rangle \\
&\quad -\frac{1}{r_{n_{km}}}\left\langle z_t - u_{n_{km}}, u_{n_{km}} - x_{n_{km}}\right\rangle + f(z_t, u_{n_{km}}) \\
&= \left\langle z_t - u_{n_{km}}, Az_t - Au_{n_{km}}\right\rangle \\
&\quad + \left\langle Au_{n_{km}} - Ax_{n_{km}}, z_t - u_{n_{km}}\right\rangle \\
&\quad -\frac{1}{r_{n_{km}}}\left\langle z_t - u_{n_{km}}, u_{n_{km}} - x_{n_{km}}\right\rangle + f(z_t, u_{n_{km}}) \\
&\geq \left\langle Au_{n_{km}} - Ax_{n_{km}}, z_t - u_{n_{km}}\right\rangle \\
&\quad -\frac{1}{r_{n_{km}}}\left\langle z_t - u_{n_{km}}, u_{n_{km}} - x_{n_{km}}\right\rangle + f(z_t, u_{n_{km}}) \\
&\geq \left\langle Au_{n_{km}} - Ax_{n_{km}}, z_t - u_{n_{km}}\right\rangle \\
&\quad -\frac{1}{b}\|z_t - u_{n_{km}}\|.\|u_{n_{km}} - x_{n_{km}}\| + f(z_t, u_{n_{km}}).
\end{aligned}$$

Using the fact that $\lim \|x_n - u_n\| = 0$, $A$ is uniformly continuous and from $A_4$, we obtain as $m \to \infty$ that

$$\langle z_t - p^*, Az_t\rangle \geq f(z_t, p^*). \tag{40}$$

From $A_1$ *and* $A_4$, we have that

$$\begin{aligned}
0 &= f(z_t, z_t) \\
&= f(z_t, ty + (1-t)p^*) \\
&\leq tf(z_t, y) + (1-t)f(z_t, p^*) \\
&\leq tf(z_t, y) + (1-t)\langle z_t - p^*, Az_t\rangle \\
&= tf(z_t, y) + t(1-t)\langle y - p^*, Az_t\rangle,
\end{aligned}$$

this implies that

$$0 \leq f(z_t, y) + (1-t)\langle y - p^*, Az_t\rangle.$$

As $t \to 0$ we obtain using $A_3$ that

$$f(p^*, y) + \langle y - p^*, Ap^*\rangle \geq 0, \quad \forall\, y \in K.$$

This implies that $p^* \in EP(f, A)$. Hence, $p^* \in \Omega$.
It follows from equation (37), Remark (1) and inequality (5) that

$$\begin{aligned}
\limsup_{n\to\infty} \langle u - x^*, x_n - x^* \rangle &= \lim_{k\to\infty} \langle (u - x^*), x_{n_k} - x^* \rangle \\
&= \lim_{i\to\infty} \left\langle u - x^*, x_{n_{k_i}} - x^* \right\rangle \\
&= \langle u - x^*, p^* - x^* \rangle \\
&\leq 0.
\end{aligned}$$

But,

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \langle x_{n+1} - x^*, x_{n+1} - x^* \rangle . \\
&= \langle \alpha_n(u - x^*) + (1 - \alpha_n)(z_n - x^*), x_{n+1} - x^* \rangle \\
&= \alpha_n \langle (u - x^*), x_{n+1} - x^* \rangle \\
&\quad + (1 - \alpha_n) \langle (z_n - x^*), x_{n+1} - x^* \rangle \\
&\leq \alpha_n \langle (u - x^*), x_{n+1} - x^* \rangle \\
&\quad + (1 - \alpha_n)\|z_n - x^*\|.\|x_{n+1} - x^*\| \\
&\leq \alpha_n \langle (u - x^*), x_{n+1} - x^* \rangle \\
&\quad + \frac{1 - \alpha_n}{2}(\|z_n - x^*\|^2 + \|x_{n+1} - x^*\|^2).
\end{aligned}$$

