Journal of the Nigerian Mathematical Society
Vol. 38, Issue 3, pp. 447-468, 2019

Special Issue: The International Conference on Non-linear Analysis (organized to mark the 70th birthday anniversary of Prof. Charles Ejike Chidume)

HYBRID STEEPEST ITERATIVE ALGORITHM FOR HIERARCHICAL FIXED POINT PROBLEMS IN UNIFORMLY SMOOTH BANACH SPACES

A. M. HAMZA, BASHIR ALI AND M. H. HARBAU[1]

ABSTRACT. The purpose of this work is to study an iterative method to approximate solutions of Hierarchical fixed point problem involving a finite family of strictly pseudocontractive mappings in uniformly smooth Banach space. The results in this paper are extensions and genelisations of some recent results.

Keywords and phrases: Strong convergence, Hierarchical Fixed Point Problem, Strictly Pseudocontractive Mappings
2010 Mathematical Subject Classification: 47H09, 47J25.

1. INTRODUCTION

Let E be a real Banach space and E^* be its dual. Let K be a nonempty closed convex subset of E. For some real number q $(1 < q < \infty)$, J_q denotes the generalized duality mapping from E into 2^{E^*} given by

$$J_q(x) = \{f \in E^* : \langle f, x\rangle = \|x\|^q \text{ and } \|f\| = \|x\|^{q-1}\}.$$

J_2 (when $q = 2$) is called the normalized duality mapping, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between the elements of E and those of E^*. It is well known that if E is uniformly smooth then J_q is norm-to-norm uniformly continuous on bounded sets (see, e.g., [2, 3]). If E is smooth (i.e., E^* is strictly convex), then J_q is single-valued (see, [5]).

For a real number $L > 0$, a mapping $T : K \to E$ is called $L-$Lipschitz

Received by the editors December 26, 2018; Revised September 02, 2019 ; Accepted: December 14, 2019
www.nigerianmathematicalsociety.org; Journal available online at https://ojs.ictp.it/jnms/
[1]Corresponding author

if

$$\|Tx - Ty\| \leq L\|x - y\| \; \forall x, y \in K. \tag{1}$$

If $L = 1$ then T is called nonexpansive. A point $x \in K$ is called a fixed point of T if

$$Tx = x.$$

We shall denote the set of all fixed points of a nonlinear mapping T by $Fix(T)$ i.e.,

$$Fix(T) = \{x \in K : Tx = x\}.$$

It is well known that (see, [4]) if K is bounded closed convex subset of a reflexive real Banach space with normal structure and T is a nonexpansive mapping of K into itself, then Fix(T) is nonempty.

A mapping $T : E \to E$ is said to be accretive if for all $x, y \in E$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \geq 0.$$

For some positive real number η the mapping T is called η-strongly accretive if

$$\langle Tx - Ty, j(x - y)\rangle \geq \eta\|x - y\|^2$$

holds $\forall \; x, y \in E$ and for some positive real number λ the mapping T is λ-strictly pseudocontractive if

$$\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2 - \lambda\|(I - T)x - (I - T)y\|^2 \tag{2}$$

holds $\forall x, y \in E$. It is also known that if T is λ-strictly pseudocontractive then it is $\frac{1+\lambda}{\lambda}$-Lipschitzian. If I denotes the identity operator, then its easy to see that (2) can be written in the form

$$\langle (I - T)x - (I - T)y, j(x - y)\rangle \geq \lambda\|(I - T)x - (I - T)y\|^2. \tag{3}$$

In Hilbert spaces, (2) (and hence (3)) for $\lambda \in (0, \frac{1}{2})$, is equivalent to the inequality

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \tag{4}$$

where $k = (1 - 2\lambda) < 1$.

The variational inequality problem (VIP) is a problem of finding a point $x^* \in K$ such that

$$\langle Tx^*, j_q(y - x^*)\rangle \geq 0 \; \forall y \in K, \tag{5}$$

where $j_q(y - x^*) \in J_q(y - x^*)$. The set of solutions of variational inequality problem, VIP with respect to K and T is denoted by $VIP(T, K)$. If the mapping T is monotone, then the VIP is called a monotone variational inequality problem ($MVIP$). The theory of variational inequalities is well established in the literature because of its applications in science, engineering, social sciences, and so forth. For further detail on variational inequalities and their applications, we refer the reader to ([6]- [9]) and [10] and the references contained therein. It is easy to see that in a real Hilbert space H, finding $x^* \in VIP(K,T)$ is equivalent to finding the solution of the fixed point equation:

$$x^* = P_K(I - \lambda T)x^*,$$

where $\lambda > 0$ and P_K is the metric projection of H onto K.

Also the variational inequality problem is considered over the set of fixed points of a nonexpansive mapping. If we assume K is a fixed point set of a nonexpansive mapping T and V is another nonexpansive mapping (not necessarily with fixed point), the problem (5) becomes the VIP of finding $x^* \in Fix(T)$ such that

$$\langle (I - V)x^*, j(y - x^*)\rangle \geq 0 \ \forall y \in Fix(T), \tag{6}$$

where we assume that $Fix(T) \neq \emptyset$. This problem is called Hierarchical fixed point problem (in short, $HFPP$). It is known that Finding x^* in the solution set of $HFPP$ is equivalent to finding the solution of the fixed point equation:

$$find\ x^* \in K\ such\ that\ x^* = P_{Fix(T)}V(x^*).$$

Methods of solving equation (6) has been studied by many authors (see, [12]- [17]) and the refrences contained therein. Ceng et al. [19] proposed explicit and implicit iterative schemes for finding a common solution for a fixed points of nonexpansive mapping. Buong and Duong in [20] also studied the explicit iterative algorithm for finding the solution of a VIP defined over the set of common fixed points of a finite number of nonexpansive mappings:

$$u_{k+1} = (1 - b_k^0)u_k + b_k^0 S_0^k S_p^k \cdots S_1^k u_k$$

where $S_i^k = (1 - b_k^i)I + b_k^i S^i$ for$1 \leq i \leq p$, $\{S_i\}_{i=1}^p$ are finite family of nonexpansive mappings of a real Hilbert space H, $S_0^k = I - \lambda k \mu F$, and F is an η-strongly monotone and L-Lipschitz continuous mapping.

Recently, Zhang and Yang [18] studied the following more general explicit iterative algorithm for solving HFPP.

$$u_{k+1} = a_k c g(u_k) + (I - \mu a_k F) S_p^k S_{p-1}^k \cdots S_1^k u_k \tag{7}$$

where g is a contraction, F is an η-strongly monotone and L-Lipschitz continuous mapping and $S_i^k = (1 - b_k^i)I + b_k^i S_i$ for $1 \leq i \leq p$. Under some assumptions, the iterative sequence $\{u_k\}$ generated by (7) strongly converges to the solution of the HFPP, i.e., some $z \in \cap_{i=1}^{p} F(S_i)$ such that

$$\langle (\mu F - \gamma g)z, v - z\rangle \geq 0, \forall v \in \cap_{i=1}^{p} F(S_i). \tag{8}$$

Very recently Husain and Singh [1] studied Hybrid steepest iterative algorithm for a hierarchical fixed point problem of a finite family of nonexpansive mappings in a real Hilbert space. They studied the algorithm defined as

$$\begin{cases} y_n = b_n x_n + (1-b_n) S_p^n S_{p-1}^n ... S_1^n x_n, \\ x_{n+1} = a_n \rho g(y_n) + c_n y_n + [(1-c_n)I - a_n \mu F] S_p^n S_{p-1}^n ... S_1^n y_n, \\ \forall n \geq 0, \end{cases} \tag{9}$$

where $S_i^n = \alpha_n^i I + (1-\alpha_n^i) S_i$, $\{S_i\}_{i=1}^{p}$ are nonexpansive mappings and proved that the sequence generated converges strongly to the solution of the HFPP, $z \in \cap_{i=1}^{p} F(S_i)$ such that

$$\langle (\mu F - \gamma g)z, v - z\rangle \geq 0, \forall v \in \cap_{i=1}^{p} F(S_i).$$

Motivated by the work of Husain and Singh [1], in this paper we studied a modified iterative scheme and prove its strong convergence to a solution of HFPP involving finite family of strictly pseudocontractive maps in a setting of Banach spaces more general than Hilbert. The results presented here improve, extend and generalised some recently annouced results.

