Journal of the Nigerian Mathematical Society
Vol. 38, Issue 3, pp. 469-489, 2019

Special Issue: The International Conference on Non-linear Analysis (organized to mark the 70th birthday anniversary of Prof. Charles Ejike Chidume)

NOTE ON A GENERALIZATION OF THE SPACE OF DERIVATIVES OF LIPSCHITZ FUNCTIONS

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ABSTRACT. In this note, we denote by $(Lip^1)'$ the space of derivatives of Lipschitz functions of order 1. We propose a generalization of the space $(Lip^1)'$ on the interval $[0, 2\pi]$ for general measures on subsets of $[0, 2\pi]$ with respect to the representation of the norm. As a byproduct, we obtain Hölder's type inequalities and duality results between the space $(Lip^1)'$ as well as its generalization, and the special atoms spaces B and $B(\mu, 1)$, spaces first introduced by De Souza in his PhD thesis. Another byproduct is a relation between the space $(Lip^1)'$ as well as its generalization, and the space L_∞. As a result we prove that the special atom space is a simple characterization of L_1.

Keywords and phrases: Lipschitz spaces; Dual spaces; Banach spaces.
2010 Mathematical Subject Classification: 3A85; 46B45; 46B07.

1. Introduction

The Lipschitz space often denoted by Lip^1 is the space of real-valued functions f defined on the interval $[0, 2\pi]$, for which $|f(x + h) - f(x)| \le Mh$ for some positive constant M. This space has been studied and generalized in several different ways. The first generalization is to replace h with h^α where $0 < \alpha \le 1$ to obtain the so called Lipschitz spaces Lip^α of order α. Another generalization has been to replace h with a positive function $\rho(h)$ playing the role of a weight (See [3],[5]). Recently, De Souza in [4] gave a generalization related to the space Lip^α for general measures on subsets of the interval $[0, 2\pi]$ for $0 < \alpha < 1$. In this note, we are concerned with a similar generalization $Lip(\mu, 1)$ to a space related

Received by the editors September 22, 2019; Accepted: October 09, 2019
www.nigerianmathematicalsociety.org; Journal available online at https://ojs.ictp.it/jnms/
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to the case $\alpha = 1$, where μ is a general measure on $[0, 2\pi]$ with certain properties. In particular, we prove that $Lip(\mu, 1)$ is the dual space of the generalization $B(\mu, 1)$ of the special atom space to general measures μ on $[0, 2\pi]$. We start with the definition of the Lipschitz condition for some functions, which the reader can find in any undergraduate or graduate text in Analysis, including [7].

Definition 1.1. A function $f : D \subset R^2 \to \mathbb{R}$ is said to satisfy the Lipschitz condition with respect to x if there exists $K > 0$ such that

$$|f(t, x_1) - f(t, x_2)| \leq K|x_1 - x_2|, \ \forall (t, x_1), (t, x_2) \in D. \tag{1}$$

Remark 1.2. We note that in Definition 1.1, if the reverse inequality also holds, that is, there exists $K > 1$ such that

$$\frac{1}{K}|x_1 - x_2| \leq |f(t, x_1) - f(t, x_2)| \leq K|x_1 - x_2|, \ \forall (t, x_1), (t, x_2) \in D,$$

then f is referred to as Bi-Lipschitz. This notion is used to define Lipschitz functions on topological manifolds, see [6].

Lipschitz functions occur almost in every aspect of Mathematics. In ordinary differential equations, the Lipschitz condition is used in the existence and uniqueness theorem: that is, if $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ satisfies the Lipschitz condition in D with respect to x, then the initial value problem

$$\begin{cases} \frac{dx}{dt} = f(t, x) & \text{for } (t, x) \in D \\ x(t_0) = x_0 \end{cases}$$

has a unique solution in D. In analysis, any function that satisfies the Lipschitz condition is said to be Lipschitz continuous or simply Lipschitz. It is known that functions with bounded derivative are Lipschitz functions, and that Lipschitz functions are almost everywhere differentiable (e.g. $f(x) = |x|$). The next definition is a generalization of Lipschitz functions, see [6], [7].

Definition 1.3. A function $f : [0, 2\pi] \to \mathbb{R}$ is said to be a Lipschitz function of order α if $\forall x \in [0, 2\pi]$, and $h > 0$,

$$\frac{|f(x + h) - f(x)|}{h^\alpha} \leq M, \quad \text{for some } M \geq 0 \text{ and } 0 < \alpha \leq 1.$$

Definition 1.4. We denote by Lip^α, $0 < \alpha \leq 1$ the set of all Lipschitz functions of order α and endow it with the norm

$$\|f\|_{Lip^\alpha} = \sup_{\substack{x \in [0, 2\pi] \\ h > 0}} \frac{|f(x + h) - f(x)|}{h^\alpha}.$$

Here the constants are identified as the zeros. It is worth noting that without this classification or some other condition(s) $\|\cdot\|_{Lip^\alpha}$ will not be a norm.

In this paper, we are concerned with the situation where $\alpha = 1$. That is, space of Lipschitz functions of order 1. The next definition is the space of derivatives of Lipschitz functions of order 1.

Definition 1.5. We define the space $(Lip^1)'$ as follows;

$$(Lip^1)' = \left\{ g' : S \subseteq [0, 2\pi] \to \mathbb{R} : g \in Lip^1 \right\}$$

where the prime denotes the derivative. We endow $(Lip^1)'$ with the "norm"

$$\|g'\|_{(Lip^1)'} := \|g\|_{Lip^1}, \text{where} \quad g \in Lip^1.$$

Remark 1.6. The space $((Lip^1)', \|\cdot\|_{(Lip^1)'})$ is equivalent to L_∞, and hence it is a Banach space.

Details of the proof of Remark 1.6 is provided in Section 3 of this paper.

The next definition is the definition of the special atom space which is a slight modification of the space introduced by De Souza in [4].

Definition 1.7. The special atom space is the space B of functions defined by

$$B = \left\{ f : [0, 2\pi] \to \mathbb{R} : f(t) = \sum_{n=1}^{\infty} c_n \frac{1}{|I_n|} \chi_{I_n}(t), \sum_{n \geq 1} |c_n| < \infty \right\}$$

where the c_n's are real numbers, χ_{I_n} is the characteristic function of the interval I_n in $[0, 2\pi]$ and $|I_n|$ denotes the length of the interval.

We endow B with the "norm" $\|f\|_B = \inf \sum_{n=1}^{\infty} |c_n|$ where the infimum is taken over all the representations of f.