Thus,

$$\begin{aligned}
2\|x_{n+1} - x^*\|^2 &\leq 2\alpha_n \langle (u - x^*), x_{n+1} - x^* \rangle \\
&\quad + (1 - \alpha_n)(\|z_n - x^*\|^2 + \|x_{n+1} - x^*\|^2),
\end{aligned}$$

which implies that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq 2\alpha_n \langle (u - x^*), x_{n+1} - x^* \rangle + (1 - \alpha_n)\|x_n - x^*\|^2 \\
&\leq \sigma_n + (1 - \alpha_n)\|x_n - x^*\|^2,
\end{aligned}$$

where $\sigma_n = max\{0, 2\alpha_n \langle (u - x^*), x_{n+1} - x^* \rangle\}$. Clearly, $\sigma_n \to 0 \; as \; n \to \infty$. Hence by Lemma 3 that $x_n \to x^* = P_\Omega u$.

**Case 2:** There exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\|x_{n_k} - x^*\| < \|x_{n_k+1} - x^*\| \; \forall \; k \geq 1$.
By Lemma 5, there exists a sequence of integers $\{\tau(n)\}$ that satisfies

(i) $\{\tau(n)\}$ is nondecreasing, $\quad \forall \; n \geq n_0$;
(ii) $\lim_{n\to\infty} \tau(n) = \infty$;
(iii) $\|x_{\tau(n)} - x^*\| < \|x_{\tau(n)+1} - x^*\|, \quad \forall \; n \geq n_0$;

(iv) $\|x_n - x^*\| < \|x_{\tau(n)+1} - x^*\|, \quad \forall\, n \geq n_0.$

Consequently,

$$\begin{aligned} 0 &\leq \liminf(\|x_{\tau(n)+1} - x^*\| - \|x_{\tau(n)} - x^*\|) \\ &\leq \limsup(\|x_{\tau(n)+1} - x^*\| - \|x_{\tau(n)} - x^*\|) \\ &\leq \limsup(\|x_{n+1} - x^*\| - \|x_n - x^*\|) \\ &= \limsup(\|\alpha_n u + (1-\alpha_n)z_n - x^*\| - \|x_n - x^*\|) \\ &\leq \limsup(\alpha_n\|u - x^*\| + (1-\alpha_n)\|z_n - x^*\| - \|x_n - x^*\|) \\ &\leq \limsup(\alpha_n\|u - x^*\| + (1-\alpha_n)\|x_n - x^*\| - \|x_n - x^*\|) \\ &= 0. \end{aligned}$$

Therefore,

$$\lim(\|x_{\tau(n)+1} - x^*\| - \|x_{\tau(n)} - x^*\|) \;=\; 0. \tag{41}$$

It follows from equation (22) that

$$\begin{aligned} \|T_{i(\tau(n))}u_(\tau(n)) \;&-\; u_(\tau(n))\|^2 \\ &\leq \frac{1}{(1-\alpha_(\tau(n)))\delta^2}\Big[\alpha_(\tau(n))\|u - x^*||^2 \\ &\quad +\|x_(\tau(n)) - x^*\|^2 - \|x_{(\tau(n))+1} - x^*\|^2\Big]. \end{aligned} \tag{42}$$

Using equation (41) and inequality (42), we obtain that

$$\lim_{n\to\infty} \|T_{i(\tau(n))}u_{\tau(n)} - u_{\tau(n)}\| = 0.$$

Following the same argument as in the proof of Case 1, we obatin that since

$$\|z_{\tau(n)} - u_{\tau(n)}\| = \beta_{\tau(n)}\|T_{i(\tau(n))}u_{\tau(n)} - u_{\tau(n)}\|$$

and $\{\beta_{\tau(n)}\}$ is bounded, we have that

$$\lim_{n\to\infty} \|z_{\tau(n)} - u_{\tau(n)}\| = 0. \tag{43}$$

Also, for some $M_0 > 0$,

$$\begin{aligned} ||x_{\tau(n)+1} - z_{\tau(n)}|| &= \alpha_{\tau(n)}||u - z_{\tau(n)}|| \\ &\leq \alpha_{\tau(n)}M_0, \end{aligned}$$

so that

$$\lim_{n\to\infty} \|x_{\tau(n)+1} - z_{\tau(n)}\| = 0. \tag{44}$$

Therefore, comparing (44) with (25), we obatin using the same pattern of computation as in Case 1 (starting from (25), but with the index $n$ replcaed with $\tau(n)$), that $x_{\tau(n)} \to x^*$. Using part $(iv)$ of Lemma 5, we have that $x_n \to x^*$ *as* $n \to \infty$. This completes the proof. $\square$

