HYBRID STEEPEST ITERATIVE ALGORITHM FOR HIERARCHICAL FIXED POINT PROBLEMS IN UNIFORMLY SMOOTH BANACH SPACES

A. M. HAMZA, BASHIR ALI AND M. H. HARBAU

ABSTRACT. The purpose of this work is to study an iterative method to approximate solutions of Hierarchical fixed point problem involving a finite family of strictly pseudocontractive mappings in uniformly smooth Banach space. The results in this paper are extensions and generalisations of some recent results.

Keywords and phrases: Strong convergence, Hierarchical Fixed Point Problem, Strictly Pseudocontractive Mappings
2010 Mathematical Subject Classification: 47H09, 47J25.

1. INTRODUCTION

Let \( E \) be a real Banach space and \( E^* \) be its dual. Let \( K \) be a nonempty closed convex subset of \( E \). For some real number \( q \) (\( 1 < q < \infty \)), \( J_q \) denotes the generalized duality mapping from \( E \) into \( 2^{E^*} \) given by

\[
J_q(x) = \{ f \in E^* : \langle f, x \rangle = \|x\|^q \text{ and } \|f\| = \|x\|^{q-1} \}\.
\]

\( J_2 \) (when \( q = 2 \)) is called the normalized duality mapping, where \( \langle \cdot, \cdot \rangle \) denotes the duality pairing between the elements of \( E \) and those of \( E^* \). It is well known that if \( E \) is uniformly smooth then \( J_q \) is norm-to-norm uniformly continuous on bounded sets (see, e.g., [2, 3]). If \( E \) is smooth (i.e., \( E^* \) is strictly convex), then \( J_q \) is single-valued (see, [5]).

For a real number \( L > 0 \), a mapping \( T : K \rightarrow E \) is called \( L \)-Lipschitz
if
\[ \|Tx - Ty\| \leq L\|x - y\| \forall x, y \in K. \] (1)

If \( L = 1 \) then \( T \) is called nonexpansive. A point \( x \in K \) is called a fixed point of \( T \) if
\[ Tx = x. \]

We shall denote the set of all fixed points of a nonlinear mapping \( T \) by \( \text{Fix}(T) \) i.e.,
\[ \text{Fix}(T) = \{ x \in K : Tx = x \}. \]

It is well known that (see, [4]) if \( K \) is bounded closed convex subset of a reflexive real Banach space with normal structure and \( T \) is a nonexpansive mapping of \( K \) into itself, then \( \text{Fix}(T) \) is nonempty.

A mapping \( T : E \to E \) is said to be accretive if for all \( x, y \in E \), there exists \( j(x - y) \in J(x - y) \) such that
\[ \langle Tx - Ty, j(x - y) \rangle \geq 0. \]

For some positive real number \( \eta \) the mapping \( T \) is called \( \eta \)-strongly accretive if
\[ \langle Tx - Ty, j(x - y) \rangle \geq \eta\|x - y\|^2 \]
holds \( \forall x, y \in E \) and for some positive real number \( \lambda \) the mapping \( T \) is \( \lambda \)-strictly pseudocontractive if
\[ \langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 - \lambda\|(I - T)x - (I - T)y\|^2 \] (2)
holds \( \forall x, y \in E \). It is also known that if \( T \) is \( \lambda \)-strictly pseudocontractive then it is \( \frac{1 + \lambda}{\lambda} \)-Lipschitzian. If \( I \) denotes the identity operator, then its easy to see that (2) can be written in the form
\[ \langle (I - T)x - (I - T)y, j(x - y) \rangle \geq \lambda\|(I - T)x - (I - T)y\|^2. \] (3)

In Hilbert spaces, (2) (and hence (3)) for \( \lambda \in (0, \frac{1}{2}) \), is equivalent to the inequality
\[ \|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \] (4)
where \( k = (1 - 2\lambda) < 1 \).

The variational inequality problem (VIP) is a problem of finding a point \( x^* \in K \) such that
\[ \langle Tx^*, j_q(y - x^*) \rangle \geq 0 \ \forall y \in K, \] (5)
where \( j_q(y - x^*) \in J_q(y - x^*) \). The set of solutions of variational inequality problem, \( \text{VIP} \) with respect to \( K \) and \( T \) is denoted by \( \text{VIP}(T, K) \). If the mapping \( T \) is monotone, then the \( \text{VIP} \) is called a monotone variational inequality problem (\( \text{MVIP} \)). The theory of variational inequalities is well established in the literature because of its applications in science, engineering, social sciences, and so
HYBRID STEEPEST ITERATIVE ALGORITHM FOR HIERARCHICAL...

forth. For further detail on variational inequalities and their applications, we refer the reader to ([6]-[9]) and [10] and the references contained therein. It is easy to see that in a real Hilbert space $H$, finding $x^* \in VIP(K,T)$ is equivalent to finding the solution of the fixed point equation:

$$x^* = P_K(I - \lambda T)x^*,$$

where $\lambda > 0$ and $P_K$ is the metric projection of $H$ onto $K$.

Also the variational inequality problem is considered over the set of fixed points of a nonexpansive mapping. If we assume $K$ is a fixed point set of a nonexpansive mapping $T$ and $V$ is another non-expansive mapping (not necessarily with fixed point), the problem (5) becomes the VIP of finding $x^* \in Fix(T)$ such that

$$\langle (I - V)x^*, j(y - x^*) \rangle \geq 0 \forall y \in Fix(T), \quad (6)$$

where we assume that $Fix(T) \neq \emptyset$. This problem is called Hierarchical fixed point problem (in short, HFPP). It is known that Finding $x^*$ in the solution set of HFPP is equivalent to finding the solution of the fixed point equation:

$$\text{find } x^* \in K \text{ such that } x^* = P_{Fix(T)}V(x^*).$$

Methods of solving equation (6) has been studied by many authors (see, [12]-[17]) and the references contained therein. Ceng et al. [19] proposed explicit and implicit iterative schemes for finding a common solution for a fixed points of nonexpansive mapping. Buong and Duong in [20] also studied the explicit iterative algorithm for finding the solution of a VIP defined over the set of common fixed points of a finite number of nonexpansive mappings:

$$u_{k+1} = (1 - b_k^0)u_k + b_k^0S_k^0S_p^k \cdots S_1^ku_k$$

where $S_i^k = (1 - b_{ki}^i)I + b_k^i S_i^k$ for $1 \leq i \leq p$, $\{S_i^k\}_{i=1}^p$ are finite family of nonexpansive mappings of a real Hilbert space $H$, $S_0^k = I - \lambda k_\mu F$, and $F$ is an $\eta$-strongly monotone and $L$-Lipschitz continuous mapping.

