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ON THE TOPOLOGICAL PICARD'S FIXED POINT ITERATIVE ALGORITHM FOR THE SIMPLEX METHOD OF OPTIMIZATION

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ABSTRACT. This research aims at generating the topological fixed point iterative scheme for the simplex method of linear programming problems in optimization as exemplified in the optimization of the flight attendants' hiring problem of the Nigerian Airways company displayed in section three. The generated main result in section two reaffirms that the traditional simplex method reviewed in section one is the Picard's fixed point iterative method which is seen illustrated in section three.

Keywords and phrases: Linear programming problem, Flight attendants', hiring problem, minimization of simplex method, maximization of Simplex method, sensitivity analysis, contraction map, iterative map
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1. INTRODUCTORY REVIEW OF THE SIMPLEX METHOD LEADING TO PICARD'S ALGORITHM

1.1 Background of study

Nigerian Airway founded in 1958 and headquartered in Lagos was owned by the Federal Government of Nigeria. It was started as part of West African Airways cooperation as a joint venture among Nigeria, Gambia, Sierra Leone and Ghana but later existed as an Independent Nigerian corporation in 1968. It abruptly ceased operation in 2002 due to corruption and mismanagement related cases but was later bought over by the virgin Nigeria Airways which up to now is the flag bearer of the former Nigerian Airways (Nwabuisi, 2008)

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In 2007, Okereke, E.C. formulated a Simplex model on the flight attendants' hiring and training problem of Nigeria Airways, the solution of which is contained in section three of this work. Classical analysis of the flight attendants' problem so far formulated and its simplex method of solution deeply created the understanding that the simplex problem of optimization is a reformulated Picard's fixed point problem with iterative method of solution (Argyros, 2005). Results of theorem 2.1 and 2.2 classically confirms this topologically.

1.2 Motivation

The flight attendants' hiring and training problem in the Nigeria Airways industry has never encountered meaningful improvement since the inception of the industry due to corruption, favouritism, Nepotism and all the likes (Ladan, Suleman, 2012), that department is yet to grow due to setbacks (Ogbeidi, 2006). Though the flagbear, the Virgin Airways has improved on the services of those flight attendants but the Nigerian factor is still working against its rapid development, hence the aim of this research including the academic aim of trying to generate its associated mathematical results.

1.3 Preliminary

Here, we introduce the primal simplex method for solutions of linear programming problems which is a classical linear programming method invented in the late forties by Danzig Gab according to (Fiacco and McCormic, 1968) and hence present the following:

Solution of A System of Linear Simultaneous Equation

Before studying the most general method of solving a linear programming problem, it will be useful to review the methods of solving a system of linear equations. Hence in the present section we review some of the elementary concepts of linear equations. Consider as in [Taha, 2005] the following system of n equations in $n-$unknowns.

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1(E_1)\\
&a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2(E_2)\\
&a_{31}x_1 + a_{32}x_2 + \ldots + a_{3n}x_n = b_3\,(E_3)\\
&\vdots\\
&a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n(E_n)
\end{aligned} \tag{1}$$

Assuming as in [Jetter, 1986] that the set of equations possesses a unique solution, a method of solving the system consists of reducing the equations to a form known as canonical form.
It is well known from elementary algebra that the solutions of equation (1) will not be altered under the following elementary operations:
(a) any equations E_r is replaced by the equations kE_r, where k is a non-zero-constant , and
(b) any equation E_r is replaced by the equation $E_r + kE_s$, where $E_s$is any other equation of the system. By making use of these elementary operations. The system of equation (1) can be reduced to a convenient equivalent form as follows. Let us select some variable x_1 and try to eliminate it from all the equations except the jth one (for which a_{ji} non zero). This can be accomplished by dividing the jth by a_{ji} and subtracting a_{ki} times the result from each of the other equations, $k = 1, 2, \ldots, j-1, j+1, \ldots, n$. The resulting system of equations can be written as in [Martin, 1999],[Murray, 1983]