Remark 1.8. It is worth noting that the space B contains all simple functions. That is, if f is a simple function with $f(t) = \sum_{n=1}^{k} c_n \chi_{I_n}(t)$ then $f \in B$. Also, every element in B is the limit of a sequence of simple functions. Thus, the space B is not the same as the space of simple functions by the definition. As, we shall show later the space B is equivalent to $L_1([0; 2\pi])$.

Theorem 1.9. *$(B, \|\cdot\|_B)$ is a Banach space.*

Note that Theorem 1.9 is a particular case of Theorem 2.7 in Section 2 and the proof is similar up to minor modifications. The next results are special cases of much general results obtained in Section 2, showing that the $(Lip^1)'$ is the dual space of the special atom space B.

Theorem 1.10 (Hölder's type inequality)**.** *If $f \in B$ and $g' \in (Lip^1)'$, then*

$$\left|\int_0^{2\pi} f(t)g'(t)dt\right| \leq \|f\|_B\|g'\|_{(Lip^1)'}.$$

Proof. Let $g \in Lip^1$, we observe that $\frac{|g(t+h)-g(t)|}{h} \leq \|g\|_{Lip^1}$, for all $t \in [0, 2\pi]$ and $h > 0$. Thus $|g'(t)| \leq \|g\|_{Lip^1}$, for almost all $t \in [0, 2\pi]$. Now let $f \in B$ with

$$f(t) = \sum_{n\geq 1} c_n \frac{1}{|I_n|}\chi_{I_n}(t), \text{ and } \sum_{n\geq 1}|c_n| < \infty,$$

we have that

$$\int_0^{2\pi} f(t)g'(t)dt = \int_0^{2\pi}\sum_{n\geq 1}\left(c_n\frac{1}{|I_n|}g'(t)\chi_{I_n}(t)\right)dt.$$

Thus

$$\begin{aligned}\left|\int_0^{2\pi} f(t)g'(t)dt\right| &\leq \sum_{n\geq 1}|c_n|\frac{1}{|I_n|}\int_{I_n}\left|g'(t)\right|dt \\ &\leq \sum_{n\geq 1}|c_n|\frac{1}{|I_n|}|I_n|\|g\|_{Lip^1}, \text{ since } \quad |g'(t)| \leq \|g\|_{Lip^1} \\ &\leq \left(\sum_{n\geq 1}|c_n|\right)\|g\|_{Lip^1}.\end{aligned}$$

Taking the infimum on the R.H.S. of the latter over all representations of f, we have

$$\left|\int_0^{2\pi} f(t)g'(t)dt\right| \leq \|f\|_B\|g'\|_{(Lip^1)'}.$$

□

We will denote the dual space of a normed space S by $S^\star$. That is, $S^\star$ is the set of all bounded linear functionals on S.

Theorem 1.11. *The dual space* $B^\star$ *of* B, *is* $(Lip^1)'$. *That is,* $\phi \in B^\star$ *if and only if there exists* $g' \in (Lip^1)'$ *so that* $\phi(f) = \int_0^{2\pi} f(t)g'(t)dt$, $\forall f \in B$ *and* $\|\phi\|_{B^\star} = \|g'\|_{(Lip^1)'}$.

Proof. $\Longleftarrow$. Fix $g' \in (Lip^1)'$ and define $\phi_g(f) = \int_0^{2\pi} f(t)g'(t)dt$ for all $f \in B$. ϕ_g is a linear map on B, and $|\phi_g(f)| \leq \|f\|_B \|g'\|_{(Lip^1)'}$, by Theorem 1.10. Hence $\phi_g \in B^\star$

$\Longrightarrow$. Consider the map $\psi : (Lip^1)' \to B^\star$ defined by $\psi(g') = \phi_g$, ϕ_g defined as above. We want to show that ψ is onto, i.e. given $\phi \in B^\star$, there exists $g' \in (Lip^1)'$ such that $\phi = \phi_g$. Let $\phi \in B^\star$, and define $g(t) = \phi\left(\chi_{(0,t]}\right)$, $t \in [0, 2\pi]$.

Claim: $g \in Lip^1$ and hence $g' \in (Lip^1)'$.

In fact, observe that

$$g(t+h) - g(t) = \phi(\chi_{(0,t+h]} - \chi_{(0,t]}) = \phi(\chi_{[t,t+h]}).$$

Thus

$$|g(t+h) - g(t)| = |\phi(\chi_{[t,t+h]})| \leq \|\phi\|_{B^\star} \|\chi_{[t,t+h]}\|_B \leq \|\phi\|_{B^\star} h.$$

It follows that

$$\frac{|g(t+h) - g(t)|}{h} \leq \|\phi\|_{B^\star} < \infty, \ \forall h > 0.$$

Hence the claim is proved. Thus, we have that $g'(t)$ exists almost everywhere.

This implies that

$$\phi\left(\chi_{(0,t]}\right) = g(t) = \int_0^t g'(s)ds = \int_0^{2\pi} g'(s)\chi_{[0,t]}(s)ds.$$

Now since

$$\chi_{[a,b]}(t) = \chi_{[0,b]}(t) - \chi_{[0,a]}(t) \ \text{ for } a < b,$$

we have that

$$\begin{aligned}
\phi(\chi_{[a,b]}) &= \phi(\chi_{[0,b]}) - \phi(\chi_{[0,a]}) \text{ since } \phi \text{ is linear} \\
&= \int_0^{2\pi} g'(t)\chi_{[0,b]}(t)dt - \int_0^{2\pi} g'(t)\chi_{[0,a]}(t)dt \\
&= \int_0^{2\pi} g'(t)\left(\chi_{[0,b]}(t) - \chi_{[0,a]}(t)\right)dt \\
&= \int_0^{2\pi} g'(t)\chi_{[a,b]}(t)dt.
\end{aligned}$$

Therefore

$$\phi\left(\frac{1}{b-a}\chi_{[a,b]}\right) = \int_0^{2\pi} \frac{1}{b-a} g'(t)\chi_{[a,b]}(t)dt.$$

For $f(t) = \sum_{n\geq 1} c_n \frac{1}{|I_n|}\chi_{I_n}(t)$ with $\sum_{n\geq 1} |c_n| < \infty$, we have

$$f(t) = \lim_{k\to\infty} f_k(t) \text{ where } \quad f_k(t) = \sum_{n=1}^{k} c_n \frac{1}{|I_n|}\chi_{I_n}(t), k \in \mathbb{N}.$$