Recently, Zhang and Yang [18] studied the following more general explicit iterative algorithm for solving HFPP.

$$u_{k+1} = a_kcg(u_k) + (I - \mu akF)S_p^kS_{p-1}^k \cdots S_1^ku_k$$

where $g$ is a contraction, $F$ is an $\eta$-strongly monotone and $L$-Lipschitz continuous mapping and $S_i^k = (1 - b_{ki}^i)I + b_k^i S_i^k$ for $1 \leq i \leq p$. Under some assumptions, the iterative sequence $\{u_k\}$ generated
by (7) strongly converges to the solution of the HFPP, i.e., some $z \in \cap_{i=1}^p F(S_i)$ such that
\[
\langle (\mu F - \gamma g)z, v - z \rangle \geq 0, \forall v \in \cap_{i=1}^p F(S_i).
\] (8)

Very recently Husain and Singh [1] studied Hybrid steepest iterative algorithm for a hierarchical fixed point problem of a finite family of nonexpansive mappings in a real Hilbert space. They studied the algorithm defined as
\[
\begin{aligned}
y_n &= b_n x_n + (1 - b_n)S_1^n S_{p-1}^n ... S_1^n x_n, \\
x_{n+1} &= a_n \rho g(y_n) + c_n y_n + [(1 - c_n)I - a_n \mu F]S_p^n S_{p-1}^n ... S_1^n y_n,
\end{aligned}
\]
\forall n \geq 0, (9)
where $S_i^n = \alpha_i^n I + (1 - \alpha_i^n)S_i$, $\{S_i\}_{i=1}^p$ are nonexpansive mappings and proved that the sequence generated converges strongly to the solution of the HFPP, $z \in \cap_{i=1}^p F(S_i)$ such that
\[
\langle (\mu F - \gamma g)z, v - z \rangle \geq 0, \forall v \in \cap_{i=1}^p F(S_i).
\]

Motivated by the work of Husain and Singh [1], in this paper we studied a modified iterative scheme and prove its strong convergence to a solution of HFPP involving finite family of strictly pseudocontractive maps in a setting of Banach spaces more general than Hilbert. The results presented here improve, extend and generalised some recently announced results.

2. PRELIMINARY

Let $K$ be a nonempty closed convex and bounded subset of a Banach space $E$ and let the diameter of $K$ be defined by $d(K) := \sup\{\|x - y\| : x, y \in K\}$. For each $x \in K$, let $r(x, K) := \sup\{\|x - y\| : y \in K\}$ and let $r(K) := \inf\{r(x, K) : x \in K\}$ denote the Chebyshev radius of $K$ relative to itself. The normal structure coefficient $N(E)$ of $E$ (see, e.g., [11]) is defined by
\[
N(E) := \inf\left\{ \frac{d(K)}{r(K)} : K \subset E \text{ with } d(K) > 0 \right\}.
\]
A space $E$ such that $N(E) > 1$ is said to have uniform normal structure. It is known that all uniformly convex and uniformly smooth Banach spaces have uniform normal structure (see, e.g., [23, 24]) and the references contained therein.

Let $E$ be a real Banach space and $U(E) = \{z \in E : \|z\| = 1\}$. The space $E$ is said to be uniformly convex if for each $\epsilon \in (0, 2)$,
there exists $\delta > 0$ such that $\frac{\|x+y\|}{2} \leq 1 - \delta$ for all $x, y \in U$ with $\|x - y\| \geq \epsilon$. The modulus of smoothness of $E$ is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(t) = \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : x \in U(E), \|y\| \leq t \right\}.$$ 

$E$ is said to be uniformly smooth, if and only if

$$\lim_{t \to 0} \frac{\rho_E(t)}{t} = 0.$$

A Banach space $E$ is said to have Fréchet differentiable norm, if for all $x, y \in U(E)$

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists and is attained uniformly for $y \in U(E)$. $E$ is said to have Gateaux differentiable norm (and $E$ is called smooth) if the limit exists for all $x, y \in U(E)$, and $E$ is said to have a uniformly Gateaux differentiable norm if for each $y \in U$ the limit is attained uniformly for $x \in U$.

It is well known that if $E$ is uniformly smooth, then the norm on $E$ is Fréchet differentiable and the normalized duality map is single-valued and norm to norm uniformly continuous on bounded subsets. A Banach space $E$ is said to be $p-$ uniformly smooth, if there exists a constant $c^* > 0$ such that $\rho_E(t) \leq c^* t^p$ for all $t > 0$. Every $p-$ uniformly smooth Banach space is uniformly smooth.

In the sequel we will make use of the following lemmas.

**Lemma 1:** [26] Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers such that

$$\alpha_{n+1} \leq (1 - w_n)\alpha_n + w_n t_n,$$

where

1. $\{w_n\} \subset [0, 1]$ , $\sum w_n = \infty$
2. $\limsup t_n \leq 0$

Then $\alpha_n \to 0$ as $n \to \infty$.