$$\begin{aligned}
&a'_{11}x_1 + a'_{12}x_2 + \ldots + a'_{1,i-1}x_{i-1} + 0x_i + a'_{1.i+1}x_{i+1} + \ldots + \ldots a'_{1n}x_n = b'_1\\
&a'_{21}x_1 + a'_{22}x_2 + \ldots + a'_{2,i-1}x_{i-1} + 0x_i + a'_{2.i+1}x_{i+1} + \ldots + \ldots a'_{2n}x_n = b'_2\\
&\vdots\\
&a'_{j-1}x_1 + a'_{j-1.}x_2 + \ldots + a'_{j-1,i-1} + 0x_i + a'_{j-i.j+1}x_{i+1} + \ldots + a'_{i-1n}x_n = b'_{j-1}\\
&a'_{j1}x_1 + a'_{j2.}x_2 + \ldots + a'_{j,i-1,}x_{i-1} + 1x_i + a'_{j,i+1}x_{i+1} + \ldots + a'_{jn}x_n = b'_j\\
&a'_{j+1,1}x_1 + a'_{j+1,2}x_2 + \ldots + a'_{j+1,i-1,}x_{i-1} + 0x_i + a'_{j+1,i+1}x_{i+1} + \ldots + a'_{j+1,n}x_n = b'_j + 1\\
&a'_{n1}x_1 + a'_{n2}x_2 + \ldots + a'_{n,i-1,}x_{i-1} + 0x_i + a'_{n,i+1}x_{i+1} + \ldots + a'_{nn}x_n = b'_n
\end{aligned} \tag{2}$$

where the primes indicate that the a'_{ij} and b'_j are changed from the original system. This procedure of eliminating a particular variable from all but one equation is called a pivot operation. The system of equation (2) produced by the pivot operation have exactly the same solution as the original set of equation (1).That is, [Murray, 1983] the vector X that satisfies equation (1) and also equation (2) and vice versa.
Next time, if we take the system of equation (2) and perform a new pivot operating by eliminating $x_s, s \neq i$, in all the equations except the tth equation, $t \neq j$,the zeros or the 1 in the ith column will not be disturbed. The pivotal operations as in [Nash and Sofar, 1996] can be repeated by using a different variable and equation each time until the system of equation (1) is reduced to the form [Lueabenger, 1984]

$$\begin{aligned}
1x_1 + 0x_2 + 0x_3 + \ldots + 0x_n &= b_1^n\\
0x_1 + 1x_2 + 0x_3 + \ldots + 0x_n &= b_2^n
\end{aligned}$$

$$
\begin{aligned}
&0x_1 + 0x_2 + 1x_3 + \ldots 0x_n = \; b_3^n \\
&\vdots \\
&0x_1 + 0x_2 + 0x_3 + \ldots + 1x_n = b_n^n
\end{aligned}
\tag{3}
$$

This system of equations (3) is said to be in canonical form and has been obtained after carrying out a pivot operations.
From the canonical form, the solution vector can be directly obtained as in [Gass, 1990]

$$x_i = b_i^n \quad i = 1, 2, \ldots, n$$

Since the set of equations (3) has been obtained from equations (1) only through elementary operations the system of equations (3) to the system of equations (1). thus the solution given by equations (4) is desired solution of equations (1).

Pivotal Reduction of a General System of Equations

Instead of a square system, let us consider a system of m equations in n variables with n≥m. this system of equations is assumed to be consistent as in [Fourier, 1999] so that it will have at least one solution.

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\
&\ldots \\
&a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m
\end{aligned}
\tag{4}
$$

The solution vector(s) X that satisfy equation (4) are not evident from the equations. However, as in [Dantzig, 1998] it is possible to reduce this system to an equivalent canonical system from which at least one solution can readily be deduced. If pivotal operations with respect to any set of m variables, say $x_1, x_2, \ldots, x_m$, are carried, the resulting set of equations (4) can be written as follows:

canonical system with pivotal variables $x_{1,}x_{2,}, \ldots, x_m$

$$
\begin{aligned}
&1x_1 + 0x_2 + \ldots + 0x_m + a''_{1.m+1}x_{m+1} + \ldots + \; a''_{1n}x_n = b_1^n \\
&0x_1 + 1x_2 + \ldots + 0x_m + a'_{2.m+1}x_{m+1} + \ldots + \; a''_{2n}x_n = \; b_2^n \\
&\vdots \\
&0x_1 + 0x_2 + \ldots 1x_m + a''_{m,m+1}x_{i+1} + \ldots + \; a''_{mn}x_n = b_m^n
\end{aligned}
\tag{5}
$$

pivotal variables	Nonpivotal or independent variables	Constants

One special solution that can always be deducted from the system of equation (5) is as in [Brent, 1973]

$$\begin{cases} b''_i, & i = 1, 2, \ldots, m \\ 0 & i = m+1, m+2, \ldots, n \end{cases} \tag{6}$$

This solution is called a basic solution since the solution vector contains no more than m nonzero terms. The pivotal variables $x_i, i = 1, 2, \ldots, m$ are called the basic variables and the other variables $x_i, i = m+1, m+2, \ldots, n$are called non basic variables.Of course,this is not the only solution,but it is the one most readily deduced from equation (5). if all $b_i^{''}, i = 1, 2, \ldots, m$, in the solution given equation (6) are non- negative, it satisfies equation (3) in addition to equation (5), and hence it can be called a basic feasible solution.