2. PRELIMINARY

Let K be a nonempty closed convex and bounded subset of a Banach space E and let the diameter of K be defined by $d(K) := sup\{\|x-y\| : x, y \in K\}$. For each $x \in K$, let $r(x, K) := sup\{\|x-y\| : y \in K\}$ and let $r(K) := inf\{r(x, K) : x \in K\}$ denote the Chebyshev radius of K relative to itself. The normal structure coefficient $N(E)$ of E (see, e.g, [11]) is defined by

$$N(E) := inf\left\{\frac{d(K)}{r(K)} : K \subset E \ with \ d(K) > 0\right\}.$$

A space E such that $N(E) > 1$ is said to have uniform normal structure. It is known that all uniformly convex and uniformly smooth Banach spaces have uniform normal structure (see, e.g, [23, 24]) and the refrences contained therein.

Let E be a real Banach space and $U(E) = \{z \in E : \|z\| = 1\}$. The space E is said to be uniformly convex if for each $\epsilon \in (0, 2)$, there exists $\delta > 0$ such that $\frac{\|x+y\|}{2} \leq 1 - \delta$ for all $x, y \in U$ with $\|x - y\| \geq \epsilon$. The *modulus of smoothness* of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(t) = sup\Big\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : x \in U(E), \|y\| \leq t \Big\}.$$

E is said to be uniformly smooth, if and only if

$$\lim_{t \to 0} \frac{\rho_E(t)}{t} = 0.$$

A Banach space E is said to have Fr*é*chet differentiable norm, if for all $x, y \in U(E)$

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists and is attained uniformly for $y \in U(E)$. E is said to have Gateax differentiable norm (and E is called smooth) if the limit exists for all $x, y \in U(E)$, and E is said to have a uniformly Gateaux differentiable norm if for each $y \in U$ the limit is attained uniformly for $x \in U$.

It is well known that if E is uniformly smooth, then the norm on E is Fr*é*chet differentiable and the normalized duality map is singled-valued and norm to norm uniformly continuous on bounded subsets. A Banach space E is said to be $p-$ uniformly smooth, if there exists a constant $c^* > 0$ such that $\rho_E(t) \leq c^* t^p$ for all $t > 0$. Every $p-$ uniformly smooth Banach space is unifomly smooth.

In the sequel we will make use of the following lemmas.

Lemma 1: [26] Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers such that

$$\alpha_{n+1} \leq (1 - w_n)\alpha_n + w_n t_n,$$

where

(1) $\{w_n\} \subset [0, 1]$, $\sum w_n = \infty$
(2) $limsup\ t_n \leq 0$

Then $\alpha_n \to 0$ as $n \to \infty$.

Lemma 2: [27] Let E be real Banach space and K be a closed convex subset of E. Let $T : K \to K$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. Then the mapping $I - T$ is demiclosed at 0, that is, if $\{x_n\}$ is a sequence converging weakly to x and $\{(I - T)x_n\}$ converges strongly to 0, then $(I - T)x = 0$.

Lemma 3: [21] Let E be a real Banach space with Frechet differentiable norm. For $x \in E$, let β^* be defined by

$$\beta^*(t) = sup\left\{\left|\frac{\|x+ty\|^2 + \|y\|^2}{t} - 2\langle y, j(x)\rangle\right| : \|y\| = 1\right\} \quad (10)$$

Then, $\lim\limits_{t\to 0}\beta^*(t) = 0$ and

$$\|x+h\|^2 \leq \|x\|^2 + 2\langle h, j(x)\rangle + \|h\|\beta^*(\|h\|)$$

for all $h \in E - \{0\}$.
If $E = L_p$ $2 \leq p < \infty$, we know that

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x)\rangle + (p-1)\|y\|^2, \ \forall x, y \in E.$$

Then β^* in (10) is estimated by $\beta^*(t) = (p-1)t$ for $t > 0$.
In our more general setting, throughout this paper, we will assume that

$$\beta^*(t) \leq ct, \ t > 0,$$

for some $c > 1$ where β^* is the function appearing in (10).

Lemma 4: [24] Suppose E is a Banach space with uniform normal structure, K is a nonempty bounded subset of E and $T : K \to K$ is uniformly $L-$ Lipschitzian mapping with $L < N(E)^{\frac{1}{2}}$. Suppose also that there exists a nonempty bounded closed convex subset C of K with the following property

$$x \in C \ implies \ W_w(x) \subset C,$$

where $W_w(x)$ is the w- limit set of $\{T^n x\}_{n=1}^{\infty}$ at x, that is,

$$W_w(x) = \Big\{y \in E : y = weak - \lim_j T^{n_j}x \ for \ some \ subsequence \ \{n_j\} \ of \ \{n\}\Big\}.$$

Then T has a fixed point in C.

Lemma 5: [25] Let $(a_0, a_1, ...) \in l_\infty$ such that $\mu_n(a_n) \leq 0$ for all Banach limit μ and $\limsup\limits_{n\to\infty} (a_{n+1} - a_n) \leq 0$. Then $\limsup\limits_{n\to\infty} a_n \leq 0$.

Lemma 6: Let E be a real normed linear space. Then the following inequality holds

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle \ \forall x, y \in E, \forall j(x+y) \in J(x+y).$$

Lemma 7: [28] Let $\{x_n\}$ and $\{y_n\}$ be a bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf \beta_n < 1$. Suppose that $x_{n+1} = \beta_n y_n + (1-\beta_n)x_n$ for all integer $n \geq 1$ and $\limsup(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $lim\|y_n - x_n\| = 0$

Lemma 8: Let E be a real Banach space with Frechet differentiable norm and let K be a nonempty closed convex subset of E. Suppose that $T : K \to K$ is $\lambda-$strictly pseudocontractive such that $F(T) \neq \emptyset$. For any $\alpha \in (0,1)$ we define $T_\alpha : K \to E$ by $T_\alpha x = \alpha x + (1-\alpha)Tx \; \forall x \in K$. Then T_α is a nonexpansive mapping such that $Fix(T_\alpha) = Fix(T)$. for $1-\alpha \in (0,\mu)$, $\mu = Min\left\{1, 1-\frac{2\lambda}{c}\right\}$ with $\frac{c}{2} \geq \lambda$ and $\lambda \in (0,1)$.