For each $k \in \mathbb{N}$,

$$\begin{aligned}
\phi(f_k) &= \phi\left(\sum_{n=1}^{k} c_n \frac{1}{|I_n|}\chi_{I_n}\right) \\
&= \sum_{n=1}^{k} c_n \frac{1}{|I_n|}\phi\left(\chi_{I_n}\right) \\
&= \sum_{n=1}^{k} c_n \frac{1}{|I_n|}\int_0^{2\pi} \chi_{I_n}(t)g'(t)dt \\
&= \int_0^{2\pi} \left(\sum_{n=1}^{k} c_n \frac{1}{|I_n|}\chi_{I_n}(t)\right) g'(t)dt \\
&= \int_0^{2\pi} f_k(t)g'(t)dt.
\end{aligned}$$

That is,

$$\phi(f_k) = \int_0^{2\pi} f_k(t)g'(t)dt.$$

Now since $\phi \in B^\star$, it follows that

$$\lim_{k\to\infty} \phi(f_k) = \phi(f).$$

On the other hand, we have that

$$\int_0^{2\pi} f_k(t)g'(t)dt \to \int_0^{2\pi} f(t)g'(t)dt.$$

To see this, let $h_k(t) = f_k(t)g'(t)$ and $p_k(t) = \sum_{n=1}^{k} |c_n| \frac{1}{|I_n|}|g'(t)|\chi_{I_n}(t)$.

We observe that

$$|h_k(t)| \leq p_k(t) \quad \text{for all} \quad k \in \mathbb{N} \quad \text{and} \quad t \in [0, 2\pi].$$

In addition,

$$0 \le p_k(t) \le p_{k+1}(t), \quad \text{for} \quad t \in [0, 2\pi]$$

and

$$p_k(t) \to p(t) := \sum_{n \ge 1} |c_n| \frac{1}{|I_n|} |g'(t)| \chi_{I_n}(t).$$

So by the Monotone convergence theorem (see [9], page 83), we have that

$$\int_0^{2\pi} p_k(t)dt \to \int_0^{2\pi} p(t)dt = \sum_{n \ge 1} |c_n| \frac{1}{|I_n|} \int_{I_n} |g'(t)|dt \le \|g\|_{Lip^1} \sum_{n \ge 1} |c_n| < \infty.$$

That is

$$\int_0^{2\pi} p_k(t)dt \to \int_0^{2\pi} p(t)dt < \infty.$$

Hence by the Dominated convergence theorem (see [9], page 89), we have that

$$\lim_{k \to \infty} \int_0^{2\pi} h_k(t)dt = \int_0^{2\pi} \lim_{k \to \infty} h_k(t)dt.$$

Thus

$$\int_0^{2\pi} f_k(t)g'(t)dt \to \int_0^{2\pi} f(t)g'(t)dt.$$

Hence

$$\phi(f) = \int_0^{2\pi} f(t)g'(t)dt.$$

That is $\phi = \phi_g$.Therefore, ψ is onto. In addition we have,

$$\|\phi\|_{B^\star} = \sup_{\|f\|_B \le 1} |\phi(f)| \le \|g'\|_{(Lip^1)'} \text{ by the Hölder's inequality.}$$

That is

$$\|\phi\|_{B^\star} \le \|g'\|_{(Lip^1)'}.$$

On the other hand, for $f_h(t) = \dfrac{1}{h}\chi_{[x,x+h]}(t),\ h > 0$, we have $f_h \in B$ with $\|f_h\|_B \le 1$ and

$$\phi(f_h) = \frac{1}{h}\int_0^{2\pi} \chi_{[x,x+h]}(t)g'(t)dt = \frac{1}{h}\int_x^{x+h} g'(t)dt = \frac{g(x+h) - g(x)}{h}.$$

This implies that

$$|\phi(f_h)| = \frac{|g(x+h) - g(x)|}{h} \le \|\phi\|_{B^\star}.$$

Taking the supremum over $x \in [0, 2\pi]$ and $h > 0$, we obtain $\|g\|_{Lip^1} \leq \|\phi\|_{B^\star}$. So that

$$\|\phi\|_{B^\star} = \|g'\|_{(Lip^1)'}$$

□

Remark 1.12. By Remark 1.6 and Theorem 1.11, we deduce that $B^\star \cong L_\infty$.

Theorem 1.13. *The special atom space B is continuously contained in L_1 and*

$$\|f\|_1 \leq C \|f\|_B \,, \text{ for } f \in B$$

.

Proof. Let $f \in B$ with $f(t) = \sum_{n\geq 1} c_n \frac{1}{|I_n|}\chi_{I_n}(t)$ and $\sum_{n\geq 1} |c_n| < \infty$ and consider

$$\int_0^{2\pi} |f(t)|dt \leq \sum_{n\geq 1} |c_n| \frac{1}{|I_n|} \int_{I_n} 1 \; dt = \sum_{n\geq 1} |c_n| < \infty.$$

So $f \in L_1$ and $\|f\|_1 \leq \|f\|_B \,,$ for $f \in B$.

□

The following theorem is a classical result in Functional Analysis, which can be found in [8] (see page 160).

Theorem 1.14. *Let X and Y be two normed linear spaces, and let $T \in \mathcal{L}(X, Y)$. Let $T^\star$ be the adjoint operator of T defined by $T^\star f = f \circ T$ for all $f \in Y^\star$. Then*

(1) $T^\star \in \mathcal{L}(Y^\star, X^\star)$ and $\|T^\star\| = \|T\|$
(2) $T^\star$ is injective if and only if the range of T is dense in Y. In addition, if X and Y are Banach spaces then $T^\star$ is invertible if and only if T is invertible.

Now, we have the following situations;

(1) $B \subseteq L_1$ with $\|f\|_1 \leq \|f\|_B \,,$ for $f \in B$ by Theorem 1.13.
(2) $B^\star \cong L_1^\star$ by Remark 1.12.
(3) B is dense in L_1. This can be verified with standard techniques and a corollary of the Hahn-Banach Theorem.

As a consequence of these facts and Theorem 1.14, the embedding operator $I : B \to L_1$ defined by $I(f) = f$ is a Banach space isomorphism. So, we have the following result;

Theorem 1.15. *$B \cong L_1$ with equivalent norms, i.e, there exist $f \in B \iff f \in L_1$ and $\alpha \|f\|_B \leq \|f\|_1 \leq \beta \|f\|_B$ form some absolute positive constants α and β.*

2. GENERALIZATION OF $(Lip^1)'$ AND B

Let $X = [0, 2\pi]$ and let $(X, \mathcal{M}, \mu)$ be a finite, non-atomic measure space. The next definition is a natural generalization of the space of derivatives of Lipschitz functions to general measures.