It is possible as in [Brent, 1973] to obtain the other basic solutions from the canonical system of equation (5).We can perform an additional pivotal operation on the system after it is in canonical form, by choosing $a_{pq}^{''}$(which is nonzero) as the pivot term, $q > m$, and using any row $p\,(among\ 1, 2, \ldots m)$.The new system will still be in canonical form but with x_q as the pivotal variable in place of x_p.The variable x_p, which was a basic variable in the original canonical form,will no longer be a basic variable in the new canonical form.This new canonical system yields a new basic solution(which may or may not be feasible) similar to that of equation (6). It is to be noted that the values of all the basic variables change, in general, as we go from one basic solution to another, but only one zero variable (which is non-basic in the original canonical form) becomes nonzero (which is basic in the new canonical system), and vice versa. [Boggs, Byrds and Schnabel, 1985]

Motivation of the Topological Simplex fixed point iterative algorithm

Given a system in canonical form corresponding to a basic solution, we have seen how to move a neighbouring basic solution by a pivot operation. Thus as in [Beveridge and Schechter, 1970] one way to find the basic solutions and pick the one that is feasible and corresponds to the optimal value of the objective function. This can be done because the optimal solution, if one exists, always occurs at an extreme point or vertex of the feasible domain. If there are m equality constraints in n variables with $n \geq m$, a basic solution can be obtained by setting any of the $n - m$ variables equal to zero. The number of basic solutions to be inspected is thus equal to the number of ways in which m variables can be selected from a set of

n variables, that is,

$$\binom{n}{m} = \frac{n!}{(n-m)!m!}$$

For example, if $n = 10$and $m = 5$, we have 252 basic solutions, and if $n = 20$ and $m = 10$, we have 184,756 basic solutions. Usually, we do not have to inspect all these basic solutions since many of them will be infeasible. However as in [Bender, 2000], for large values of n and m, this is still a very large number to inspect one by one. Hence what we really need is a computational scheme that examines a sequence of basic feasible solutions, each of which corresponds to a lower value of f until a minimum is reached. The simplex method of Dantzig is a powerful scheme for obtaining a basic feasible solution; if the solution is not optimal, the method provides for finding a neighbouring basic feasible solution that has a lower or equal value of f. The process is repeated until, in a finite number of steps, an optimum is found.

The first step involved in the simplex method as in [Gass, 1990] is to construct an auxiliary problem by introducing certain variables known as artificial variables into the standard form of the linear programming problem. The primary aim of adding the artificial variables is to bring the resulting auxiliary problem into a canonical form from which the basic feasible solution can be obtained immediately. Starting from the canonical form, the optimal solution of the original linear programming problem is to sought in two phases. The first phase is intended to find a basic feasible solution to the original linear programming problem. It consists of a sequence of PIVOT operations that produces a succession of different canonical forms from which the optimal solution of the auxiliary problem can be found. This also enables us to find a basic feasible solution, if one exists, of the original linear programming problem. The second phase as in [Nocedal and Wright, 1999] is intended to find the optimal solution of the original linear programming problem; it consists of a second sequence of pivot operations that enables us to move from one basic feasible solution to the next of the original linear programming problem. In this process, the optimal solution of the problem, if one exists, will be identified. The sequence of different canonical forms that is necessary in both the phases of the simplex method is generated according to the simplex algorithm described in the next section. That is, [Schittkawski, 1987] the simplex algorithm forms the main subroutine of the simplex method.