Proof: In view of Lemma 3, , we have the following estimate

$$\begin{aligned}
&\|T_\alpha x - T_\alpha y\|^2 = \|\alpha x + (1-\alpha)Tx - \alpha y - (1-\alpha)Ty\|^2 \\
&= \|\alpha x - \alpha y + (1-\alpha)(Tx - Ty)\|^2 \\
&= \|\alpha(x-y) - (x-y) + (x-y) + (1-\alpha)(Tx - Ty)\|^2 \\
&= \|x - y - (1-\alpha)(x - y - (Tx - Ty))\|^2 \\
&\leq \|x-y\|^2 - 2(1-\alpha)\langle (I-T)x - (I-T)y, j(x-y)\rangle + \\
&\quad (1-\alpha)\|x - y - (Tx - Ty)\|\beta^*((1-\alpha)\|x - y - (Tx - Ty)\| \\
&\leq \|x-y\|^2 - 2\lambda(1-\alpha)\|x - y - (Tx - Ty)\|^2 + c(1-\alpha)^2 \\
&\quad \|x - y - (Tx - Ty)\|^2 \\
&= \|x-y\|^2 - [2\lambda(1-\alpha) - c(1-\alpha)^2]\|x - y - (Tx - Ty)\|^2 \\
&= \|x-y\|^2 - (1-\alpha)[2\lambda - c(1-\alpha)]\|x - y - (Tx - Ty)\|^2 \\
&\leq \|x-y\|^2.
\end{aligned}$$

3. MAIN RESULTS

Lemma 9: Let $\{T_i\}_{i=1}^p$ be λ_i- strictly pseudocontractive mappings and let $\lambda = \max_i\{\lambda_i\}$ and $\{\alpha_i\}_{i=1}^p$ be a finite real sequences in $(0,1)$. Define $T_{\alpha_i}x = \alpha_i x + (1-\alpha_i)T_i x$ and $T := T_{\alpha_1} \circ T_{\alpha_2} \circ \cdots \circ T_{\alpha_p}$. Then T_{α_i} for each $i = 1, 2, ...p$ and T are strictly pseudocontrative mappings.

Proof: Consider the following computation

$$\begin{aligned}
&\langle T_{\alpha_i}x - T_{\alpha_i}y, j(x-y)\rangle \\
&= \langle \alpha_i x + (1-\alpha_i)T_i x - (\alpha_i y + (1-\alpha_i)T_i y), j(x-y)\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \alpha_i\langle(x-y), j(x-y)\rangle + (1-\alpha_i)\langle(T_ix-T_iy), j(x-y)\rangle \\
&\leq \alpha_i\|x-y\|^2 + (1-\alpha_i)[\|x-y\|^2 - \lambda_i\|(I-T_i)x-(I-T_i)y\|^2] \\
&= \|x-y\|^2 - \lambda_i(1-\alpha_i)\|(I-T_i)x-(I-T_i)y\|^2 \\
&= \|x-y\|^2 - \lambda_i(1-\alpha_i)\|x - \frac{(T_{\alpha_i}x-\alpha_ix)}{(1-\alpha_i)} - \left(y - \frac{(T_{\alpha_i}y-\alpha_iy)}{(1-\alpha_i)}\right)\|^2 \\
&= \|x-y\|^2 - \left(\frac{\lambda_i}{1-\alpha_i}\right)\|(I-T_{\alpha_i})x-(I-T_{\alpha_i})y\|^2.
\end{aligned}
$$

Therefore T_{α_i} is $\frac{\lambda_i}{1-\alpha_i}$ strictly pseudocontractive. Also

$$
\begin{aligned}
&\langle Tx-Ty, j(x-y)\rangle \\
&\quad= \langle T_{\alpha_1}\circ T_{\alpha_2}\circ\cdots\circ T_{\alpha_p}x - T_{\alpha_1}\circ T_{\alpha_2}\circ\cdots\circ T_{\alpha_p}y, j(x-y)\rangle \\
&\quad\leq \|T_{\alpha_2}\circ T_{\alpha_3}\circ\cdots\circ T_{\alpha_p}x - T_{\alpha_2}\circ T_{\alpha_3}\circ\cdots\circ T_{\alpha_p}y\|^2 - \\
&\qquad \frac{\lambda_1}{1-\alpha_1}\|(I-T_{\alpha_1}\circ T_{\alpha_2}\circ\cdots\circ T_{\alpha_p})x - \\
&\qquad (I-T_{\alpha_1}\circ T_{\alpha_2}\circ\cdots\circ T_{\alpha_p})y\|^2 \\
&\quad\leq \|x-y\|^2 - \frac{\lambda_1}{1-\alpha_1}\|(I-T_{\alpha_1}\circ T_{\alpha_2}\circ\cdots\circ T_{\alpha_p})x - \\
&\qquad (I-T_{\alpha_1}\circ T_{\alpha_2}\circ\cdots\circ T_{\alpha_p})y\|^2
\end{aligned}
$$

which implies T is $\frac{\lambda_1}{1-\alpha_1}$ strictly pseudocontractive.

Lemma 10: Let E be a real Banach space with Fr$\acute{e}$chet differentiable norm and let K be a nonempty closed convex subset of E. Suppose that $T_i : E \to E$ is λ_i- stricly pseudocontractive for $i = 1, 2...p$, $\lambda_i \in (0, 1)$ and $F : E \to E$ be an $\eta-$ strongly accretive mapping which is also $L-$Lipschitz for some $L > 0$. Assume $c > 1$, $\eta \in (0, \frac{2(1-d)+bcL^2\mu}{2(1-d)})$ and $\mu \in (0, \frac{2(1-d)}{acL^2})$, $a_n \in (a, b) \subset (0, 1)$. Define $H : E \to E$ by $Hx = [I - \frac{a_n\mu F}{(1-d)}]T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x$ $\forall x \in E$. Then H satisfies the following inequality $\|Hx - Hy\| \leq \left[1 - \frac{bh}{(1-d)}\right]\|x - y\|$, where $h = \mu\left(\eta - \frac{L^2bc\mu}{2(1-d)}\right)$.

Proof: For any $x, y \in E$ and Lemma 3, we have

$$
\begin{aligned}
\|Hx - Hy\|^2 &= \|[I - \frac{a_n\mu F}{(1-d)}]T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x \\
&- [I - \frac{a_n\mu F}{(1-d)}]T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}y\|^2 \\
&= \Bigg\|(T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x - T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}y) - \\
&\left(\frac{a_n\mu F(T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x)}{(1-d)} - \frac{a_n\mu F(T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}y)}{(1-d)}\right)\Bigg\|^2 \\
&\leq \|T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x - T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}y\|^2 - 2\left(\frac{a_n\mu}{(1-d)}\right) \\
&\Big\langle F(T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x) - F(T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}y), \\
&j(T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x - T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}y)\Big\rangle \\
&+ \left(\frac{a_n\mu}{(1-d)}\right)\Big\|F(T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x) - F(T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}y)\Big\| \\
&\beta^*\Bigg(\left(\frac{a_n\mu}{(1-d)}\right)\Big\|F(T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x) \\
&-F(T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}y)\Big\|\Bigg) \\
&\leq \|T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x - T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}y\|^2 \\
&- 2\left(\frac{a_n\mu}{(1-d)}\right)\eta\Big\|T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x - T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}y\Big\|^2 \\
&+ c\left(\frac{a_n\mu}{(1-d)}\right)^2\Big\|F(T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x) \\
&-F(T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}y)\Big\|^2 \\
&\leq \|T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x - T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}y\|^2 \\
&-2\left(\frac{a_n\mu}{(1-d)}\right)\eta\Big\|T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x - T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}y\Big\|^2 \\
&+c\left(\frac{a_n\mu}{(1-d)}\right)^2L^2\Big\|T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x \\
&-T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}y\Big\|^2 \\
&= \left(1 - \frac{2a_n\mu}{(1-d)}\left(\eta - \frac{L^2c}{2}\left(\frac{a_n\mu}{(1-d)}\right)\right)\right) \\
&\Big\|T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x - T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}y\Big\|^2
\end{aligned}
$$

$$\leq \left[1 - \frac{a_n\mu}{(1-d)}\left(\eta - \frac{L^2c}{2}\frac{a_n\mu}{(1-d)}\right)\right]^2 \|x-y\|^2$$
$$\leq \left[1 - \frac{a_n h}{(1-d)}\right]^2 \|x-y\|^2.$$