**Lemma 2:** [27] Let $E$ be real Banach space and $K$ be a closed convex subset of $E$. Let $T : K \to K$ be a nonexpansive mapping with $\text{Fix}(T) \neq \emptyset$. Then the mapping $I - T$ is demiclosed at 0, that is, if $\{x_n\}$ is a sequence converging weakly to $x$ and $\{(I - T)x_n\}$ converges strongly to 0, then $(I - T)x = 0$. 
Lemma 3: [21] Let $E$ be a real Banach space with Frechet differentiable norm. For $x \in E$, let $\beta^*$ be defined by

$$\beta^*(t) = \sup \left\{ \frac{\|x + ty\|^2 + \|y\|^2}{t} - 2\langle y, j(x) \rangle : \|y\| = 1 \right\}$$

Then, $\lim_{t \to 0} \beta^*(t) = 0$ and

$$\|x + h\|^2 \leq \|x\|^2 + 2\langle h, j(x) \rangle + \|h\|\beta^*(\|h\|)$$

for all $h \in E - \{0\}$. If $E = L^p_2$, we know that

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x) \rangle + (p - 1)\|y\|^2, \forall x, y \in E.$$ 

Then $\beta^*$ in (10) is estimated by $\beta^*(t) = (p - 1)t$ for $t > 0$. In our more general setting, throughout this paper, we will assume that

$$\beta^*(t) \leq ct, t > 0,$$

for some $c > 1$ where $\beta^*$ is the function appearing in (10).

Lemma 4: [24] Suppose $E$ is a Banach space with uniform normal structure, $K$ is a nonempty bounded subset of $E$ and $T : K \to K$ is uniformly $L$–Lipschitzian mapping with $L < N(E)^{1/2}$. Suppose also that there exists a nonempty bounded closed convex subset $C$ of $K$ with the following property

$x \in C$ implies $W_w(x) \subset C,$

where $W_w(x)$ is the $w$-limit set of $\{T^n x\}_{n=1}^\infty$ at $x$, that is,

$$W_w(x) = \left\{ y \in E : y = \text{weak} - \lim_j T^{n_j} x \text{ for some subsequence } \{n_j\} \text{ of } \{n\} \right\}.$$

Then $T$ has a fixed point in $C$.

Lemma 5: [25] Let $(a_0, a_1, ...) \in l_\infty$ such that $\mu_n(a_n) \leq 0$ for all Banach limit $\mu$ and $\limsup_{n \to \infty} (a_{n+1} - a_n) \leq 0$. Then $\limsup_{n \to \infty} a_n \leq 0$.

Lemma 6: Let $E$ be a real normed linear space. Then the following inequality holds

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle \forall x, y \in E, \forall j(x + y) \in J(x + y).$$
Lemma 7: [28] Let \( \{x_n\} \) and \( \{y_n\} \) be a bounded sequences in a Banach space \( E \) and let \( \{\beta_n\} \) be a sequence in \([0, 1]\) with \( 0 < \lim inf \beta_n < 1 \). Suppose that \( x_{n+1} = \beta_n y_n + (1 - \beta_n) x_n \) for all integer \( n \geq 1 \) and \( \lim sup(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0 \). Then, \( \lim\|y_n - x_n\| = 0 \).

Lemma 8: Let \( E \) be a real Banach space with Frechet differentiable norm and let \( K \) be a nonempty closed convex subset of \( E \). Suppose that \( T : K \rightarrow K \) is \( \lambda \)-strictly pseudocontractive such that \( F(T) \neq \emptyset \). For any \( \alpha \in (0, 1) \) we define \( T_\alpha : K \rightarrow E \) by \( T_\alpha x = \alpha x + (1 - \alpha)Tx \ \forall x \in K \). Then \( T_\alpha \) is a nonexpansive mapping such that \( Fix(T_\alpha) = Fix(T) \). for \( 1 - \alpha \in (0, \mu) \), \( \mu = \min \left\{ 1, 1 - \frac{2\lambda}{c} \right\} \) with \( \frac{\epsilon}{2} \geq \lambda \) and \( \lambda \in (0, 1) \).

Proof: In view of Lemma 3, we have the following estimate
\[
\|T_\alpha x - T_\alpha y\|^2 = \|\alpha x + (1 - \alpha)Tx - \alpha y - (1 - \alpha)Ty\|^2
\]
\[
= \|\alpha x - \alpha y + (1 - \alpha)(Tx - Ty)\|^2
\]
\[
= \|\alpha(x - y) - (x - y) + (1 - \alpha)(Tx - Ty)\|^2
\]
\[
= \|x - y - (1 - \alpha)(x - y - (Tx - Ty))\|^2
\]
\[
\leq \|x - y\|^2 - 2(1 - \alpha)<(I - T)x - (I - T)y, j(x - y)> + (1 - \alpha)\|x - y - (Tx - Ty)\|^2
\]
\[
\leq \|x - y\|^2 - 2\lambda(1 - \alpha)\|x - y - (Tx - Ty)\|^2 + c(1 - \alpha)^2\|x - y - (Tx - Ty)\|^2
\]
\[
= \|x - y\|^2 - [2\lambda(1 - \alpha) - c(1 - \alpha)^2]\|x - y - (Tx - Ty)\|^2
\]
\[
= \|x - y\|^2 - (1 - \alpha)[2\lambda - c(1 - \alpha)]\|x - y - (Tx - Ty)\|^2
\]
\[
\leq \|x - y\|^2.
\]

3. MAIN RESULTS

Lemma 9: Let \( \{T_i\}_{i=1}^p \) be \( \lambda_i \)-strictly pseudocontractive mappings and let \( \lambda = \max \left\{ \lambda_i \right\} \) and \( \{\alpha_i\}_{i=1}^p \) be a finite real sequences in \((0, 1)\).
Define \( T_\alpha x = \alpha_i x + (1 - \alpha_i)T_i x \) and \( T := T_\alpha \circ T_{\alpha_2} \circ \cdots \circ T_{\alpha_p} \). Then \( T_\alpha \), for each \( i = 1, 2, \ldots p \) and \( T \) are strictly pseudocontractive mappings.