Simplex Algorithm

The starting point of the simplex algorithm is always a set of equations, which includes the objective function along with the equality constraints of the problem in canonical form. Thus as in [Schrijter, 1990] the objective of the simplex algorithm is to find the vector $X \geq 0$ that minimizes the function $f(X)$and satisfies the equation:

$$\begin{aligned}
&1x_1 + 0x_2 + \ldots + 0x_m + a''_{1.m+1}x_{m+1} + \ldots + a''_{1n}x_n = b''_1 \\
&0x_1 + 1x_2 + \ldots + 0x_m + a''_{2.m+1}x_{m+1} + \ldots + a''_{2n}x_n = b''_2 \\
&\vdots \\
&0x_1 + 0x_2 + \ldots + 1x_m + a''_{m.m+1}x_{m+1} + \ldots + a''_{mn}x_n = b''_m \\
&0x_1 + 0x_2 + \ldots + 0x_m - f \\
&+c''_{m+1}x_{m+1} + \ldots + c''_{mn}x_n = -f''_0
\end{aligned} \tag{7}$$

Where a''_{ij}, c''_j, b''_i and f''_0 are constants. Notice that $(-f)$ is treated as a basic variable in the canonical form of equation (7). The basic solution which can readily be deduced from equation (7) is

$$\begin{aligned}
x_1 &= b''_i, \ i = 1, 2, \ldots, m \\
f &= f^n_0 \\
x_i &= 0, \ i = m+1, m+2, \ldots, n
\end{aligned} \tag{8}$$

If the basic solution is also feasible, the values of $x_i, \ i = 1, 2, \ldots, n$, are non-negative and hence

$$b''_i \geq 0, \ i = 1, 2, \ldots, m \tag{9}$$

In phase I of the simplex method as in [Vanderbei, 1999], the basic solution corresponding to the canonical form obtained after the introduction of the artificial variables will be feasible for the auxiliary problem. As stated earlier, phase II of these simplex method starts with a basic feasible solution of the original linear programming problem. Hence the initial canonical form at the start of the simplex algorithm will always be a basic feasible solution.

We know that as in [Wesley, 1983] the optimal solution of linear programming problem lies at one of the basic feasible solutions. Since the simplex algorithm is intended to move from one basic feasible solution to the other through pivotal operations, before moving to the next basic feasible solution is not the optimal solution. By merely glancing at the numbers.

$$c''_j, \ j = 1, 2, \ldots, n \tag{10}$$

We can tell whether or not the present basic feasible solution is optimal. Theorem 1 provides a means of identifying the optimal point.

Identifying an Optimal Point

Theorem 1 [Charbonnier, Banc-Feraud, Arthur and Bahaud, 1994]: A basic feasible solution is an optimal solution with a minimum objective function $f_0^{''}$ if all the cost coefficients $c_j^{''},\ j = m+1, m+2\ldots, n$ in equation (7) are nonnegative.

Proof: from the last row of Eqs. (1.8), we can write that

$$f_0^{''} + \sum_{i=m+1}^{n} c_i^{''} x_i = f \tag{11}$$

Since the variables$x_{m+1}, x_{m+2}, \ldots, x_n$ are presently zero and are constrained to be nonnegative, the only way one of any of them can change is to become positive. But if $c_i^{''} > 0$ for $i = m+1, m+2, \ldots, n$, then increasing any x_i cannot decrease the value of the objective function f. Since no change in the nonbasic variables can cause f to decrease, the present solution must be optimal with the optimal value of $f$equal to $f_0^{''}$.

A glance over $c_i^{''}$ can also tell us if there are multiple optima. Let all $c_i^{''} > 0, i = m+1, m+2, \ldots, k-1, k+1, \ldots, n$, and let $c_k^{''} = 0$ for some nonbasic variablex_k. Then if the constraints allow that variable to be made positive (from its present value of zero), no change in f results, and there are multiple optima. It is possible, however, that the variable may not be allowed by the constraints to become positive; this may occur in the case of degenerate solutions. Thus, as a corollary to the discussion above, we can as in [Aubert and Vese, 1997] state that a basic feasible solution is the unique optimal feasible solution $c_i^{''} > 0$ for all nonbasic variables $x_j,\ j = m+1, m+2, \ldots, n$.If, after testing for optimality, the current basic feasible solution is found to be non-optimal, an improved basic solution is obtained from the present canonical form as follows.