Hence, we have

$$\left\|Hx - Hy\right\| \leq \left[1 - \frac{bh}{(1-d)}\right]\|x-y\|.$$

We now prove the following Theorem

Theorem 1: Let E be a real uniformly smooth Banach space and let K be a nonempty closed convex subset of E. Suppose that $T_i : E \to E$ is λ_i- stricly pseudocontractive mappings for $i = 1, 2, ..., p$ such that $\Omega = \cap_{n=1}^{p} Fix(T_i) \neq \emptyset$ and let $F : E \to E$ be an $\eta-$ strongly accretive mapping which is also $L-$ Lipschitz. Let $g : E \to E$ be a contraction with contractive constant $\tau \in (0, 1)$. For an arbitrary $x_0 \in E$, define a sequence $\{x_n\}$ by

$$\begin{cases} y_n = b_n x_n + (1-b_n)T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x_n, \\ x_{n+1} = a_n\rho g(y_n) + dx_n + [(1-d)I - a_n\mu F]T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}y_n \\ \forall n \geq 0, \end{cases} \tag{11}$$

where $T_{\alpha_i} = \alpha_i x + (1-\alpha_i)T_i x$ and $\alpha_i \in (0, 1-\lambda)$ where $\lambda = \underset{i}{Max}\{\lambda_i\}$ for $i = 1, 2, ...p$, and the parameters ρ, h, d satisfy $0 < \rho < \frac{h}{\tau}$, $\mu \in (0, \frac{2(1-d)}{acL^2})$ and $h = \mu\left(\eta - \frac{L^2c}{2}\frac{c\mu}{(1-d)}\right)$, $d \in (0, 1)$, and $\{a_n\}, \{b_n\}$ are sequences in $(0, 1)$ satisfying the following condition:

(1) $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n = \infty$ and $\sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty$
(2) $b_n \in (\frac{2d}{1+d}, 1)$ and $\lim_{n\to\infty} |b_{n+1} - b_n| = 0$
(3) $\frac{2\lambda_1}{1-\alpha_1} > \frac{2(1-d)}{1+d}$

Then the sequence generated by algorithm (11) converges strongly to $z \in \cap_{n=1}^{p} Fix(T_i)$, which is also a solution of the $HFPP$:

$$\langle(\mu F - \rho g)z, j(x-z)\rangle \geq 0 \ \forall x \in \cap_{i=1}^{p} Fix(T_i). \tag{12}$$

Proof: We split the proof into four steps.
STEP 1 We show that $\{y_n\}, \{x_n\}$ are bounded.

Let $x^* \in \Omega$. From the scheme (11) and Lemma 3, we have

$$\begin{aligned}
&\|y_n - x^*\|^2 = \|b_n x_n + (1-b_n)T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x_n - x^*\|^2 \\
&= \|x_n - x^* - (1-b_n)(x_n - T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x_n)\|^2 \\
&\leq \|x_n - x^*\|^2 - 2(1-b_n)\langle x_n - T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x_n, j(x_n - x^*)\rangle \\
&\quad + (1-b_n)\|x_n - T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x_n\| \times \\
&\quad \beta^*((1-b_n)\|x_n - T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x_n\|) \\
&\leq \|x_n - x^*\|^2 - 2(1-b_n) \\
&\quad \times\langle x_n - x^* + x^* - T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x_n, j(x_n - x^*)\rangle \\
&\quad + c(1-b_n)^2\|x_n - T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x_n\|^2 \\
&= \|x_n - x^*\|^2 - 2(1-b_n)\langle x_n - x^*, j(x_n - x^*)\rangle + 2(1-b_n) \\
&\quad \langle T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x_n - x^*, j(x_n - x^*)\rangle + \\
&\quad c(1-b_n)^2\|x_n - T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x_n\|^2 \\
&= \|x_n - x^*\|^2 - 2(1-b_n)\langle x_n - x^*, j(x_n - x^*)\rangle + 2(1-b_n) \\
&\quad \times\langle T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x_n - T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x^*, j(x_n - x^*)\rangle \\
&\quad + c(1-b_n)^2\|x_n - T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x_n\|^2 \\
&\leq \|x_n - x^*\|^2 - 2(1-b_n)\|x_n - x^*\|^2 + 2(1-b_n)[\|x_n - x^*\|^2 - \\
&\quad \frac{\lambda_1}{1-\alpha_1}\|(I - T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1})x_n - (I - T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1})x^*\|^2] \\
&\quad + c(1-b_n)^2\|x_n - T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x_n\|^2 \\
&= \|x_n - x^*\|^2 - \frac{2\lambda_1}{1-\alpha_1}(1-b_n)\|x_n - T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x_n\|^2 + \\
&\quad c(1-b_n)^2\|x_n - T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x_n\|^2 \\
&= \|x_n - x^*\|^2 - (1-b_n)\left(\frac{2\lambda_1}{1-\alpha_1} - c(1-b_n)\right) \\
&\quad \times\|x_n - T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x_n\|^2.
\end{aligned}$$

So

$$\|y_n - x^*\|^2 \leq \|x_n - x^*\|^2. \tag{13}$$

From the algorithm (11), Lamma 10 and equation (13), we have

$$\begin{aligned}
\|x_{n+1}-x^*\| &= \|a_n\rho g(y_n)+dx_n \\
&\quad +[(1-d)I-a_n\mu F]T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}y_n - x^*\| \\
&= \|a_n(\rho g(y_n)-\mu F(x^*))+d(x_n-x^*)+ \\
&\quad [(1-d)I-a_n\mu F]T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}y_n-[(1-d)I-a_n\mu F]x^*\| \\
&\leq \|a_n(\rho g(y_n)-\mu F(x^*))\|+\|d(x_n-x^*)\|+\|[(1-d)I-a_n\mu F] \\
&\quad \times T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}y_n-[(1-d)I-a_n\mu F]x^*\| \\
&= \|a_n(\rho g(y_n)-\mu F(x^*))\|+d\|(x_n-x^*)\| \\
&+ (1-d)\|[I-\frac{a_n\mu F}{(1-d)}]T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}y_n \\
&\quad -[I-\frac{a_n\mu F}{(1-d)}]T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x^*\| \\
&\leq \|a_n(\rho g(y_n)-\mu F(x^*))\|+d\|x_n-x^*\| \\
&\quad +(1-d)\left[1-\frac{a_nh}{(1-d)}\right]\|y_n-x^*\| \\
&\leq \|a_n(\rho g(y_n)-\mu F(x^*))\|+d\|x_n-x^*\| \\
&\quad +(1-d)\|x_n-x^*\|-a_nh\|x_n-x^*\| \\
&= \|a_n\rho g(y_n)-a_n\mu F(x^*)-a_n\rho g(x^*)+a_n\rho g(x^*)\| \\
&\quad +(1-a_nh)\|x_n-x^*\| \\
&\leq a_n\rho\tau\|y_n-x^*\|+a_n\|\rho g(x^*)-\mu F(x^*)\|+(1-a_nh)\|x_n-x^*\| \\
&\leq a_n\rho\tau\|x_n-x^*\|+a_n\|\rho g(x^*)-\mu F(x^*)\|+(1-a_nh)\|x_n-x^*\| \\
&= (1-a_n(h-\rho\tau)\|x_n-x^*\|+a_n\|\rho g(x^*)-\mu F(x^*)\| \\
&= (1-a_n(h-\rho\tau)\|x_n-x^*\|+\frac{a_n(h-\rho\tau)\|\rho g(x^*)-\mu F(x^*)\|}{(h-\rho\tau)} \\
&\leq Max\left\{\|x_n-x^*\|,\frac{\|\rho g(x^*)-\mu F(x^*)\|}{(h-\rho\tau)}\right\}.
\end{aligned}$$