Proof: Consider the following computation
\[
\langle T_\alpha x - T_\alpha y, j(x - y) \rangle
\]
\[
= \langle \alpha_i x + (1 - \alpha_i)T_i x - (\alpha_i y + (1 - \alpha_i)T_i y), j(x - y) \rangle
\]
\[
\begin{align*}
= & \quad \alpha_i \langle x - y, j(x - y) \rangle + (1 - \alpha_i) \langle (T_i x - T_i y), j(x - y) \rangle \\
\leq & \quad \alpha_i \|x - y\|^2 + (1 - \alpha_i) \|x - y\|^2 - \lambda_i \| (I - T_i)x - (I - T_i)y \|^2 \\
= & \quad \|x - y\|^2 - \lambda_i (1 - \alpha_i) \| (I - T_i)x - (I - T_i)y \|^2 \\
= & \quad \|x - y\|^2 - \lambda_i (1 - \alpha_i) \| x - \left( \frac{(T_{\alpha_1} x - \alpha_i x)}{(1 - \alpha_i)} - \left( y - \frac{(T_{\alpha_i} y - \alpha_i y)}{(1 - \alpha_i)} \right) \right) \|^2 \\
= & \quad \|x - y\|^2 - \left( \frac{\lambda_i}{1 - \alpha_i} \right) \| (I - T_{\alpha_i}) x - (I - T_{\alpha_i}) y \|^2.
\end{align*}
\]

Therefore \( T_{\alpha_i} \) is \( \frac{\lambda_i}{1 - \alpha_i} \) strictly pseudocontractive. Also
\[
\langle Tx - Ty, j(x - y) \rangle = \langle T_{\alpha_1} \circ T_{\alpha_2} \circ \cdots \circ T_{\alpha_p} x - T_{\alpha_1} \circ T_{\alpha_2} \circ \cdots \circ T_{\alpha_p} y, j(x - y) \rangle \\
\leq \quad \| T_{\alpha_1} \circ T_{\alpha_2} \circ \cdots \circ T_{\alpha_p} x - T_{\alpha_1} \circ T_{\alpha_2} \circ \cdots \circ T_{\alpha_p} y \|^2 - \frac{\lambda_1}{1 - \alpha_1} \| (I - T_{\alpha_1} \circ T_{\alpha_2} \circ \cdots \circ T_{\alpha_p}) x - (I - T_{\alpha_1} \circ T_{\alpha_2} \circ \cdots \circ T_{\alpha_p}) y \|^2 \\
\leq \quad \|x - y\|^2 - \frac{\lambda_1}{1 - \alpha_1} \| (I - T_{\alpha_1} \circ T_{\alpha_2} \circ \cdots \circ T_{\alpha_p}) x - (I - T_{\alpha_1} \circ T_{\alpha_2} \circ \cdots \circ T_{\alpha_p}) y \|^2
\]

which implies \( T \) is \( \frac{\lambda_i}{1 - \alpha_i} \) strictly pseudocontractive.

**Lemma 10**: Let \( E \) be a real Banach space with Fréchet differentiable norm and let \( K \) be a nonempty closed convex subset of \( E \). Suppose that \( T_i : E \rightarrow E \) is \( \lambda_i \) strictly pseudocontractive for \( i = 1, 2, \ldots, p \), \( \lambda_i \in (0, 1) \) and \( F : E \rightarrow E \) be an \( \eta \)-strongly accretive mapping which is also \( L \)-Lipschitz for some \( L > 0 \). Assume \( c > 1 \), \( \eta \in (0, \frac{2(1-d)+bcL^2 \mu}{2(1-d)}) \) and \( \mu \in (0, \frac{2(1-d)}{acL^2}) \), \( a_n \in (a, b) \subset (0, 1) \). Define \( H : E \rightarrow E \) by \( Hx = [I - \frac{a_n F}{(1-d)} T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1}] x \ \forall x \in E \). Then \( H \) satisfies the following inequality \( \| Hx - Hy \| \leq \left[ 1 - \frac{b h}{(1-d)} \right] \| x - y \|, \) where \( h = \mu \left( \eta - \frac{L^2 bc \mu}{2(1-d)} \right) \).
**Proof:** For any \( x, y \in E \) and Lemma 3, we have

\[
\| Hx - Hy \|^2 = \| \left( I - \frac{a_n \mu F}{1 - d} \right) T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x \|
\]

\[
- \left[ I - \frac{a_n \mu F}{(1 - d)} \right] T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} y \|
\]

\[
= \left\| (T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x - T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} y) - \right. \\
\left. \left( a_n \mu F( T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x) - a_n \mu F( T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} y) \right) \right\|^2 \\
\leq \| T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x - T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} y \|^2 - 2 \left( \frac{a_n \mu}{1 - d} \right) \\
\left\langle F( T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x) - F( T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} y), \\
j( T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x - T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} y) \right\rangle \\
+ \left( \frac{a_n \mu}{1 - d} \right) \| F( T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x) - F( T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} y) \|
\]

\[
= \| T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x - T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} y \|^2 \\
- 2 \left( \frac{a_n \mu}{1 - d} \right) \eta \| T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x - T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} y \|^2 \\
+ c \left( \frac{a_n \mu}{1 - d} \right) \| F( T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x) \\
- F( T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} y) \|^2 \\
\leq \| T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x - T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} y \|^2 \\
- 2 \left( \frac{a_n \mu}{1 - d} \right) \eta \| T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x - T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} y \|^2 \\
+ c \left( \frac{a_n \mu}{1 - d} \right) L^2 \| T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x \\
- T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} y \|^2 \\
= \left( 1 - \frac{2a_n \mu}{(1 - d)} \left( \eta - \frac{L^2 c}{2} \left( \frac{a_n \mu}{1 - d} \right) \right) \right) \\
\| T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x - T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} y \|^2 
\]
\[
\left[ 1 - \frac{a_n \mu}{(1 - d)} \right] \left[ \eta - \frac{L^2 c}{2 (1 - d)} a_n \mu \right] \leq \frac{1}{2} \| x - y \|^2.
\]

Hence, we have
\[
\| Hx - Hy \| \leq \left[ 1 - \frac{bh}{(1 - d)} \right] \| x - y \|.
\]

We now prove the following Theorem

**Theorem 1:** Let \( E \) be a real uniformly smooth Banach space and let \( K \) be a nonempty closed convex subset of \( E \). Suppose that \( T_i : E \rightarrow E \) is \( \lambda_i \)-strictly pseudocontractive mappings for \( i = 1, 2, ..., p \) such that \( \Omega = \cap_{n=1}^p \text{Fix}(T_i) \neq \emptyset \) and let \( F : E \rightarrow E \) be an \( \eta \)-strongly accretive mapping which is also \( L \)-Lipschitz. Let \( g : E \rightarrow E \) be a contraction with contractive constant \( \tau \in (0, 1) \).