Improving a Non-optimal Basic Feasible Solution

From the last row of equation (11), we can as in [Nikolova and Nar, 2004] write the objective function as

$$f = f_0^{''} + \sum_{i=1}^{m} c_i^{''} x_i + \sum_{j=m+1}^{n} c_j^{''} x_j = f_0^{''} \tag{12}$$

for the solution given by equation (8).
If at least one $c_j^{''}$ is negative, the value of f can be reduced by making the corresponding $x_j > 0$. In other words, the non-basic variable x_j, for which the cost coefficient $c_j^{''}$ is negative,is to made a basic variable in order to reduce the value of the objective function. At the same time, due to the pivotal operation, one of the current basic variables will become non-basic and hence the values of the new basic variables are to be adjusted in order to bring the value of f less than $f_0^{'}$. If there are more than one $c_j^{''} < 0$,the index $s$of the nonbasic variable $x_s$which is to made basic is chosen such that

$$c_s^{''} = \text{minimum } \; c_j^{''} < 0 \tag{13}$$

The chance of r in the case of a tie, assuming that all $b_i^{''} > 0$,is arbitrary by any $b_i^{''}$ for which $a_i^{''} > 0$ is zero in equation (10), x_s cannot be increased by any amount. Such a solution is called a degenerate solution.
In the case of a non-degenerate basic feasible solution, a new basic feasible solution can as in [Dantzig, 1998] be constructed with a lower value of the objective function as follows. By substituting the value of x_s^* given by equation (13) into equation (11) and (12), we obtain

$$x_s = x_s^*$$

$$x_i = b_i^{''} - a_{is}^{''} x_s^* \geq 0, \; i = 1, 2, \ldots, m \text{ and } i \neq r \tag{14}$$

$x_r = 0$
$x_j = 0, \; j = m+1, m+2, \ldots, n$ and $j \neq s$

$$f = f_0^{''} + c_s^{''} x_s^* \leq f_0^{''} \tag{15}$$

which can readily be seen to be feasible solution different from the previous one. Since $a_{rs}^{''} > 0$ in equation (12), a single pivot operation on the element $a_{rs}^{''}$ in the system of equation (15) will lead to a new canonical form from which the basic feasible solution of equation (14) can easily be deduced. Also, equation (14) shows that this basic feasible solution corresponds to a lower objective function value compared to that of equation (15).This basic feasible solution [Hofmann, 2011],[Ladan, 2002] can again be tested for optimality by seeing whether all $c_i^{''} > 0$ in the new canonical form. If the solution is not optimal, the entire procedure of moving to another basic feasible solution from the present one has to be repeated. In the simplex algorithm [Mikairu and Eteghe, 2012],[Ogbedi, 2006], this procedure is repeated in an iterative manner until the algorithm finds either (a) a class of feasible solutions for which $f \to -\infty$ or (b) an optimal basic feasible solutions with all $c_i^{''} \geq 0, i = 1, 2, \ldots, n$. Since there are only a finite number of ways to choose a set of m basic variables out of n variables, the iterative process of the simplex algorithm will terminate in a finite number of cycles.

2. MAIN RESULTS ON TOPOLOGICAL ANALYSIS OF THE ASSOCIATED PICARD'S FIXED POINT ITERATIVE ALGORITHM

Review of the Picards iterative method. The Picards Lindeloff theorem which embodies the Picards iterative method indicates that if we consider the initial value problem

$$x'(t) = f(t, x(t)), x(t_0) = x_0$$

Then for the ordinary differential equation on $[t_0 - c, t_0 + c]$, the Picards iterative method of solution becomes $x_{n+1} = x_0 + \int_0^t f(t, x(t))\, dt$; where t is the variable of integration. Hence we are looking for a curve which satisfies the differential equation and passes through (t_0, p) with $p = p_0$ and at the end points, $t = t_0 \pm c, x'(t)$ and one sided derivative.

Also, the Picards formula for solving a linear system of equations, $Ax + b$ which is diagonal dominant is given by

$$x_{n+1} = f(x_n), x(0) = x_0$$

where $f(x_n) = A_{ij}x_i + b_j, 1 \leq i \leq n; 1 \leq j \leq n$. The Picards iterative method has always been in use over ages primarily for solving initial value problems and system of equations and by this research has been found equivalent to the traditional simplex method of optimization Having discussed enough on the Basic theory of the simplex method, it becomes important to shift into developing more compact theorems that topologically generalizes what we have earlier discussed in the previous section. To achieve progress in this direction of establishing a suitable iterative algorithm for the simplex method of optimization, we assume the following:

a.: That the domain of existence of the simplex method is the metric space (X, ρ)

b.: That the solution of that method converges in the metric space

c.: That the simplex method an initial value problem (16) in the complete metric spaces is complete

d.: That the simplex iterative method is exactly a reformulated Picard's iterative method for systems of linear equations the above facts make the existence of the following theorem