Hence by induction,

$$\|x_n-x^*\|\leq Max\left\{\|x_0-x^*\|,\frac{\|\rho g(x^*)-\mu F(x^*)\|}{(h-\rho\tau)}\right\}\ \forall n\geq 0.$$

Therefore,$\{x_n\}$, $\{y_n\}$ are bounded.

STEP 2 We show that
(i) $\lim\limits_{n\to\infty}\|x_{n+1}-x_n\|=0$ and
(ii) $\lim\limits_{n\to\infty}\|x_n-T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x_n\|=0.$

Define

$$z_n = \frac{x_{n+1} - dx_n}{1-d}.$$

Thus,

$$\begin{aligned}
\|z_{n+1} - z_n\| &= \left\| \frac{x_{n+2} - dx_{n+1}}{1-d} - \frac{x_{n+1} - dx_n}{1-d} \right\| \\
&= \left\| \frac{a_{n+1}\rho g(y_{n+1}) + dx_{n+1}}{(1-d)} \right. \\
&\quad + \frac{[(1-d)I - a_{n+1}\mu F]T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1} y_{n+1}}{(1-d)} \\
&\quad - \frac{dx_{n+1} - (a_n\rho g(y_n)}{(1-d)} \\
&\quad \left. + \frac{dx_n + [(1-d)I - a_n\mu F]T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1} y_n - dx_n)}{(1-d)} \right\| \\
&= \left\| \frac{a_{n+1}\rho g(y_{n+1}) + [(1-d)I - a_{n+1}\mu F]T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1} y_{n+1}}{(1-d)} \right. \\
&\quad \left. - \frac{a_n\rho g(y_n) - [(1-d)I - a_n\mu F]T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1} y_n}{(1-d)} \right\| \\
&= \left\| \frac{a_{n+1}\rho g(y_{n+1})}{(1-d)} - \frac{a_n\rho g(y_n)}{(1-d)} \right. \\
&\quad + \frac{T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1} y_{n+1} - T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1} y_n}{(1-d)} \\
&\quad \left. + \frac{a_n\mu F T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1} y_n}{1-d} - \frac{a_{n+1}\mu F T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1} y_{n+1}}{(1-d)} \right\| \\
&\leq \left\| \frac{a_{n+1}\rho g(y_{n+1})}{(1-d)} - \frac{a_n\rho g(y_n)}{(1-d)} \right\| \\
&\quad + \left\| T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1} y_{n+1} - T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1} y_n \right\| \\
&\quad + \left\| \frac{a_n\mu F T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1} y_n}{(1-d)} - \frac{a_{n+1}\mu F T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1} y_{n+1}}{(1-d)} \right\| \\
&\leq \left\| \frac{a_{n+1}\rho g(y_{n+1})}{(1-d)} - \frac{a_n\rho g(y_n)}{(1-d)} \right\| + \left\| y_{n+1} - y_n \right\| \\
&\quad + \left\| \frac{a_n\mu F T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1} y_n}{(1-d)} - \frac{a_{n+1}\mu F T_{\alpha_p}T_{\alpha_{p-1}} \cdots T_{\alpha_1} y_{n+1}}{(1-d)} \right\|
\end{aligned}$$

$$
\begin{aligned}
\leq\ & \frac{a_{n+1}}{1-d}\Big\|\rho g(y_{n+1})-\rho g(y_n)\Big\| \\
&+\Big|\frac{a_{n+1}-a_n}{1-d}\Big|\Big\|\rho g(y_n)\Big\|+\Big\|y_{n+1}-y_n\Big\| \\
&+\frac{a_n\mu}{1-d}\Big\|FT_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}y_n-FT_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}y_{n+1}\Big\|+ \\
&\Big|\frac{a_n-a_{n+1}}{1-d}\Big|\mu\Big\|FT_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}y_{n+1}\Big\| \\
\leq\ & \frac{a_{n+1}}{1-d}\rho\tau\Big\|y_{n+1}-y_n\Big\|+\Big|\frac{a_{n+1}-a_n}{1-d}\Big|\Big\|\rho g(y_n)\Big\|+\Big\|y_{n+1}-y_n\Big\| \\
&+\frac{a_n\mu}{1-d}L\Big\|y_n-y_{n+1}\Big\| \\
&+\Big|\frac{a_n-a_{n+1}}{1-d}\Big|\mu\Big\|FT_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}y_{n+1}\Big\|
\end{aligned}
$$

But

$$
\begin{aligned}
\|y_{n+1}-y_n\| &= \|b_{n+1}x_{n+1}+(1-b_{n+1})T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x_{n+1} \\
&- \quad b_nx_n-(1-b_n)T_{\alpha_p}T_{\alpha_{p-1}\cdots T_{\alpha_1}x_n\|} \\
&= \quad \|(1-b_{n+1})(T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x_{n+1}-T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x_n) \\
&\qquad -(b_{n+1}-b_n)T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x_n \\
&\qquad +b_{n+1}(x_{n+1}-x_n)-(b_n-b_{n+1})x_n\| \\
&= \quad \|(1-b_{n+1})(T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x_{n+1}-T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x_n) \\
&\qquad -(b_{n+1}-b_n)T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x_n \\
&\qquad +(b_{n+1}-b_n)x_n+b_{n+1}(x_{n+1}-x_n)\| \\
&\leq \quad (1-b_{n+1})\|T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x_{n+1}-T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x_n\| \\
&\qquad +|b_{n+1}-b_n|\|T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x_n-x_n\| \\
&\qquad +b_{n+1}\|x_{n+1}-x_n\| \\
&\leq \quad b_{n+1}\|x_{n+1}-x_n\|+(1-b_{n+1})\|x_{n+1}-x_n\| \\
&\qquad +|b_{n+1}-b_n|\|T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x_n-x_n\| \\
&= \quad \|x_{n+1}-x_n\|+|b_{n+1}-b_n|\|T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x_n-x_n\|
\end{aligned}
$$

So

$$\begin{aligned}\|z_{n+1}-z_n\| \leq{}& \frac{a_{n+1}}{1-d}\rho\tau\Big\|y_{n+1}-y_n\Big\| \\ &+ \Big|\frac{a_{n+1}-a_n}{1-d}\Big|\Big\|\rho g(y_n)\Big\| + \Big\|y_{n+1}-y_n\Big\| \\ &+ \frac{a_n\mu}{1-d}L\Big\|(y_n)-(y_{n+1})\Big\| \\ &+ \Big|\frac{a_n-a_{n+1}}{1-d}\Big|\mu\Big\|FT_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}y_{n+1}\Big\|\end{aligned}$$