For an arbitrary \( x_0 \in E \), define a sequence \( \{x_n\} \) by

\[
\begin{align*}
\{ y_n \} &= b_n x_n + (1 - b_n) T_{\alpha_1} \cdots T_{\alpha_p} \cdots T_{\alpha_1} x_n, \\
x_{n+1} &= a_n \rho g(y_n) + dx_n + [(1 - d)I - a_n \mu F] T_{\alpha_1} \cdots T_{\alpha_p} \cdots T_{\alpha_1} x_n, \\
\forall n \geq 0,
\end{align*}
\]

(11)

where \( T_{\alpha_i} = \alpha_i x + (1 - \alpha_i) T_i x \) and \( \alpha_i \in (0, 1 - \lambda) \) where \( \lambda = Max \{ \lambda_i \} \)

and the parameters \( \rho, h, d \) satisfy

\[ 0 < \rho < \frac{h}{\tau}, \]

\[ \mu \in (0, \frac{2(1-d)}{a c L^2}) \]

and \( h = \mu \left( \eta - \frac{2L^2 c}{2(1-d)} \right), d \in (0, 1), \) and \( \{a_n\}, \{b_n\} \)

are sequences in \((0, 1)\) satisfying the following condition:

\[
\begin{align*}
(1) \quad &\lim_{n \rightarrow \infty} a_n = 0, \quad \sum_{n=1}^{\infty} a_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty, \\
(2) \quad &b_n \in \left( \frac{2d}{1 + d}, 1 \right) \quad \text{and} \quad \lim_{n \rightarrow \infty} |b_{n+1} - b_n| = 0, \\
(3) \quad &\frac{2 \lambda_1}{1 - \alpha_1} > \frac{2(1-d)}{1 + d}
\end{align*}
\]

Then the sequence generated by algorithm (11) converges strongly to \( z \in \cap_{n=1}^p \text{Fix}(T_i) \), which is also a solution of the HFPP:

\[
\langle (\mu F - \rho g) z, j(x - z) \rangle \geq 0 \quad \forall x \in \cap_{i=1}^p \text{Fix}(T_i).
\]

(12)

**Proof:** We split the proof into four steps.

**STEP 1** We show that \( \{y_n\}, \{x_n\} \) are bounded.
Let \( x^* \in \Omega \). From the scheme (11) and Lemma 3, we have
\[
\|y_n - x^*\|^2 = \|b_n x_n + (1 - b_n) T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x_n - x^*\|^2
\]
\[
= \|x_n - x^* - (1 - b_n)(x_n - T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x_n)\|^2
\]
\[
\leq \|x_n - x^*\|^2 - 2(1 - b_n) \langle x_n - x^*, j(x_n - x^*) \rangle + (1 - b_n) \|x_n - T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x_n\| \times
\]
\[
\beta^*((1 - b_n)\|x_n - T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x_n\|)
\]
\[
\leq \|x_n - x^*\|^2 - 2(1 - b_n)
\times \langle x_n - x^* + x - T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x_n, j(x_n - x^*) \rangle + c(1 - b_n)^2 \|x_n - T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x_n\|^2
\]
\[
= \|x_n - x^*\|^2 - 2(1 - b_n) \langle x_n - x^*, j(x_n - x^*) \rangle + 2(1 - b_n)
\times \langle T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x_n - x^*, j(x_n - x^*) \rangle + c(1 - b_n)^2 \|x_n - T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x_n\|^2
\]
\[
\leq \|x_n - x^*\|^2 - 2(1 - b_n) \|x_n - x^*\|^2 + 2(1 - b_n)\|x_n - x^*\|^2 - \frac{\lambda_1}{1 - \alpha_1} \|(I - T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1}) x_n - (I - T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1}) x^*\|^2
\]
\[
+ c(1 - b_n)^2 \|x_n - T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x_n\|^2
\]
\[
= \|x_n - x^*\|^2 - 2\frac{\lambda_1}{1 - \alpha_1} (1 - b_n) \|x_n - T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x_n\|^2 + c(1 - b_n)^2 \|x_n - T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x_n\|^2
\]
\[
= \|x_n - x^*\|^2 - (1 - b_n) \left(\frac{2\lambda_1}{1 - \alpha_1} - c(1 - b_n)\right)
\times \|x_n - T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x_n\|^2.
\]
So
\[
\|y_n - x^*\|^2 \leq \|x_n - x^*\|^2. \tag{13}
\]
We show that
$$\lim_{n \to \infty} x_n = x^*$$

Hence by induction,
$$\lim_{n \to \infty} \left( x_n - x^* \right) = 0$$

Therefore, \( \{ x_n \} \) and \( \{ y_n \} \) are bounded.

**STEP 2** We show that
(i) \( \lim_{n \to \infty} \| x_{n+1} - x_n \| = 0 \) and
(ii) \( \lim_{n \to \infty} \| x_n - T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x_n \| = 0 \).
Define
\[ z_n = \frac{x_{n+1} - dx_n}{1 - d}. \]