Theorem 2.1. Let the topological domain of existence of the simplex method be the complete metric space (X, ρ) for a diagonal dominant system of equations. Then for a strongly contractive method of solution, the simplex initial value problem becomes

$$\begin{aligned} &\text{minimize or maximize} \\ &\qquad Z = \sum_{j=1}^{n} c_j x_j \\ &\text{subject to} \\ &\qquad \sum_{j=1}^{n} \bar{P}_j x_j = \bar{b} x_j \geq 0 \;\; j = 1, 2, \ldots, n \end{aligned} \tag{16}$$

for any given basic vector $\bar{X}_j$ so that its corresponding basic $\bar{B}$ and objective vector $\bar{C}_j$ with the simplex iterative method becomes

$$\begin{aligned} z + \sum_{j=1}^{n} (z_j - c_j)\, x_j &= \bar{C}_B B^{-1} b \\ (\bar{x}) + \sum_{j=1}^{n} \left(\bar{B}^{-1} \bar{p}_j\right) x &= \left(\bar{B}^{-1} \bar{b}\right) \end{aligned} \tag{17}$$

where

$$z - c_j = \bar{C}_B \bar{B}^{-1} \bar{p}_j - z_j$$

$(\bar{V}_j)$ Represent the *ith* element of the vector $\bar{V}$ then (17) converges to a unique solution $x^* \in X$ such that $Tx^* = x^*$ and the sequence $\{x_n\}$ of successive approximations generated by the above iteration method is the simplex algorithm

$$\begin{aligned} &x_{j+1} = Tx_j = z + \sum_{j=1}^{n} (z_j - c_j)\, x_j = \bar{C}_B B^{-1} \\ &\bar{x}_B + \sum_{j=1}^{n} \left(\bar{B}^{-1} \bar{P}_j\right) x_j = \bar{B}^{-1} \bar{b} \\ &z - c_j = \bar{C}_B B^{-1} \bar{P}_j - z_j \end{aligned} \tag{18}$$

represented below.

(1) Choose a basic and non basic partition (B, N) such that $(x_B^0, X_N^0) = (B^{-1}b, 0) \geq 0 \quad k := 0$
(2) $y^k := B^{-T} c_B$
(3) If $\exists j^k \in N : s^k j^k = c_{j^k} - A_{j^k}^T y^k > 0$ then continue: else exit because x^k is an optimal solution

(4) Let $\left[\begin{array}{c} dx_B \\ dx_N \end{array}\right] := \left[\begin{array}{c} -B^{-1}Ne_{j^k} \\ e_{j^k} \end{array}\right]$

(5) If $dx_B \geq 0$, then terminate because (P) is unbounded

(6) Let $a^k := \min\left\{-x_B^k/dx_{B_i} : dx_{B_i} < 0\right\}$ and choose an $i^k e\left\{i : dx_{B_i} < 0, \alpha^k := -x_B^k/dx_B^k\right\}$

(7) $x^{k+1} = x^k + \alpha^k dx$

(8) $B := (B\backslash\{B_{i^k}\})\bigcup\left\{j^k\right\}, N := \left(N\backslash j^k\right)\bigcup B_{i^k}$

(9) $k = k+1$

(10) Go to 2

Proof. If $x^*$is the unique fixed point, $x^* = x_0 = T(x_0)$ by the contraction principle.
But let $x_1 = T(x_0)$, then

$$\begin{aligned} x_2 &= \mathrm{T}(x_1) = \mathrm{T}(\mathrm{T}(x_0)) = \mathrm{T}^2(x_0) \\ x_3 &= \mathrm{T}(x_2) = \mathrm{T}^2(\mathrm{T}(x_2)) = \mathrm{T}^3(x_0) \\ &\vdots \\ x_n &= \mathrm{T}^{\mathrm{n}-1}(\mathrm{T}(x_0) = \mathrm{T}^{\mathrm{n}}(x_0) \end{aligned} \tag{19}$$