Therefore

$$\begin{aligned}&\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\| \\ &\leq \frac{a_{n+1}}{1-d}\rho\tau\Big\|y_{n+1}-y_n\Big\| \\ &+ \Big|\frac{a_{n+1}-a_n}{1-d}\Big|\Big\|\rho g(y_n)\Big\| \\ &+ |b_{n+1}-b_n|\Big\|T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x_n-x_n\Big\| \\ &+ \frac{a_n\mu}{1-d}\Big\|F(y_n)-F(y_{n+1})\Big\| + \\ &\quad \Big|\frac{a_n-a_{n+1}}{1-d}\Big|\mu\Big\|FT_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}y_{n+1}\Big\|.\end{aligned}$$

Hence,

$$\limsup_{n\to\infty}\Big(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|\Big) \leq 0.$$

Thus by Lemma 7

$$\mathop{lim}_{n\to\infty}\|z_n-x_n\| = 0.$$

This implies

$$\mathop{lim}_{n\to\infty}\|x_{n+1}-x_n\| = (1-d)\mathop{lim}_{n\to\infty}\|z_n-x_n\| = 0. \tag{14}$$

Now

$$\begin{aligned}&\|x_n-T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x_n\| \\ &= \|x_n-x_{n+1}+x_{n+1}-T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x_n\| \\ &\leq \|x_n-x_{n+1}\|+\|x_{n+1}-T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}x_n\|\end{aligned}$$

$$
\begin{aligned}
&= \|x_n - x_{n+1}\| + \|a_n \rho g(y_n) + dx_n + \\
&\quad [(1-d)I - a_n \mu F] T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} y_n - T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x_n\| \\
&\leq \|x_n - x_{n+1}\| + a_n \|\rho g(y_n) - \mu F T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} y_n\| \\
&\quad + d\|x_n - T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} y_n\| + \\
&\quad \|T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} y_n - T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x_n\| \\
&\leq \|x_n - x_{n+1}\| + a_n \|\rho g(y_n) - \mu F T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} y_n\| \\
&\quad + d\|x_n - T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x_n + T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x_n \\
&\quad - T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} y_n\| + \|y_n - x_n\|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \|x_n - x_{n+1}\| + a_n \|\rho g(y_n) - \mu F T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} y_n\| \\
&\quad + d\|x_n - T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x_n\| \\
&\quad + d\|T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x_n - T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} y_n\| + \|y_n - x_n\| \\
&\leq \|x_n - x_{n+1}\| + a_n \|\rho g(y_n) - \mu F T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} y_n\| \\
&\quad + d\|x_n - T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x_n\| + (1+d)\|x_n - y_n\|.
\end{aligned}
$$

Notice that

$$\|y_n - x_n\| = (1-b_n)\|T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x_n - x_n\|.$$

Therefore

$$
\begin{aligned}
&\|x_n - T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x_n\| \leq \|x_n - x_{n+1}\| \\
&\qquad + a_n \|\rho g(y_n) - \mu F T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} y_n\| \\
&\qquad + d\|x_n - T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x_n\| + (1+d)(1-b_n) \times \\
&\qquad \|T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x_n - x_n\| \\
&\quad = \|x_n - x_{n+1}\| + a_n \|\rho g(y_n) - \mu F T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} y_n\| \\
&\qquad (1 + 2d - b_n(1+d))\|T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x_n - x_n\|.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&(b_n(1+d) - 2d)\|x_n - T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x_n\| \\
&\quad \leq \|x_n - x_{n+1}\| + a_n \|\rho g(y_n) - \mu F T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} y_n\|
\end{aligned}
$$

From condition (1), (2) and equation (14), we obtain

$$\lim_{n\to\infty} \|x_n - T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x_n\| = 0.$$

STEP 3 Next we show that for $z \in \Omega$,

$$\limsup_{n\to\infty} \langle \rho g(z) - \mu F(z), j(x_n - z) \rangle \leq 0,$$

Define a map $\phi : E \to \mathbb{R}$ by

$$\phi(y) = \mu_n \|x_n - y\|^2 \ \forall y \in E.$$

Then $\phi(y) \to \infty$ as $\|y\| \to \infty$, ϕ is continuous and convex, so as E is reflexive, there exists $q \in E$ such that $\phi(q) = \min_{u \in E} \phi(u)$. Hence the set

$$K^* := \left\{ y \in E : \phi(y) = \min_{u \in E} \phi(u) \right\} \neq \emptyset$$

Since $\underset{n\to\infty}{lim} \|x_n - T_{\alpha p} T_{\alpha p-1} \cdots T_{\alpha 1} x_n\| = 0$ and by Lemma 4 we obtain $K^* \cap \cap_{i=1}^{p} Fix(T_i) \neq \emptyset$.

Without loss of generality, assume that $y^* = z \in K^* \cap \Omega$. let $t \in (0, 1)$, then it follows that

$$\phi(z) \leq \phi(z + t(\rho g - \mu F)z)$$

and using Lemma 6

$$\begin{aligned} \|x_n - z - t(\rho g - \mu F)z\|^2 &\leq \|x_n - z\|^2 \\ - \quad & 2t\langle(\rho g - \mu F)z, j(x_n - z + t(\rho g - \mu F)z)\rangle. \end{aligned}$$

Thus, taking Banach limit of both sides, we have

$$\begin{aligned} \mu_n\|x_n - z - t(\rho g - \mu F)z\|^2 &\leq \mu_n\|x_n - z\|^2 \\ - \quad & 2t\mu_n\langle(\rho g - \mu F)z, j(x_n - z + t(\rho g - \mu F)z)\rangle. \end{aligned}$$

This implies

$$\begin{aligned} & 2t\mu_n\langle(\rho g - \mu F)z, j(x_n - z + t(\rho g - \mu F)z)\rangle \\ \leq \quad & \mu_n\|x_n - z\|^2 - \mu_n\|x_n - z - t(\rho g - \mu F)z\|^2 \\ = \quad & \phi(z) - \phi(z + t(\rho g - \mu F)z) \leq 0. \end{aligned}$$

Therefore,

$$\mu_n\langle(\rho g - \mu F)z, j(x_n - z + t(\rho g - \mu F)z)\rangle \leq 0 \tag{15}$$

Thus, by equation (15), we have

$$\begin{aligned} & \mu_n\langle(\rho g - \mu F)z, j(x_n - z)\rangle \\ = \quad & \mu_n\langle(\rho g - \mu F)z, j(x_n - z) - j(x_n - z + t(\rho g - \mu F)z)\rangle \\ & +\mu_n\langle(\rho g - \mu F)z, j(x_n - z - t(\rho g - \mu F)z)\rangle \\ \leq \quad & \mu_n\langle(\rho g - \mu F)z, j(x_n - z) - j(x_n - z + t(\rho g - \mu F)z)\rangle \end{aligned}$$