Definition 2.1. Define the space $Lip(\mu, 1)$ as

$$Lip(\mu,1) = \left\{ g : [0, 2\pi] \to \mathbb{R} : \frac{1}{\mu(A)} \left| \int_A g(t) d\mu(t) \right| \leq C < \infty, \ \forall A \in \mathcal{M}, \ \mu(A) \neq 0 \right\}.$$

Endow $Lip(\mu, 1)$ with the "norm"

$$\|g\|_{Lip(\mu,1)} = \sup_{\mu(A) \neq 0} \frac{1}{\mu(A)} \left| \int_A g(t) d\mu(t) \right|.$$

Remark 2.2. Here, we consider the equivalence classes of integrable functions that are equal almost everywhere. The space $Lip(\mu, 1)$ as we shall show later is also $L_\infty(X)$ under the condition that μ is finite and nonatomic. However, the representation of the norm provides an easy way to see the connection between the derivatives of Lipschitz functions and L_∞ functions, and also obtain the dual of the generalized special atom space. Indeed, we observe that if we take μ =Lebesgue measure and the measurable sets to be intervals then we have a more general representation of the norm on $(Lip^1)'$. That is if $g \in Lip^1$ and $A = [x, x + h]$ then $\frac{1}{\mu(A)} \left| \int g'(t) d\mu(t) \right| = \frac{|g(x+h) - g(x)|}{h}$.

Lemma 2.3. *If $g \in Lip(\mu, 1)$ and $A \in \mathcal{M}$ then*

$$\int_A |g(t)| d\mu(t) \leq \mu(A) \|g\|_{Lip(\mu,1)}.$$

Proof. Let $g \in Lip(\mu, 1)$ and $A \in \mathcal{M}$. Now let $A_+ = \{t \in A : g(t) \geq 0\}$ and $A_- = \{t \in A : g(t) < 0\}$. We have that $A_-, A_+ \in$

$\mathcal{M}$, $A = A_+ \cup A_-$ and $A_+ \cap A_- = \emptyset$. Now consider

$$\begin{aligned}
\int_A |g(t)| d\mu(t) &= \int_{A_+} g(t) d\mu(t) - \int_{A_-} g(t) d\mu(t) \\
&\leq \left| \int_{A_+} g(t) d\mu(t) \right| + \left| \int_{A_-} g(t) d\mu(t) \right| \\
&\leq \mu(A_+) \|g\|_{Lip(\mu,1)} + \mu(A_-) \|g\|_{Lip(\mu,1)} \\
&= \mu(A) \|g\|_{Lip(\mu,1)}
\end{aligned}$$

That is,

$$\int_A |g(t)| d\mu(t) \leq \mu(A) \|g\|_{Lip(\mu,1)}.$$

Hence the Lemma is proved. $\square$

Theorem 2.4.

(a) $\|\cdot\|_{Lip(\mu,1)}$ *is a norm on* $Lip(\mu, 1)$.
(b) $\big(Lip(\mu, 1), \|\cdot\|_{Lip(\mu,1)}\big)$ *is Banach space.*

Proof.
(a) To show that $\|\cdot\|_{Lip(\mu,1)}$ is a norm, first observe that $\|g\|_{Lip(\mu,1)} \geq 0$, $\forall g \in Lip(\mu, 1)$. Now suppose $\|g\|_{Lip(\mu,1)} = 0$. Then $\left| \int_A g(t) d\mu(t) \right| = 0$, $\forall A \in \mathcal{M}$ with $\mu(A) \neq 0$. Thus $\int_A g(t) d\mu(t) = 0$, $\forall A \in \mathcal{M}$ with $\mu(A) \neq 0$. This implies that $g = 0$, $\mu - a.e.$
For $\alpha \in \mathbb{R}$ and $g \in Lip(\mu, 1)$, we have

$$\begin{aligned}
\|\alpha g\|_{Lip(\mu,1)} &= \sup_{\mu(A)\neq 0} \frac{1}{\mu(A)} \left| \int_A \alpha g(t) d\mu(t) \right| \\
&= \sup_{\mu(A)\neq 0} |\alpha| \frac{1}{\mu(A)} \left| \int_A g(t) d\mu(t) \right| \\
&= |\alpha| \sup_{\mu(A)\neq 0} \frac{1}{\mu(A)} \left| \int_A g(t) d\mu(t) \right| \\
&= |\alpha| \, \|g\|_{Lip(\mu,1)}.
\end{aligned}$$

Finally, for $f, g \in Lip(\mu, 1)$ and $A \in \mathcal{M}$ with $\mu(A) \neq 0$, we have

$$\begin{aligned}
\frac{1}{\mu(A)} \left| \int_A (f(t) + g(t)) \, d\mu(t) \right| &\leq \frac{1}{\mu(A)} \left| \int_A f(t) d\mu(t) \right| + \frac{1}{\mu(A)} \left| \int_A g(t) d\mu(t) \right| \\
&\leq \|f\|_{Lip(\mu,1)} + \|g\|_{Lip(\mu,1)}
\end{aligned}$$

Taking the supremum on the L.H.S of the above inequality, we get

$$\|f + g\|_{Lip(\mu,1)} \leq \|f\|_{Lip(\mu,1)} + \|g\|_{Lip(\mu,1)}.$$

Thus $\|\cdot\|_{Lip(\mu,1)}$ is a norm on $Lip(\mu,1)$. To complete the proof, we need to proved that $Lip(\mu,1)$ is a complete space. In order to do so, it is sufficient to prove that for any sequence $(g_n)_{n\in\mathbb{N}} \subseteq Lip(\mu,1)$ such that $\sum\limits_{n\geq 1} \|g_n\|_{Lip(\mu,1)} \leq C < \infty$, we have

$$\sum_{n\geq 1} g_n \in Lip(\mu,1) \quad \text{and} \quad \left\|\sum_{n\geq 1} g_n\right\|_{Lip(\mu,1)} \leq \sum_{n\geq 1} \|g_n\|_{Lip(\mu,1)}.$$

To do this, let $(g_n)_{n\in\mathbb{N}} \subseteq Lip(\mu,1)$ such that $\sum\limits_{n\geq 1} \|g_n\|_{Lip(\mu,1)} \leq C < \infty$. Let $G = \sum\limits_{n\geq 1} |g_n|$, we observe that

$$\begin{aligned}
\int_X G(t)d\mu(t) &= \int_X \sum_{n\geq 1} |g_n(t)|d\mu(t) \\
&= \sum_{n\geq 1} \int_X |g_n(t)|d\mu(t) \text{ by the Monotone Convergence Theorem} \\
&\leq \sum_{n\geq 1} \mu(X)\|g_n\|_{Lip(\mu,1)} \text{ by Lemma 2.3}
\end{aligned}$$