Thus,
\[
\|z_{n+1} - z_n\| = \left\| \frac{x_{n+2} - dx_{n+1}}{1 - d} - \frac{x_{n+1} - dx_n}{1 - d} \right\|
\]
\[
= \left\| \frac{a_{n+1} \rho g(y_{n+1}) + dx_{n+1}}{(1 - d)} - \frac{[(1 - d)I - a_{n+1} \mu F]T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}y_{n+1}}{(1 - d)} \right.
\]
\[
- \frac{a_n \rho g(y_n) - [(1 - d)I - a_n \mu F]T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}y_n}{(1 - d)}
\]
\[
+ \left. \frac{dx_n + [(1 - d)I - a_n \mu F]T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}y_n - dx_n}{(1 - d)} \right\|
\]
\[
= \left\| \frac{a_{n+1} \rho g(y_{n+1}) + [(1 - d)I - a_{n+1} \mu F]T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}y_{n+1}}{(1 - d)} \right.
\]
\[
- \frac{a_n \rho g(y_n) - [(1 - d)I - a_n \mu F]T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}y_n}{(1 - d)}
\]
\[
+ \frac{T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}y_{n+1} - T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}y_n}{(1 - d)}
\]
\[
+ \frac{a_n \mu FT_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}y_n}{1 - d} - \frac{a_{n+1} \mu FT_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}y_{n+1}}{(1 - d)}
\]
\[
\bigg\| \frac{a_{n+1} \rho g(y_{n+1})}{(1 - d)} - \frac{a_n \rho g(y_n)}{(1 - d)} \bigg\|
\]
\[
+ \frac{T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}y_{n+1} - T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}y_n}{(1 - d)}
\]
\[
+ \frac{a_n \mu FT_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}y_n}{(1 - d)} - \frac{a_{n+1} \mu FT_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}y_{n+1}}{(1 - d)}
\]
\[
\bigg\| \frac{a_{n+1} \rho g(y_{n+1})}{(1 - d)} - \frac{a_n \rho g(y_n)}{(1 - d)} \bigg\|
\]
\[
+ \frac{T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}y_{n+1} - T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}y_n}{(1 - d)}
\]
\[
+ \frac{a_n \mu FT_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}y_n}{(1 - d)} - \frac{a_{n+1} \mu FT_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}y_{n+1}}{(1 - d)}
\]
\[
\begin{align*}
&\leq \frac{a_{n+1}}{1-d} \left| \rho g(y_{n+1}) - \rho g(y_n) \right| \\
&\quad + \frac{a_{n+1} - a_n}{1-d} \left| \rho g(y_n) \right| + \left| y_{n+1} - y_n \right| \\
&\quad + \frac{a_n \mu}{1-d} \left| \mu \right| \left| FT_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} y_n - FT_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} y_{n+1} \right| \\
&\leq \frac{a_{n+1}}{1-d} \rho^T \left| y_{n+1} - y_n \right| + \frac{a_{n+1} - a_n}{1-d} \left| \rho g(y_n) \right| + \left| y_{n+1} - y_n \right| \\
&\quad + \frac{a_n \mu}{1-d} \left| y_n - y_{n+1} \right| \\
&\quad + \frac{a_n - a_{n+1}}{1-d} \left| \mu \right| \left| FT_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} y_{n+1} \right|
\end{align*}
\]

But
\[
\left| y_{n+1} - y_n \right| = \left| b_{n+1} x_{n+1} + (1 - b_{n+1}) T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x_n \right|
\]
\[
= \left| (1 - b_{n+1})(T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x_{n+1}) - T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x_n \right|
\]
\[
= \left| (1 - b_{n+1})(T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x_{n+1}) - T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x_n \right|
\]
\[
\left| b_{n+1} - b_n \right| \left| T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x_n \right|
\]
\[
\leq \left| b_{n+1} \right| \left| x_{n+1} - x_n \right| + \left| b_{n+1} - b_n \right| \left| T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x_n \right|
\]
\[
= \left| x_{n+1} - x_n \right| + \left| b_{n+1} - b_n \right| \left| T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x_n - x_n \right|
\]
So
\[
\|z_{n+1} - z_n\| \leq \frac{a_{n+1}}{1-d}\rho r\|y_{n+1} - y_n\|
\]
\[+ \frac{a_{n+1} - a_n}{1-d}\|\rho g(y_n)\| + \|y_{n+1} - y_n\|
\]
\[+ \frac{a_n\mu}{1-d}L\|y_n - (y_{n+1})\|
\]
\[+ \frac{a_n - a_{n+1}}{1-d}\|\mu\|FT_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}y_{n+1}\|
\]
Therefore
\[
\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|
\]
\[\leq \frac{a_{n+1}}{1-d}\rho r\|y_{n+1} - y_n\|
\]
\[+ \frac{a_{n+1} - a_n}{1-d}\|\rho g(y_n)\|
\]
\[+ |b_{n+1} - b_n|\|T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}x_n - x_n\|
\]
\[+ \frac{a_n\mu}{1-d}\|F(y_n) - F(y_{n+1})\| + \frac{a_n - a_{n+1}}{1-d}\|\mu\|FT_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}y_{n+1}\|
\]
Hence,
\[
\limsup_{n \to \infty}\left(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|\right) \leq 0.
\]
Thus by Lemma 7
\[
\lim_{n \to \infty}\|z_n - x_n\| = 0.
\]
This implies
\[
\lim_{n \to \infty}\|x_{n+1} - x_n\| = (1-d)\lim_{n \to \infty}\|z_n - x_n\| = 0. \quad (14)
\]
Now
\[
\|x_n - T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}x_n\|
\]
\[= \|x_n - x_{n+1} + x_{n+1} - T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}x_n\|
\]
\[\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}x_n\|.
\]
Thus,

\[
\begin{align*}
\|x_n - x_{n+1}\| &+ \|a_n \rho g(y_n) + dx_n + \\
[(1-d)I - a_n \mu F]T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}y_n - T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}x_n\|
\end{align*}
\]

\[
\leq \|x_n - x_{n+1}\| + a_n \|\rho g(y_n) - \mu F T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}y_n\|
\]

\[
+ d\|x_n - T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}x_n\|
\]

\[
+ d\|T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}y_n - T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}x_n\|
\]

\[
\leq \|x_n - x_{n+1}\| + a_n \|\rho g(y_n) - \mu F T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}y_n\|
\]

\[
+ d\|x_n - T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}x_n + T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}x_n - T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}x_n\| + \|y_n - x_n\|
\]

Notice that

\[
\|y_n - x_n\| = (1 - b_n)\|T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}x_n - x_n\|
\]

Therefore

\[
\|x_n - T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}x_n\| \leq \|x_n - x_{n+1}\|
\]

\[
+ a_n \|\rho g(y_n) - \mu F T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}y_n\|
\]

\[
+ d\|x_n - T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}x_n\| + (1 + d)(1 - b_n) \times
\]

\[
\|T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}x_n - x_n\|
\]

\[
= \|x_n - x_{n+1}\| + a_n \|\rho g(y_n) - \mu F T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}y_n\|
\]

\[
(1 + 2d - b_n(1 + d))\|T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}x_n - x_n\|
\]