Since j is norm to norm uniformly continuous on bounded subset of E, we have

$$\mu_n\langle(\rho g - \mu F)z, j(x_n - z)\rangle \leq 0.$$

Furthermore since $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$, we have

$$\limsup_{n\to\infty}\Big[\langle(\rho g-\mu F)z, j(x_n-z)\rangle - \langle(\rho g-\mu F)z, j(x_{n+1}-z)\rangle\Big] \leq 0$$

and so we obtain by Lemma 5

$$\limsup_{n\to\infty}\ \langle(\rho g-\mu F)z, j(x_n-z)\rangle \leq 0.$$

STEP 4 Finally we show that $x_n \to z$ as $n \to \infty$.

$$\begin{aligned}
\|x_{n+1}-z\|^2 &= \|a_n\rho g(y_n) \\
&+ \quad dx_n + [(1-d)I - a_n\mu F]T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}y_n - z\|^2 \\
&= \quad \langle a_n\rho g(y_n) + dx_n + [(1-d)I - a_n\mu F]\times \\
&\quad\quad T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}y_n - z, j(x_{n+1}-z)\rangle \\
&= \quad \Big\langle a_n(\rho g(y_n)-\mu F(z)) + d(x_n-z) \\
&\quad\quad +(1-d)\times\Bigg[\left(I-\frac{a_n\mu F}{1-d}\right)T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}y_n \\
&\quad\quad -\left(I-\frac{a_n\mu F}{1-d}\right)z\Bigg], j(x_{n+1}-z)\Big\rangle \\
&= \quad a_n\langle(\rho g(y_n)-\mu F(z)), j(x_{n+1}-z)\rangle + \langle d(x_n-z), j(x_{n+1}-z)\rangle \\
&\quad +\Big\langle(1-d)\Bigg[\left(I-\frac{a_n\mu F}{1-d}\right)T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}y_n - \\
&\quad \left(I-\frac{a_n\mu F}{1-d}\right)z\Bigg], j(x_{n+1}-z)\Big\rangle \\
&= \quad a_n\rho\langle(g(y_n)-g(z)), j(x_{n+1}-z)\rangle \\
&\quad +a_n\langle(\rho g(z)-\mu F(z)), j(x_{n+1}-z)\rangle + d\langle(x_n-z), j(x_{n+1}-z)\rangle \\
&\quad +(1-d)\Big\langle\left(I-\frac{a_n\mu F}{1-d}\right)T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}y_n \\
&\quad -\left(I-\frac{a_n\mu F}{1-d}\right)T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}z, j(x_{n+1}-z)\Big\rangle \\
&\leq \quad a_n\rho\|g(y_n)-g(z)\|\|x_{n+1}-z\| + \\
&\quad\quad a_n\langle(\rho g(z)-\mu F(z)), j(x_{n+1}-z)\rangle + d\|x_n-z\| \\
&\quad\quad \|x_{n+1}-z\| + (1-d)\left\|\left(I-\frac{a_n\mu F}{1-d}\right)T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}y_n - \right. \\
&\quad\quad \left.\left(I-\frac{a_n\mu F}{1-d}\right)T_{\alpha_p}T_{\alpha_{p-1}}\cdots T_{\alpha_1}z\right\|\|x_{n+1}-z\|
\end{aligned}$$

$$\begin{aligned}
&\leq a_n\rho\tau\|y_n - z\|\|x_{n+1} - z\| + \langle a_n\rho g(z) - \mu F(z)), j(x_{n+1} - z)\rangle \\
&\quad +d\|x_n - z\|\|x_{n+1} - z\| \\
&\quad +(1-d)(1 - \frac{a_n h}{1-d})\|y_n - z\|\|x_{n+1} - z\| \\
&= a_n\rho\tau\|y_n - z\|\|x_{n+1} - z\| + a_n\langle(\rho g(z) - \mu F(z)), j(x_{n+1} - z)\rangle \\
&\quad +d\|x_n - z\|\|x_{n+1} - z\| + [1-d]\|y_n - z\|\|x_{n+1} - z\| \\
&\quad -a_n h\|y_n - z\|\|x_{n+1} - z\| \\
&= [1 + a_n\rho\tau - d - a_n h]\|y_n - z\|\|x_{n+1} - z\| \\
&\quad +d\|x_n - z\|\|x_{n+1} - z\| \\
&\quad +a_n\langle(\rho g(z) - \mu F(z)), j(x_{n+1} - z)\rangle \\
&\leq [1 + a_n\rho\tau - d - a_n h]\|x_n - z\|\|x_{n+1} - z\| + \\
&\quad d\|x_n - z\|\|x_{n+1} - z\| \\
&\quad +a_n\langle(\rho g(z) - \mu F(z)), j(x_{n+1} - z)\rangle \\
&= [1 - a_n(h - \rho\tau)]\|x_n - z\|\|x_{n+1} - z\| \\
&\quad +a_n\langle(\rho g(z) - \mu F(z)), j(x_{n+1} - z)\rangle \\
&\leq \frac{1 - a_n(h - \rho\tau)}{2}\Big(\|x_n - z\|^2 + \|x_{n+1} - z\|^2\Big) \\
&\quad +a_n\langle(\rho g(z) - \mu F(z)), j(x_{n+1} - z)\rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left[1 - \frac{(1 - a_n(h - \rho\tau)}{2}\right]\|x_{n+1} - z\|^2 \\
&\quad \leq \frac{1 - a_n(h - \rho\tau)}{2}\|x_n - z\|^2 + \\
&\quad\quad a_n\langle(\rho g(z) - \mu F(z)), j(x_{n+1} - z)\rangle,
\end{aligned}$$

that is,

$$\begin{aligned}
&\left[\frac{(1 + a_n(h - \rho\tau)}{2}\right]\|x_{n+1} - z\|^2 \\
&\quad \leq \frac{(1 - a_n(h - \rho\tau)}{2}\|x_n - z\|^2 + \\
&\quad\quad a_n\langle(\rho g(z) - \mu F(z)), j(x_{n+1} - z)\rangle
\end{aligned}$$

which implies

$$\begin{aligned}
\|x_{n+1} - z\|^2 \le & \left[\frac{(1 - a_n(h-\rho\tau))}{(1 + a_n(h-\rho\tau))}\right]\|x_n - z\|^2 \\
& \left[\frac{2a_n}{(1 + a_n(h-\rho\tau))}\right]\langle(\rho g(z) - \mu F(z)), j(x_{n+1} - z)\rangle \\
\le \quad & \left[1 - \frac{(2a_n(h-\rho\tau))}{(1 + a_n(h-\rho\tau))}\right]\|x_n - z\|^2 + \\
& \left[\frac{2a_n(h-\rho\tau)}{(1 + a_n(h-\rho\tau))}\right]\frac{1}{(h-\rho\tau)} \times \\
& \langle(\rho g(z) - \mu F(z)), j(x_{n+1} - z)\rangle
\end{aligned}$$

and by Lemma 1 , we have that $x_n \to z$ as $n \to \infty$. This completes the proof.
If E is a real Hilbert space H, Theorem 1 reduces to the following corollary.