Thus $\int_X G(t)d\mu(t) < \infty$, since μ is a finite measure. Hence, $G(t) < \infty$ a.e, which implies that the series $\sum\limits_{n\geq 1} g_n$ converges a.e and also integrable. Now, to show that $\sum\limits_{n\geq 1} g_n \in Lip(\mu,1)$, let $A \in \mathcal{M}$ with $\mu(A) \neq 0$. We have that

$$\begin{aligned}
\left|\int_A \sum_{n\geq 1} g_n(t)d\mu(t)\right| &\leq \int_A \left|\sum_{n\geq 1} g_n(t)\right| d\mu(t) \\
&\leq \int_A \sum_{n\geq 1} |g_n(t)|\, d\mu(t) \\
&= \sum_{n\geq 1} \int_A |g_n(t)|d\mu(t) \\
&\leq \sum_{n\geq 1} \mu(A)\|g_n\|_{Lip(\mu,1)} \text{ by Lemma 2.3}
\end{aligned}$$

That is, $\dfrac{1}{\mu(A)}\left|\int_A \sum_{n\geq 1} g_n d\mu\right| \leq \sum_{n\geq 1} \|g_n\|_{Lip(\mu,1)} \leq C < \infty$. Hence $\sum_{n\geq 1} g_n \in Lip(\mu,1)$. Taking the supremum on the L.H.S of the latter over all $A \in \mathcal{M}$ with $\mu(A) \neq 0$, we have

$$\left\|\sum_{n\geq 1} g_n\right\|_{Lip(\mu,1)} \leq \sum_{n\geq 1} \|g_n\|_{Lip(\mu,1)}.$$

□

The next definition is a natural extension of the special atom space B to general measures μ defined on $[0, 2\pi]$ first proposed by G. de Souza in [4].

Definition 2.5. We define the space $B(\mu, 1)$ as

$$B(\mu,1) = \left\{ f : [0,2\pi] \to \mathbb{R} : f(t) = \sum_{n\geq 1} c_n \frac{1}{\mu(A_n)} \chi_{A_n}(t) \text{ and } \sum_{n\geq 1} |c_n| < \infty \right\},$$

where the c_n's are real numbers, and the $A_n \in \mathcal{M}$ for each $n \geq 1$. We endow $B(\mu, 1)$ with the "norm" $\|f\|_{B(\mu,1)} = \inf \sum_{n\geq 1} |c_n|$, where the infimum is taken over all possible representations of f.

Remark 2.6. Similar to Remark1.8, we have that every simple function belongs to $B(\mu, 1)$. That is, if f is a simple function with $f(t) = \sum_{n=1}^{k} c_n \chi_{A_n}(t)$ then $f \in B(\mu, 1)$. Also, every element in $B(\mu, 1)$ is the limit of a sequence of simple functions. Similarly, $B(\mu, 1)$ is not the same as the space of simple functions by the definition. As, we shall show later the space $B(\mu, 1)$ is equivalent to $L^1(X)$.

Theorem 2.7.

(a) $\|\cdot\|_{B(\mu,1)}$ *is a norm of* $B(\mu, 1)$.
(b) $\big(B(\mu, 1), \|\cdot\|_{B(\mu,1)}\big)$ *is a Banach space.*

Proof. (a) To show that $\|\cdot\|_{B(\mu,1)}$ is a norm, observe that $\|f\|_{B(\mu,1)} \geq 0, \forall f \in B(\mu, 1)$ and $f = 0$ implies that $\|f\|_{B(\mu,1)} = 0$. On the other hand, suppose $\|f\|_{B(\mu,1)} = 0$. We want to show that $f = 0$, μ *a.e.* Let $(c_{nk})_{n,k\in\mathbb{N}}$ be a sequence of real numbers and $(A_{nk})_{n,k\in\mathbb{N}}$ be a sequence of measurable subsets of X such that $f(t) = \sum_{n\geq 1} c_{nk} \frac{1}{\mu(A_{nk})} \chi_{A_{nk}}(t)$

with $\sum_{n\geq 1}|c_{nk}|<\infty$ for each $k\in\mathbb{N}$ and $\sum_{n\geq 1}|c_{nk}|\to 0$ as $k\to\infty$. So we have for each $n\in\mathbb{N}$, $|c_{nk}|\to 0$ as $k\to\infty$. Thus, the coefficients of the representations of f converges to zero and hence $f=0$, μ *a.e.*

For $\alpha\in\mathbb{R}$ and $f\in B(\mu,1)$ with $f(t)=\sum_{n\geq 1}c_n\frac{1}{\mu(A_n)}\chi_{A_n}(t)$ and $\sum_{n\geq 1}|c_n|<\infty$, we have $(\alpha f)(t)=\sum_{n\geq 1}\alpha c_n\frac{1}{\mu(A_n)}\chi_{A_n}(t)$ and this implies that

$$\begin{aligned}\|\alpha f\|_{B(\mu,1)} &= \inf\sum_{n\geq 1}|\alpha c_n|\\ &= |\alpha|\inf\sum_{n\geq 1}|c_n|\\ &= |\alpha|\,\|f\|_{B(\mu,1)}.\end{aligned}$$

Finally, for $f,g\in B(\mu,1)$, to show that $\|f+g\|_{B(\mu,1)}\leq\|f\|_{B(\mu,1)}+\|g\|_{B(\mu,1)}$, let $\epsilon>0$ be given, and let $(c_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\mathbb{N}}$ be sequences of real numbers such that $f(t)=\sum_{n\geq 1}c_n\frac{1}{\mu(A_n)}\chi_{A_n}(t)$ and $g(t)=\sum_{n\geq 1}b_n\frac{1}{\mu(B_n)}\chi_{B_n}(t)$, for some sequence $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$, and such that $\sum_{n\geq 1}|c_n|<\|f\|_{B(\mu,1)}+\epsilon/2$, $\sum_{n\geq 1}|b_n|<\|g\|_{B(\mu,1)}+\epsilon/2$. Note that we can write

$$(f+g)(t)=\sum_{n\geq 1}d_n\frac{1}{\mu(D_n)}\chi_{D_n}(t),$$

with $\sum_{n\geq 1}|d_n|=\sum_{n\geq 1}|c_n|+|b_n|$ where

$$d_n=\begin{cases}c_{\frac{n}{2}} & \text{if } n \text{ is even}\\ b_{\frac{n+1}{2}} & \text{if } n \text{ is odd}\end{cases}$$

and

$$D_n=\begin{cases}A_{\frac{n}{2}} & \text{if } n \text{ is even}\\ B_{\frac{n+1}{2}} & \text{if } n \text{ is odd}\end{cases}$$