Thus,

\[
(b_n(1 + d) - 2d)\|x_n - T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}x_n\|
\]

\[
\leq \|x_n - x_{n+1}\| + a_n \|\rho g(y_n) - \mu F T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}y_n\|
\]

From condition (1), (2) and equation (14), we obtain

\[
\lim_{n \to \infty} \|x_n - T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}x_n\| = 0.
\]

STEP 3 Next we show that for \(z \in \Omega\),

\[
limsup_{n \to \infty} \langle \rho g(z) - \mu F(z), j(x_n - z) \rangle \leq 0,
\]
Define a map \( \phi : E \to \mathbb{R} \) by
\[
\phi(y) = \mu_n \| x_n - y \|^2 \quad \forall y \in E.
\]
Then \( \phi(y) \to \infty \) as \( \| y \| \to \infty \), \( \phi \) is continuous and convex, so as \( E \) is reflexive, there exists \( q \in E \) such that \( \phi(q) = \min_{u \in E} \phi(u) \). Hence the set
\[
K^* := \left\{ y \in E : \phi(y) = \min_{u \in E} \phi(u) \right\} \neq \emptyset
\]
Since \( \lim_{n \to \infty} \| x_n - T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x_n \| = 0 \) and by Lemma 4 we obtain \( K^* \cap \bigcap_{i=1}^{p} \text{Fix}(T_i) \neq \emptyset \).

Without loss of generality, assume that \( y^* = z \in K^* \cap \Omega \). let \( t \in (0, 1) \), then it follows that
\[
\phi(z) \leq \phi(z + t(\rho g - \mu F)z)
\]
and using Lemma 6
\[
\| x_n - z - t(\rho g - \mu F)z \|^2 \leq \| x_n - z \|^2 - 2t(\langle \rho g - \mu F, z \rangle, j(x_n - z + t(\rho g - \mu F)z)).
\]
Thus, taking Banach limit of both sides, we have
\[
\mu_n \| x_n - z - t(\rho g - \mu F)z \|^2 \leq \mu_n \| x_n - z \|^2 - 2t\mu_n \langle (\rho g - \mu F), j(x_n - z + t(\rho g - \mu F)z) \rangle.
\]
This implies
\[
2t\mu_n \langle (\rho g - \mu F), j(x_n - z + t(\rho g - \mu F)z) \rangle \\
\leq \mu_n \| x_n - z \|^2 - \mu_n \| x_n - z - t(\rho g - \mu F)z \|^2 \\
= \phi(z) - \phi(z + t(\rho g - \mu F)z) \leq 0.
\]
Therefore,
\[
\mu_n \langle (\rho g - \mu F), j(x_n - z + t(\rho g - \mu F)z) \rangle \leq 0 \quad (15)
\]
Thus, by equation (15), we have
\[
\mu_n \langle (\rho g - \mu F)z, j(x_n - z) \rangle \\
= \mu_n \langle (\rho g - \mu F)z, j(x_n - z) - j(x_n - z + t(\rho g - \mu F)z) \rangle \\
+ \mu_n \langle (\rho g - \mu F)z, j(x_n - z - t(\rho g - \mu F)z) \rangle \\
\leq \mu_n \langle (\rho g - \mu F)z, j(x_n - z) - j(x_n - z + t(\rho g - \mu F)z) \rangle \\
\]
Since \( j \) is norm to norm uniformly continuous on bounded subset of \( E \), we have
\[
\mu_n \langle (\rho g - \mu F)z, j(x_n - z) \rangle \leq 0.
\]
Furthermore since \( \| x_{n+1} - x_n \| \to 0 \) as \( n \to \infty \), we have
\[
\limsup_{n \to \infty} \left[ \langle (\rho g - \mu F)z, j(x_n - z) \rangle - \langle (\rho g - \mu F)z, j(x_{n+1} - z) \rangle \right] \leq 0
\]
and so we obtain by Lemma 5
\[
\limsup_{n \to \infty} \langle (\rho g - \mu F)z, j(x_n - z) \rangle \leq 0.
\]

**STEP 4** Finally we show that \( x_n \to z \) as \( n \to \infty \).

\[
\|x_{n+1} - z\|^2 = \|a_n \rho g(y_n) + dx_n + [(1 - d)I - a_n \mu F]T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}y_n - z\|^2
\]
\[
= \langle a_n \rho g(y_n) + dx_n + [(1 - d)I - a_n \mu F] \times T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}y_n - z, j(x_{n+1} - z) \rangle
\]
\[
= \langle a_n (\rho g(y_n) - \mu F(z)) + d(x_n - z) + (1 - d) \times \left[ \left( I - \frac{a_n \mu F}{1 - d} \right) T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}y_n - \left( I - \frac{a_n \mu F}{1 - d} \right) z, j(x_{n+1} - z) \right] \rangle
\]
\[
= a_n \langle (\rho g(y_n) - \mu F(z)), j(x_{n+1} - z) \rangle + \langle d(x_n - z), j(x_{n+1} - z) \rangle + (1 - d) \left\langle \left( I - \frac{a_n \mu F}{1 - d} \right) T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}y_n - \left( I - \frac{a_n \mu F}{1 - d} \right) z, j(x_{n+1} - z) \right\rangle
\]
\[
\leq a_n \rho \|g(y_n) - g(z)\| \|x_{n+1} - z\| + a_n \langle (\rho g(z) - \mu F(z)), j(x_{n+1} - z) \rangle + d \|x_n - z\| \|x_{n+1} - z\| + (1 - d) \left\langle \left( I - \frac{a_n \mu F}{1 - d} \right) T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1}y_n - \left( I - \frac{a_n \mu F}{1 - d} \right) z, \|x_{n+1} - z\| \right\rangle
\]
HYBRID STEEPEST ITERATIVE ALGORITHM FOR HIERARCHICAL . .