Hence, we have constructed a sequence $\{x_n\}_{n=0}$ of linear operators for that linear programming simplex matrix problem defined in the metric space(X, ρ).
We now prove that the above generated sequence is Cauchy. First, we compute $\rho(x_n, x_{n+1}) = \rho(T(x_n, x_{n+1}))$ using (19) $\leq KT(x_{n-2}, x_{n-1})$.
Since T is a contraction

$$\begin{aligned} &= KT(x_{n-2}, x_{n-1}) \text{ since } K \text{ is a contraction} \\ &= K^2T(x_{n-2}, x_{n-1}) \\ &\vdots \\ &= \mathrm{K}^n\mathrm{T}(x_0, x_1) \end{aligned} \tag{20}$$

$$\text{i.e. } KT(x_n, x_{n-1}) \leq K^nT(x_0, x_1) \tag{21}$$

We now show that x_{n} is Cauchy.
Let $m > n$, then

$$\begin{aligned} \rho(x_n, x_m) &\leq \rho(x_n, x_m) + \rho(x_{n-1,m-1}) + \ldots + \rho(x_{n-k-1}, x_{m-k-1}) \\ &\leq K^nT(x_0, x_1)\ (1 + K + K^2 + \ldots + K^{n-m-1} + K^n \end{aligned}$$

Since the series on the right hand side is a geometric progression with common ratio ¡1, its sum to infinity is$\frac{1}{1-k}$. So, we have from above that

$$\rho(x_n, x_m) \le K^n T(x_0, x_1)\left(\frac{1}{1-k}\right) \to 0 \text{ as } n \to \infty \text{ since } k < 1$$

Hence the sequence $\{x_n\}$ is Cauchy in (X, ρ) since X is complete and $\{x_n\}$ converges to point in X. Let

$$x_n \to x^* \text{ as } n \to \infty \tag{22}$$

Since T is a contraction and continuous, it follows from (22) that $(Tx_n) \to T(x^*)$ as $n \to \infty$.
But $T(x_n) = x_{n+1}$ from (21). So

$$x_{n+1} = T(x_n) = T(x^*) \tag{23}$$

But limits are unique in a metric space, so from (22) and (23), we obtain that

$$T(x^*) = x^* \tag{24}$$

Hence T has a unique fixed point in (X, ρ). We shall now prove that this fixed point is unique suppose for the contraction there exists $y^* \in X$ such that

$$y^* = x^* \text{ and } T(y^*) = y^* \tag{25}$$

Then from (23) and (24)

$$\rho(x^*, y^*) = \rho(T(x^*), T(y^*)) \le KT(x^*, y^*)$$

so that

$$(k-1)\ T(x^*, y^*) \ge 0 \text{ and } T(x^*, y^*) = 0,\ T = \rho$$

We can divide by it to get $k-1 \ge 0$ i.e $k \ge 1$ which is contradiction.
Hence $x^* =$ y*and the fixed point is unique.
Therefore

$$\begin{aligned} z + \sum_{j=1}^{n} (z_j - c_j)\, x_j &= \bar{C}_B B^{-1} b \\ \bar{x}_B + \sum_{j=1}^{n} \left(\bar{B}^{-1}\bar{P}_j\right) x_j &= \bar{B}^{-1}\bar{b} \end{aligned} \tag{26}$$

where $Z - C_j = x_j + \bar{C}_B \bar{B}^{-1} \bar{P}_j - Z_j$ and V_j represent the ith element of the vector $\bar{V}$ is by the Banach fixed point method, the simplex iterative formula for the linear programming problem

Minimize or Maximize

$$z = \sum_{j=1}^{n} c_j x_j$$
$$\text{Subject to} \quad \sum_{j=1}^{n} p_j x_j = \bar{b} x_j \geq 0 \;\; j = 1, 2, \ldots n \tag{27}$$

for any given vector $\bar{x}_{\mathrm{j}}$ with corresponding basis $\bar{B}$ and objective vector C_j it is worthy of note that the Banach fixed point method (25) satisfying the condition $K < 1$.

Theorem 2.2: The necessary and sufficient condition for the linear programming problem (26) to have a unique fixed point is that in the matrix of linear Transformation

$$A = \sum_{i=1}^{n} \begin{bmatrix} z - \sum_{j=1}^{n} c_j \\ \sum_{j=1}^{n} P_{j-} \bar{b}_j \geq 0 \end{bmatrix}$$

for any given vector x_{j} with corresponding basis $\bar{B}$ and objective vector C_j, the original matrix A is diagonal dominant and that $A_\infty = max \left\{ |\propto_{ij}|, 1 \leq i, j \leq n \right\} < 1$ in this case, the Banach method called the Picard's method becomes satisfied for use in solving the said problem.