It follows that

$$\begin{aligned} \|f+g\|_{B(\mu,1)} &\leq \sum_{n\geq 1}|d_n| \\ &= \sum_{n\geq 1}|c_n| + \sum_{n\geq 1}|b_n| \\ &< \|f\|_{B(\mu,1)} + \|g\|_{B(\mu,1)} + \epsilon \end{aligned}$$

Thus since ϵ is arbitrary, we have

$$\|f+g\|_{B(\mu,1)} \leq \|f\|_{B(\mu,1)} + \|g\|_{B(\mu,1)}.$$

(b) To prove completeness, it suffices to show that for any sequence $(f_m)_{m\geq 1} \subseteq B(\mu,1)$, we have

$$\|\sum_{m\geq 1} f_m\|_{B(\mu,1)} \leq \sum_{m\geq 1}\|f_m\|_{B(\mu,1)}.$$

Note that given $\epsilon > 0$ and for each $m \geq 1$, there are sequence real numbers (c_{m_n}) and sequence of sets $A_{m_n} \in \mathcal{M}$ such that $f_m(t) = \sum_{n\geq 1} \frac{c_{m_n}}{\mu(A_{m_n})}\chi_{A_{m_n}}(t)$ with $\sum_{n\geq 1}|c_{m_n}| < \|f_m\|_{B(\mu,1)} + \frac{\epsilon}{2^m}$. It follows that

$$\sum_{m\geq 1}\sum_{n\geq 1}|c_{m_n}| < \sum_{m\geq 1}\|f_m\|_{B(\mu,1)} + \epsilon\sum_{m\geq 1}\frac{1}{2^m} = \sum_{m\geq 1}\|f_m\|_{B(\mu,1)} + \epsilon.$$

Since ϵ is arbitrary, it follows that

$$\|\sum_{m\geq 1} f_m\|_{B(\mu,1)} \leq \sum_{m\geq 1}\|f_m\|_{B(\mu,1)}.$$

□

Theorem 2.8 (Hölder's Type Inequality)**.** *If $f \in B(\mu,1)$ and $g \in Lip(\mu,1)$, then*

$$\left|\int_X f(t)g(t)d\mu(t)\right| \leq \|f\|_{B(\mu,1)}\|g\|_{Lip(\mu,1)}.$$

Proof. Let $g \in Lip(\mu,1)$ and $f \in B(\mu,1)$ with $f(t) = \sum_{n\geq 1} c_n \frac{1}{\mu(A_n)}\chi_{A_n}(t)$ and $\sum_{n\geq 1}|c_n| < \infty$, we have

$$\int_X f(t)g(t)d\mu(t) = \int_X \sum_{n\geq 1}\left(c_n\frac{1}{\mu(A_n)}\chi_{A_n}(t)g(t)\right)d\mu(t).$$

It follows that

$$\begin{aligned}\left|\int_X f(t)g(t)d\mu(t)\right| &\leq \sum_{n\geq 1}|c_n|\frac{1}{\mu(A_n)}\int_X |g(t)|\,d\mu(t)\\ &\leq \sum_{n\geq 1}|c_n|\frac{1}{\mu(A_n)}\mu(A_n)\|g\|_{Lip(\mu,1)},\ \text{by Lemma 2.3}\\ &= \|g\|_{Lip(\mu,1)}\left(\sum_{n\geq 1}|c_n|\right).\end{aligned}$$

Taking the infimum over all possible representations of f, we have

$$\left|\int_X f(t)g(t)d\mu(t)\right| \leq \|f\|_{B(\mu,1)}\|g\|_{Lip(\mu,1)}.$$

□

The following result is a classical result in real analysis. (See for example [8], page 55 for a proof.)

Theorem 2.9. *Suppose that $\{f_n\}$ is a sequence in $L_1(X)$ such that $\sum_{n\geq 1}\int_X |f_n|d\mu < \infty$. Then $\sum_{n\geq 1} f_n$ converges a.e to a function in $L_1(X)$, and $\int_X \sum_{n\geq 1} f_n d\mu = \sum_{n\geq 1}\int_X f_n d\mu$.*

Lemma 2.10. *Let $(c_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such $\sum_{n\geq 1}|c_n| < \infty$, $(A_n)_{n\in\mathbb{N}}$ be a sequence of measurable subsets of X and $g \in Lip(\mu,1)$. For each $n\in\mathbb{N}$, define $h_n : X \to \mathbb{R}$ by $h_n(t) := c_n\frac{1}{\mu(A_n)}\chi_{A_n}(t)g(t)$. Then $\sum_{n\geq 1} h_n$ converges a.e. to a function in $L_1(X)$, and $\int_X \sum_{n\geq 1} h_n(t)d\mu(t) = \sum_{n\geq 1}\int_X h_n(t)d\mu(t)$.*

Proof. Let $n\in\mathbb{N}$ and consider

$$\begin{aligned}\int_X |h_n(t)|d\mu(t) &= \int_X |c_n|\frac{1}{\mu(A_n)}\chi_{A_n}(t)|g(t)|\mu(t)\\ &= |c_n|\frac{1}{\mu(A_n)}\int_{A_n}|g(t)|d\mu(t)\\ &\leq |c_n|\|g\|_{Lip(\mu,1)} < \infty,\ \text{by Lemma 2.3}\end{aligned}$$

Thus, $h_n \in L_1(X)$ and

$$\sum_{n\geq 1}\int_X |h_n(t)|d\mu(t) \leq \|g_n\|_{Lip(\mu,1)}\left(\sum_{n\geq 1}|c_n|\right) < \infty.$$

The conclusion follows from Theorem 2.9 $\square$

Theorem 2.11 (Duality)**.** *$B^\star(\mu,1) \equiv Lip(\mu,1)$ with equivalent norms, that is, $\varphi \in B^\star(\mu,1)$ if and only if there exists $g \in Lip(\mu,1)$ such that $\varphi(f) = \int_X f(t)g(t)d\mu(t),\ \forall f \in B(\mu,1)$. Moreover,*

$$\|\varphi\| = \|g\|_{Lip(\mu,1)}.$$

Proof.
"$\Longleftarrow$:" Fix $g \in Lip(\mu,1)$ and define $\varphi_g : B(\mu,1) \to \mathbb{R}$ by

$$\varphi_g(f) = \int_X f(t)g(t)d\mu(t),\ \forall f \in B(\mu,1). \tag{2}$$

Then clearly φ_g is a linear map and by Theorem 2.8, we have

$$|\varphi_g(f)| \leq \|f\|_{B(\mu,1)}\|g\|_{Lip(\mu,1)}. \text{ Thus } \varphi_g \in B^\star(\mu,1).$$