**Corollary 0.4.** *Let $K$ be a nonempty, closed and convex subset of a real Hilbert space $H$ and $f : K \times K \to \mathbb{R}$ be a bifunction satisfying $A_1 - A_4$ and $A$ be $\alpha$-inverse strongly monotone mapping of $K$ into $H$. Let $T_i : K \to K$ be a finite family of nonexpansive mappings $i = 1, 2, \cdots, N$ Assume that $\Omega = \bigcap_{i=1}^{N} F(T_i) \bigcap GEP(f, A) \neq \emptyset$. Let $\{x_n\}, \{u_n\}$ be sequences defined from arbitrary elements $x_1, u \in K$ by*

$$\begin{cases} f(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall\, y \in K, \\ z_n = (1 - \beta_n)u_n + \beta_n T_{i(n)}u_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)z_n, \end{cases} \tag{45}$$

*where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ satisfy the folowing conditions:*

*(i)* $\lim \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty, \ \ \{\alpha_n\} \subseteq (0, 1)$;
*(ii)* $0 < \delta \leq \beta_n \leq 1 - k - \delta < 1, \quad \{\beta_n\} \subseteq (0, 1)$;
*(iii)* $0 < b \leq r_n \leq c \leq 2\alpha$.

*Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $x^* = P_\Omega(u)$*

*Proof.* Taking $k_i = 0, \ \forall i = 1, 2, \cdots, N$ in theorem 0.3, Then, the result follows. $\square$

## 4. APPLICATION

In this section, we will apply our main result to approximate the solution of the generalized mixed equilibrium problems and the common fixed point of a finite family of strict pseudocontractions. Precisely, we prove the following theorem:

**Theorem 0.5** (Generalised Mixed Equilibrium Problem)**.** *Let $K$ be a nonempty closed and convex subset of a real Hilbert space $H$ and $f : K \times K \to \mathbb{R}$ be a bifunction satisfying $A_1 - A_4$, $A$ be $\alpha -$ inverse strongly monotone mapping of $K$ into $H$ and $\phi$ be a lower semicontinuous and convex functional . Let $T_i : K \to K$ be finite family of $k_i$-strictly pseudocontractive mapping $i = 1, 2, \cdots, N$ , $0 \leq k_i < 1$ and $k := \sup\{k_i : 1 \leq i \leq N\}$. Assume that $\Omega = \bigcap_{i=1}^{N} F(T_i) \bigcap GEP(f, A) \neq \emptyset$. Let $\{x_n\}, \{u_n\}$ be a sequences* *generated by arbitrary element $x_1, u \in K$ and then by*

$$\begin{cases} f(u_n, y) + \phi(y) - \phi(u_n)) + \langle Ax_n, y - u_n \rangle \\ +\frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0 \; \forall \; y \in K \\ z_n = (1 - \beta_n) z_n + \beta_n T_{i(n)} z_n \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) z_n \end{cases} \tag{46}$$

*where $\{\alpha_n\}$, $\{\beta_n\}$, $\{r_n\}\{\alpha_n\}$ satisfy the folowing conditions:*

*(i)* $\lim \alpha_n = 0, \; \sum_{n=1}^{\infty} \alpha_n = \infty, \; \{\alpha_n\}, \; \{\alpha_n\} \subseteq (0, 1);$

*(ii)* $0 < \delta \leq \beta_n \leq 1 - k - \delta < 1;$

*(iii)* $0 < b \leq r_n \leq c \leq 2\alpha.$

*Then $\{x_n\}, \{z_n\}$ converges strongly to $x^* = P_F(u)$*

*Proof.* Define $\Gamma : K \times K \to \mathbb{R}$ by

$$\Gamma(x, y) = f(x, y) + \phi(y) - \phi(x), \tag{47}$$

then $\Gamma$ satisfies conditions $A_1 - A_4$ since

$$\begin{aligned} \Gamma(x, x) &= f(x, x) + \phi(x) - \phi(x) \\ &= 0. \end{aligned}$$