Corollary 1: Let H be a real Hilbert space and let K be a nonempty closed convex subset of E. Suppose that $T_i : H \to H$ is λ_i- stricly pseudocontractive mappings for $i = 1, 2, ..., p$ such that $\Omega = \cap_{n=1}^{p} Fix(T_i) \neq \emptyset$ and let $F : H \to H$ be an $\eta-$ strongly monotone mapping which is also $L-$ Lipschitz. Let $g : H \to H$ be a contraction with contractive constant $\tau \in (0, 1)$. For an arbitrary $x_0 \in H$, define a sequence $\{x_n\}$ by

$$\begin{cases} y_n = b_n x_n + (1 - b_n)T_p T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} x_n, \\ x_{n+1} = a_n \rho g(y_n) + d x_n + [(1 - d)I - a_n \mu F]T_{\alpha_p} T_{\alpha_{p-1}} \cdots T_{\alpha_1} y_n \\ \forall n \ge 0, \end{cases} \tag{16}$$

where $T_{\alpha_i} = \alpha_i x + (1-\alpha_i)T_i x$ and $\alpha_i \in (0, 1-\lambda)$ for $i = 1, 2, ...p$, let the parameters ρ, h, d satisfy $0 < \rho < \frac{h}{\tau}$ and $h = \mu\left(\eta - \frac{L^2}{2}(\frac{a_n \mu}{(1-d)})\right)$, $d \in (0, 1)$, and $\{a_n\}, \{b_n\}$ are sequences in $(0, 1)$ satisfying the following condition:

(1) $\lim\limits_{n\to\infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n = \infty$ and $\sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty$
(2) $b_n \in (\frac{2d}{1+d}, 1)$ and $\lim\limits_{n\to\infty} |b_{n+1} - b_n| = 0$
(3) $\frac{2\lambda_1}{1-\alpha_1} > \frac{2(1-d)}{1+d}$

Then the sequence generated by algorithm (16) converges strongly to $z \in \cap_{n=1}^{p} Fix(T_i)$, which is also solution of the $HFPP$:

$$\langle(\rho g - \mu F)z, x - z\rangle \ge 0 \ \forall x \in \cap_{i=1}^{p} Fix(T_i).$$

4. CONCLUDING REMARKS

Theorem 1 genralizes the results of Husain and Singh [1] from finite family of nonexpansive mappings in real Hilbert space to finite family of strictly pseudocontractive mappings in real uniformly smooth Banach space.

REFERENCES

[1] S. Husain and N. Singh, Hybrid Steepest Iterative Algorithm for a Hierarchical Fixed Point Problem, Fixed Point Theory and Appl. **2017**:25, Doi:10.1186/s13663-017-0618-8, (2017).

[2] C. E. Chidume, *Geometric properties of Banach Spaces and Nonlinear Iterations,* Springer Verlag, Lecture Notes Math. 1965 (2009).

[3] Z. B. Xu, G. F. Roach,*Characteristic inequalities of uniformly smooth Banach spaces,* J. Math. Anal. Appl. **157** (1991) 189-210.

[4] W. A. Kirk *A fixed point theorem for mappings which do not increase distances,* The Amer. Math. Monthly, **72** (1965), pp. 1004- 1006.

[5] W. Takahashi, *Nonlinear Functional Analysis, Fixed Point Theory and Its Applications,* Yokahama, Japan 2000.

[6] W. F. Ames, *Numerical Methods for Partial Differential Equations,* Third Edit., Academic Press, New York, (1992).

[7] A. Nagurney, *Network economics a variational inequality approach,* Academic Publishers, Boston, 1993.

[8] M. Pappalardo, M. Passacantando, *Stability for equilibrium problems from variational inequalities to dynamical systems,* J.O.T.A. **113**, (2002.)

[9] P. T. Harker, J. S. Pang, *Finite-dimensional variational inequality and non-linear complementarity problems: A survey of theory, algorithms and applications*, Math. Programming, **48**, (1990) 161-220.

[10] H. Stark, *Image Recovery Theory and Applications*, Academic Press, New York (1987).

[11] W. L. Bynum, *Normal structure coefficients for Banach spaces,* Pacific J. Math., **86**, (2), (1980), 427 - 436.

[12] L. C. Ceng, Q. H. Ansari, J.C. Yao, *Iterative methods for hierarchical variational inequalities in Hilbert spaces,* J. Optim. Theory Appl. **151**, (2011), 489-512.

[13] H. K. Xu, *Viscosity method for hierarchical fixed point approach to variational inequalities,* Taiwan. J. Math. 14(2), (2010), 463-478.

[14] Y. Wang, W. Xu, *Strong convergence of a modified iterative algorithm for hierarchical fixed point problems and variational inequalities,* Fixed Point Theory Appl. **2013**, Article ID 121 (2013)

[15] Q. H. Ansari, L. C. Ceng, H. Gupta, *Triple hierarchical variational inequalities.* In: Q.H. Ansari, (ed.) Nonlinear Analysis: Approximation Theory, Optimization and Applications, Springer, New York (2014), 231-280.

[16] A. Bnouhachem, Q. H. Ansari, J. C. Yao, *An iterative algorithm for hierarchical fixed point problems for a finite family of nonexpansive mappings,* Fixed Point Theory Appl. **2015**, Article ID 111 (2015)

[17] A. Bnouhachem, Q. H. Ansari, J. C. Yao, *Strong convergence algorithm for hierarchical fixed point problems of a finite family of nonexpansive mappings,* Fixed Point Theory **17**, (2016) 47-62.

[18] C. Zhang, C. Yang, *A new explicit iterative algorithm for solving a class of variational inequalities over the common fixed point set of a finite family of nonexpansive mappings,* Fixed Point Theory Appl. **2014**, Article ID 60 (2014)

[19] L. C. Ceng, A. R. Khan, Q. H. Ansari, J. C.Yao, *Viscosity approximation methods for strongly positive and monotone operators,* Fixed Point Theory **10**(1), (2009), 35-71.

[20] N. Buong, L. T. Duong, *An explicit iterative algorithm for a class of variational inequalities in Hilbert spaces,* J. Optim. Theory Appl. **151**, (2011), 513-524.

[21] P. Cholamjiak, S. Suantai,*Weak convergence theorems for a countable family of strict pseudocontractions in Banach spaces,* Fixed Point Theory Appl. **2010**, (2010), Article ID 632137, 17 pages.

[22] S. Yekini, *An iterative approximation of fixed points of strictly pseudocontractive mappings in Banach spaces,* MATEMATHYKN BECHNK **2**, (2015) 79- 91.

[23] C. E. Chidume, J. Li , A. Udomene, *Convergence of paths and approximation of fixed points of asymptotically nonexpansive mappings,* Proceedings of the Amer. Math. Soc., **133**, (2), (2005), 473 -480.

[24] T. C. Lim, H. K. Xu, *Fixed point theorems for asymptotically nonexpansive mapping,* Nonlinear Analysis; Theory, Method and Applications, **22** (11), (1994), 1345-1355.

[25] N. Shioji, W. Takahashi, *Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces,* Proceedings Amer. Math. Soc., **125** (12), (1997), 3641 -3645.

[26] H. K. Xu, *Iterative algorithms for nonlinear operator,* J. London Math. Soc., **66** (1), (2002), 240-256.

[27] K. Geobel, W. A Kirk, *Topics in Metric Fixed Point Theory,* Cambridge University Press, Cambridge (1990).

[28] T. Suzuki, *Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals,* J. Math. Anal. **305** (2005) 227-239.

SCHOOL OF CONTINUING EDUCATION, BAYERO UNIVERSITY, KANO, NIGERIA
E-mail address: hamzahadi84@yahoo.com

DEPARTMENT OF MATHEMATICAL SCIENCES, BAYERO UNIVERSITY, KANO, NIGERIA
E-mail address: bashiralik@yahoo.com

DEPARTMENT OF SCIENCE AND TECHNOLOGY EDUCATION, BAYERO UNIVERSITY, KANO NIGERIA
E-mail address: murtalaharbau@yahoo.com