"$\Longrightarrow$:" Consider the map $\psi : Lip(\mu,1) \to B^\star(\mu,1)$ define by $\psi(g) = \varphi_g$ where φ_g is defined as in (2). We want to show that ψ in onto. Let $\varphi \in B^\star(\mu,1)$. Define $\lambda : \mathcal{M} \to \mathbb{R}$ by $\lambda(A) = \varphi(\chi_A), \forall A \in \mathcal{M}$. For any disjoint sequence of measurable subsets $(A_n)_{n=1}^\infty$ in $\mathcal{M}$, let $A = \cup_{n=1}^\infty A_n$. Without loss of generality assume that for each n, $\mu(A_n) \neq 0$. We have that

$$\chi_A = \sum_{n=1}^\infty \chi_{A_n} = \sum_{n=1}^\infty \mu(A_n)\frac{1}{\mu(A_n)}\chi_{A_n}$$

and $\sum_{n=1}^\infty \mu(A_n) = \mu(\cup_{n=1}^\infty A_n) < \infty$, since μ is a finite measure. Hence $\chi_A \in B(\mu,1)$. In addition, the series converges in $B(\mu,1)$, since

$$\left\|\chi_A - \sum_{n=1}^k \chi_{A_n}\right\|_{B(\mu,1)} = \left\|\sum_{n=k+1}^\infty \chi_{A_n}\right\|_{B(\mu,1)} \leq \sum_{n=k+1}^\infty \mu(A_n) = \mu(\cup_{n=k+1}^\infty A_n)$$

which approaches to 0 as $k \to \infty$. Hence, since φ is linear and continuous,

$$\lambda(\cup_{n=1}^\infty A_n) = \lambda(A) = \sum_{n=1}^\infty \varphi(\chi_{A_n}) = \sum_{n=1}^\infty \lambda(A_n).$$

This proves that λ is a σ additive and hence a (signed) measure. We observe that

$$|\lambda(A)| = |\varphi(\chi_A)| \leq \|\varphi\| \|\chi_A\|_{B(\mu,1)} \leq \|\varphi\|\mu(A).$$

Thus if $\mu(A) = 0$ then $\lambda(A) = 0$ and thus $\lambda << \mu$. Hence by the Radon-Nikodym Theorem, we have that $\lambda(A) = \int_A g d\mu$ for some $g \in L_1(X)$. In particular, $g \in Lip(\mu, 1)$ since $\int_A g d\mu = \varphi(\chi_A)$ implies $\left|\int_A g d\mu\right| \leq \|\varphi\|\mu(A)$. Thus

$$\frac{1}{\mu(A)}\left|\int_A g d\mu\right| \leq \|\varphi\| < \infty, \quad \forall A \in \mathcal{M} \text{ with } \mu(A) \neq 0.$$

So we have

$$\varphi(\chi_A) = \int_A g d\mu = \int_X g\chi_A d\mu.$$

That is,

$$\varphi(\chi_A) = \int_X \chi_A(t)g(t)d\mu(t).$$

Now given $f \in B(\mu, 1)$ with $f(t) = \sum_{n\geq 1} c_n \frac{1}{\mu(A_n)}\chi_{A_n}(t)$ and $\sum_{n\geq 1} |c_n| < \infty$, we have that $\varphi(f) = \varphi\left(\sum_{n\geq 1} c_n \frac{1}{\mu(A_n)}\chi_{A_n}\right) = \sum_{n\geq 1} c_n \frac{1}{\mu(A_n)}\varphi(\chi_{A_n})$, since $\varphi \in B^\star(\mu, 1)$. So we get,

$$\begin{aligned}
\varphi(f) &= \sum_{n\geq 1} c_n \frac{1}{\mu(A_n)} \int_X \chi_{A_n}(t)g(t)d\mu(t) \\
&= \sum_{n\geq 1} \int_X \left(c_n \frac{1}{\mu(A_n)}\chi_{A_n}(t)g(t)\right) d\mu(t) \\
&= \int_X \sum_{n\geq 1} \left(c_n \frac{1}{\mu(A_n)}\chi_{A_n}(t)g(t)\right) d\mu(t), \text{ by Lemma 2.10} \\
&= \int_X \left(\sum_{n\geq 1} c_n \frac{1}{\mu(A_n)}\chi_{A_n}(t)\right) g(t)d\mu(t) \\
&= \int_X f(t)g(t)d\mu(t).
\end{aligned}$$

Thus

$$\varphi(f) = \int_X f(t)g(t)d\mu(t).$$

This shows that $\varphi_g(f) = \varphi(f)$, $\forall f \in B(\mu,1)$ and for some $g \in Lip(\mu,1)$. That is, $\psi(g) = \varphi_g = \varphi$. It follows that the inclusion map $i : Lip(\mu,1) \to B^\star(\mu,1)$ is a bijection. Moreover, it follows from the inequality $|\varphi(f)| \leq \|f\|_{B(\mu,1)}\|g\|_{Lip(\mu,1)}$ that

$$\|\varphi\| = \sup_{\|f\|_{B(\mu,1)}\leq 1} |\varphi(f)| \leq \|g\|_{Lip(\mu,1)}.$$

Let $A \in \mathcal{M}$ with $\mu(A) \neq 0$, and let $f = \dfrac{1}{\mu(A)}\chi_A$. We have that $f \in B(\mu,1)$ with $\|f\|_{B(\mu,1)} \leq 1$ and

$$\varphi(f) = \frac{1}{\mu(A)}\int_X \chi_A(t)g(t)d\mu(t) = \frac{1}{\mu(A)}\int_A g(t)d\mu(t).$$

Thus

$$|\varphi(f)| = \frac{1}{\mu(A)}\left|\int_A g(t)d\mu(t)\right| \leq \|\varphi\|.$$

Taking the supremum on the L.H.S over all $A \in \mathcal{M}$ with $\mu(A) \neq 0$, we have

$$\|g\|_{Lip(\mu,1)} \leq \|\varphi\|, \text{ and hence } \|\varphi\| = \|g\|_{Lip(\mu,1)}.$$

□

Remark 2.12. By Remark 2.2 and Theorem 2.11, we deduce that $B^\star(\mu,1) \cong L_\infty(X)$.