\[ a_n \rho \tau \| y_n - z \| \| x_{n+1} - z \| + \langle a_n \rho g(z) - \mu F(z), j(x_{n+1} - z) \rangle \\
+ d \| x_n - z \| \| x_{n+1} - z \| \\
+(1 - d)(1 - \frac{a_n h}{1 - d}) \| y_n - z \| \| x_{n+1} - z \| \\
= a_n \rho \tau \| y_n - z \| \| x_{n+1} - z \| + a_n \langle (\rho g(z) - \mu F(z)), j(x_{n+1} - z) \rangle \\
+ d \| x_n - z \| \| x_{n+1} - z \| + [1 - d] \| y_n - z \| \| x_{n+1} - z \|
\]

\[ n h \| y_n - z \| \| x_{n+1} - z \| \\
= [1 + a_n \rho \tau - d - a_n h] \| y_n - z \| \| x_{n+1} - z \|
\]

\[ d \| x_n - z \| \| x_{n+1} - z \| \\
+ a_n \langle (\rho g(z) - \mu F(z)), j(x_{n+1} - z) \rangle \\
\leq [1 + a_n \rho \tau - d - a_n h] \| x_n - z \| \| x_{n+1} - z \| + \\
2 \| x_n - z \| \| x_{n+1} - z \|
\]

\[ a_n \langle (\rho g(z) - \mu F(z)), j(x_{n+1} - z) \rangle \\
\leq \left[ 1 - a_n(h - \rho \tau) \right] \| x_n - z \| \| x_{n+1} - z \|
\]

\[ a_n \langle (\rho g(z) - \mu F(z)), j(x_{n+1} - z) \rangle \\
\leq \left[ 1 - \frac{(1 - a_n(h - \rho \tau)}{2} \right] \| x_{n+1} - z \|^2 + \\
\frac{(1 - a_n(h - \rho \tau))}{2} \| x_n - z \|^2 + \\
a_n \langle (\rho g(z) - \mu F(z)), j(x_{n+1} - z) \rangle \\
\]

Therefore,

\[ 1 - \frac{(1 - a_n(h - \rho \tau)}{2} \right] \| x_{n+1} - z \|^2 \\
\leq \frac{1 - a_n(h - \rho \tau)}{2} \| x_n - z \|^2 + \\
a_n \langle (\rho g(z) - \mu F(z)), j(x_{n+1} - z) \rangle,
\]

that is,

\[ \left[ 1 + a_n(h - \rho \tau) \right] \| x_{n+1} - z \|^2 \\
\leq \frac{(1 - a_n(h - \rho \tau))}{2} \| x_n - z \|^2 + \\
a_n \langle (\rho g(z) - \mu F(z)), j(x_{n+1} - z) \rangle \\
\]
which implies
\[
\|x_{n+1} - z\|^2 \leq \left[ \frac{1 - a_n(h - \rho \tau)}{1 + a_n(h - \rho \tau)} \right] \|x_n - z\|^2 + \left( \frac{2a_n}{(1 + a_n(h - \rho \tau))} \right) \langle \rho g(z) - \mu F(z), j(x_{n+1} - z) \rangle
\]

and by Lemma 1, we have that \( x_n \to z \) as \( n \to \infty \). This completes the proof.

If \( E \) is a real Hilbert space \( H \), Theorem 1 reduces to the following corollary.

**Corollary 1:** Let \( H \) be a real Hilbert space and let \( K \) be a nonempty closed convex subset of \( E \). Suppose that \( T_i : H \to H \) is \( \lambda_i \)-strictly pseudocontractive mappings for \( i = 1, 2, ..., p \) such that \( \Omega = \cap_{i=1}^{p} Fix(T_i) \neq \emptyset \) and let \( F : H \to H \) be an \( \eta \)-strongly monotone mapping which is also \( L \)-Lipschitz. Let \( g : H \to H \) be a contraction with contractive constant \( \tau \in (0, 1) \). For an arbitrary \( x_0 \in H \), define a sequence \( \{x_n\} \) by

\[
\begin{align*}
  y_n &= b_n x_n + (1 - b_n) T_p T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x_n, \\
  x_{n+1} &= a_n \rho g(y_n) + d x_n + [(1 - d) I - a_n \mu F] T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} y_n \\
  \forall n \geq 0,
\end{align*}
\]

(16)

where \( T_{\alpha_i} = \alpha_i x + (1 - \alpha_i) T_i x \) and \( \alpha_i \in (0, 1 - \lambda) \) for \( i = 1, 2, ..., p \), let the parameters \( \rho, h, d \) satisfy \( 0 < \rho < \frac{h}{\tau} \) and \( h = \mu \left( \eta - \frac{L^2}{2} \frac{\alpha \mu}{(1 - d)} \right) \), \( d \in (0, 1) \), and \( \{a_n\}, \{b_n\} \) are sequences in \((0,1)\) satisfying the following condition:

1. \( \lim_{n \to \infty} a_n = 0 \), \( \sum_{n=1}^{\infty} a_n = \infty \) and \( \sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty \)
2. \( b_n \in \left( \frac{2d}{1+d}, 1 \right) \) and \( \lim_{n \to \infty} |b_{n+1} - b_n| = 0 \)
3. \( \frac{2\alpha_i}{1 - \alpha_i} > \frac{2(1 - d)}{1+d} \)

Then the sequence generated by algorithm (16) converges strongly to \( z \in \cap_{i=1}^{p} Fix(T_i) \), which is also solution of the HFPP:

\[
\langle (\rho g - \mu F) z, x - z \rangle \geq 0 \quad \forall x \in \cap_{i=1}^{p} Fix(T_i).
\]
4. CONCLUDING REMARKS

Theorem 1 generalizes the results of Husain and Singh [1] from finite family of nonexpansive mappings in real Hilbert space to finite family of strictly pseudocontractive mappings in real uniformly smooth Banach space.

REFERENCES


SCHOOL OF CONTINUING EDUCATION, BAYERO UNIVERSITY, KANO, NIGERIA
E-mail address: hamzahadi84@yahoo.com

DEPARTMENT OF MATHEMATICAL SCIENCES, BAYERO UNIVERSITY, KANO, NIGERIA
E-mail address: bashiralik@yahoo.com

DEPARTMENT OF SCIENCE AND TECHNOLOGY EDUCATION, BAYERO UNIVERSITY, KANO NIGERIA
E-mail address: murtalabarbau@yahoo.com