Also,

$$\begin{aligned} \Gamma(x, y) + \Gamma(y, x) &= (f(x, y) + \phi(y) - \phi(x)) + (f(y, x) + \phi(x) - \phi(y)) \\ &= f(x, y) + f(y, x) \\ &\leq 0. \end{aligned}$$

Moreover,

$$\begin{aligned} \limsup_{t \to 0^+} \Gamma(x, ty + (1 - t)w) &= \limsup_{t \to 0^+} \Big( f(x, ty + (1 - t)w) \\ &\quad + \phi(ty + (1 - t)w) - \phi(x) \Big) \\ &\leq \limsup_{t \to 0^+} \Big( t f(x, y) + (1 - t) f(x, w) \\ &\quad + t\phi(y) + (1 - t)\phi(w) - \phi(x) \Big) \\ &= f(x, w) + \phi(w) - \phi(x) \\ &= \Gamma(x, w). \end{aligned}$$

Besides, $\Gamma(x, .)$ is a sum of lower semicontinuous functions, and thus lower semicontinuous. So , it remains to show that $\Gamma(x, .)$ is a convex function. Let $t \in [0,1]$ and $y, w \in K$ be arbitrary, then

$$\begin{aligned}
\Gamma(x, ty + (1-t)w)) &= f(x, ty + (1-t)w)) \\
&\quad + \phi(ty + (1-t)w)) - \phi(x) \\
&\le tf(x,y) + (1-t)f(x,w) \\
&\quad + t\phi(y) + (1-t)\phi(w) - \phi(x) \\
&= t\left(f(x,y) + \phi(y) - \phi(x)\right) \\
&\quad + (1-t)\left(f(x,w) + \phi(w) - \phi(x)\right) \\
&= t\Gamma(x,y) + (1-t)\Gamma(x,w).
\end{aligned}$$

Hence, if we repalce the bifunction $f$ in Theorem 0.3 with $\Gamma$ and repeat the argument displayed in the proof of Theorem 0.3, then the result follows. □

## 5. CONCLUDING REMARKS

It is worthy to note that theorem 0.3 improves the corresponding result of He [7] in the following ways: First, Theorem 0.3 approximates solution of *generalized equilibrium problems* and a common fixed point of a finite family of strict pseudocontractions while Theorem 0.1 (He [7]) approximates solution of *classical equilibrium problems* and a common fixed point of a finite family of strict pseudocontractions. Secondly, in each iterate of Theorem 0.3, we are faced with only subminimal problem of computing $z_n$ while Theorem 0.1 (He [7]) poses two subminimal problems of computing $z_n$ and $y_n$, hence each iterate in Theorem 0.1 requires more processing time. Furthermore, the the scheme introduced and studied by He [7] seems difficult to implement due to the summation involved, and the difficulty may be experienced if $N$ is large enough and the mappings $T_i, i = 1, 2, \cdots, N$ are not simple enough.

Furthermore, we observe that Theorem 0.3 complements and improves Theorem 0.2 (Huang and Ma [9]) in the sense that the choice of $\{\alpha_n\}$ in Theorem 0.2 excluded the cannonical choice $\{\frac{1}{n}\}$ and concluded *weak convergence*, which is not that important considering application point of view. Our main result, Theorem 0.3, however, concluded strong convergence which is more applicable than weak convergence. Furthermore, our iterative sequence in theorem 0.3 accommodates approximation of common fixed point of finite family of strict pseudocontractive mappings which is more general than what Huang and Ma [9] studied.

As an application (see Theorem 0.5), the iterative algorithm studied in this paper was shown to be suitable for approximation of solution of *generalized mixed equilibrium problems* which is clearly more general than classical equilibrium problems, variational inequality problem and convex optimization problems. Our theorems therefore improve, complement and unify the results of He [7], Huang and Ma [9] and several other results announced recently.

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## 7. NOMENCLATURE

$\Omega := \bigcap_{i=1}^{N} F(T_i) \bigcap GEP(f, A)$ - the set of common solution fixed point and generalized equilibrium problems studied in this paper.

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