Now, with arguments similar to those in Section 1, we have the following result;

Theorem 2.13. *$B(\mu,1) \cong L_1(\mu)$ with $m\,\|f\|_{B(\mu,1)} \leq \|f\|_1 \leq M\,\|f\|_{B(\mu,1)}$ for some absolute positive constants m and M.*

3. COMMENTS

One interesting thing about the space $(Lip^1)'$ and and $Lip(\mu,1)$, is their relation to $L_\infty(X)$.

(1) $(Lip^1)'$ is $L_\infty(X)$ (with respect to the Lebesgue measure). To see, let $g' \in (Lip^1)'$. We have that $g \in Lip^1$ with $\|g\|_{Lip^1} \leq C < \infty$. Thus, we have

$$\frac{|g(t+h) - g(t)|}{h} \leq C, \text{ for all } \quad t \in [0,2\pi] \text{ and } \quad h > 0.$$

Hence, $|g'(t)| \leq C$, for almost all $t \in [0, 2\pi]$. Thus, $g' \in L_\infty(X)$ and $\|g'\|_\infty \leq \|g'\|_{(Lip^1)'}$. To see the converse, let $g \in L_\infty \subseteq L_1$, i.e $g \in L_1$ and define $G : [0, 2\pi] \to \mathbb{R}$ by $G(t) = \displaystyle\int_0^t g(s)ds$, for $t \in [0, 2\pi]$. G is well-defined and Lipschitz with $G'(t) = g(t)$ a.e. Thus, $g \in (Lip^1)'$ and $\|g\|_{(Lip^1)'} \leq \|g\|_\infty$.

(2) Also, $Lip(\mu, 1)$ is $L_\infty(X)$. To see this, we first observe that if $g \in L_\infty(X)$ then, we have for any $A \in \mathcal{M}$ with $\mu(A) \neq 0$,

$$\frac{1}{\mu(A)} \left| \int_A g(t) d\mu(t) \right| \leq \frac{1}{\mu(A)} \int_A |g(t)| d\mu(t) \leq \|g\|_\infty \frac{1}{\mu(A)} \int_A 1 d\mu(t) \leq \|g\|_\infty.$$

So, $g \in Lip(\mu, 1)$ and $\|g\|_{Lip(\mu,1)} \leq \|g\|_\infty$. On the other hand, given $g \in Lip(\mu, 1)$ and $A \in \mathcal{M}$ with $\mu(A) \neq 0$, we obtain from Lemma 2.3 that

$$\int_A |g(t)| d\mu(t) \leq \mu(A) \|g\|_{Lip(\mu,1)}.$$

Since μ is a monoatomic finite measure, by a result in [10], page 65, there exist a measurable subset $\tilde{A}$ of X with $\mu(\tilde{A}) = \mu(A)$ such that

$$\int_0^{\mu(A)} g^*(t) d\mu(t) = \int_{\tilde{A}} |g(t)| d\mu(t),$$

where g^* denotes the decreasing rearrangement of $|g|$. Thus,

$$\int_0^{\mu(A)} g^*(t) dt \leq \mu(A) \|g\|_{Lip(\mu,1)}.$$

Since g^* is a decreasing function on $[0, \infty)$, we have that $g^*(\mu(A)) \leq g^*(t)$ for all $t \in [0, \mu(A)]$. Hence,

$$g^*(\mu(A))\mu(A) \leq \int_0^{\mu(A)} g^*(t) dt \leq \mu(A) \|g\|_{Lip(\mu,1)}.$$

So, $g^*(\mu(A)) \leq \|g\|_{Lip(\mu,1)}$. Let $\mu(A) \to 0$ to obtain $g^*(0) \leq \|g\|_{Lip(\mu,1)}$. But $\|g\|_\infty = g^*(0)$. It follows that

$$g \in L_\infty \text{ and hence } \|g\|_\infty = \|g\|_{Lip(\mu,1)}.$$

(3) If we let

$$\mathcal{S} = \left\{ g : [0, 2\pi] \to \mathbb{R} : \|g\|_{\mathcal{S}} = \sup_{\mu(A) \neq 0} \frac{1}{\mu(A)} \int_0^{\mu(A)} g^*(t) dt < \infty \right\},$$

then we can deduce that L_∞, $Lip(\mu,1)$, $\mathcal{S}$ and $M(1)$ (i.e the case $\alpha = 1$ in [11] for the space
$M(\alpha) = \left\{ g : [0,2\pi] \to \mathbb{R} : \|g\|_{M(\alpha)} = \sup_{\mu(A)\neq 0} \frac{1}{\mu(A)^\alpha} \int_A |g(t)| d\mu(t) < \infty \right\}$
introduced by G. G Lorentz) are all equivalent with equal norms.

(4) Although these observations shows that $(Lip^1)'$ and $Lip(\mu,1)$ are not new spaces, their representations provides an easy way to obtain the dual of the special atom spaces.

(5) From Theorem 1.15 and Theorem 2.13 we have that both B and $B(\mu,1)$ are equivalent to $L_1(X)$ for the Lebesgue measure and finite nonatomic measures respectively. It is however worth noting that the Lebesgue measure is also a nonatomic measure and thus for the decomposition of L_1 with the Lebesgue measure it is enough to consider only intervals.

4. CONCLUDING REMARKS

In 1980, De Souza in his Ph.D. thesis [1] introduced the now well-known special atom spaces, which consists of real-valued functions f defined on the interval $X = [0,2\pi]$ and which can be written as $f(t) = \sum_{n\geq 1} c_n b_n(t)$ where the c_n's are real numbers so that $\sum_{n\geq 1} |c_n| < \infty$ whereas the b_n's are functions defined as $b_n(t) = \frac{1}{|I_n|^\alpha} [\chi_{L_n}(t) - \chi_{R_n}(t)]$, for $0 < \alpha \leq 1$, $I_n = L_n \cup R_n$, $L_n \cap R_n = \emptyset$, I_n's are intervals, and L_n, R_n are respectively the left and right half sides of I_n. Considering also the space of functions f defined as above but with $b_n(t) = \frac{1}{|I_n|^\alpha}\chi_{I_n}(t)$, we observe that the atom space and this space are equivalent for $0 < \alpha < 1$ but yield two completely different spaces when $\alpha = 1$. Indeed, De Souza showed in [1] that the special atom space with $b_n(t) = \frac{1}{|I_n|} [\chi_{L_n}(t) - \chi_{R_n}(t)]$ is strictly contained in $L_1(X)$. However, in this note, we obtained that the special atom space with $b_n(t) = \frac{1}{|I_n|}\chi_{I_n}(t)$ which we denoted by B is equivalent to $L_1(X)$ and extended it to arbitrary measures. The space L_1 has been in existence for a long time, however we are not aware of this characterization in the literature. This therefore makes our study noteworthy.

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