Convergence Analysis

Given the general Linear Programming Problem Minimize or maximize

$$z = \sum_{j=1}^{n} c_j x_j$$

Subject to

$$\sum_{j=1}^{n} P_j x_j = b_j x_j \geq 0, \quad j = 1, 2, 3, \ldots, n$$

and for a given basis vector $\bar{X}_B$ and its corresponding basis $\bar{B}_j$ and objective vector $\bar{C}_B$, the general simplex iteration formula given by

$$z + \sum z_j - c_j x_j = \bar{C}_B \bar{B}^{-1} \bar{b}$$
$$\bar{X}_B + \sum B^{-1} P_j x_j = \left(\bar{B}^{-1} \bar{b} \right)$$

where $Z_j - c_j = \bar{C}_B \bar{B}^{-1} \bar{p}_j - C_j \ (\bar{V}_j)$ represent the jth element of the vector V;
then the Linear Programming problem above is convergent to

$$x_j = \min\left\{ \frac{\bar{B}^{-1}\bar{b}}{\bar{B}^{-1}P_j} B^{-1}P_j > 0 \right\}$$

and the basic variable responsible for minimum ratio leaves the basic solution to become non basic at zero level provided $\left(\bar{B}^{-1}\bar{b}\right) - (\bar{B}^{-1}\bar{p}_j)x_j \geq 0, \ \forall \ j$. This condition became realized when from the $Z-$equation above, an increase in non-basic $\mathrm{x_j}$ in the current zero value resulted in an improvement in the value of the Z relative to the current value $\bar{C}_B\bar{B}\bar{b}$ provided $Z_j - C_j$ is strictly negative in the case of maximization and strictly positive in the case of minimization otherwise, X_j cannot improve the solution and must remain non basic at zero level. This condition in optimization is referred to as the optimality and feasibility condition.

3 APPLICATION OF THE PICARD'S TOPOLOGICAL SIMPLEX METHOD TO THE NIGERIAN AIRWAYS' HIRING AND TRAINING OF FLIGHT ATTENDANTS PROBLEM SOLUTION

The Hiring and Training Problem.

Introduction. Air hostesses or stewardesses are members of a flight crew employed by the airline to ensure the safety and comfort of the passengers aboard a commercial flight.
The primary responsibility of these stewardesses is passenger's safety. However, they are tasked with secondary function of seeing to the care and comfort of passengers. They are often perceived by the flying public as waitresses or servants because only this latter function is normally seen on the outside and not the extremely rare events of in-flight emergency. Outside the exceptional case of in-flight emergency stewardesses usually provide courtesy services for passengers such as preparation of in-flight entertainment systems, sale of duty free and other merchandise and the like.
They are normally trained in hub or headquarters city of the airways over a period of one month on how to carry out services mentioned above. But it is seen that often times, some of these stewardesses may quit the job after training because of one reason or the other which include; being pregnant etc. this affects the airways finance so much because the cost of training becomes a waste since they cannot render the services after training. Hence the purpose of this project work which will be highlighted below.

Problem Statement. The Nigerian airways must decide how many stewardesses to hire and train over the next six months. The requirement expressed as number of stewardess flight hours are 8000 in January, 9000 in February, 8000 in March, 1000 in April, 9000 in May and 12000 in June.
It takes one month of training before a stewardess can be put on a regular flight. So a trainee must be hired at least one month before she is actually needed. Also a trainee requires 100 (hundred) hours of actual in-flight experience during the month of training. Hence for each trainee, 100 less hours are available for flight service by regular stewardess.
Each experienced stewardess can work up to 150 hours in a month and there are 60 regular stewardesses available at the beginning of January. If the maximum time available for an experienced stewardess exceeds a month flying and training requirement, the regular stewardess work fewer than 150 hours, none is laid off. Each month, approximately 10% of the experienced stewardess quit their jobs to get married or for other reasons. An experienced stewardess cost the airline #80,000 a trainee #40,000 a month in salary and other benefits.

Model Formulation. Let $x_i\,(i = 1, 2, 3, 4, 5, 6)$ be the number of trainees at the beginning of each month i.e
x_1 = Number of trainees at the beginning of January
x_2 = Number of trainees at the beginning of February
x_3 = Number of trainees at the beginning of March
x_4 = Number of trainees at the beginning of April
x_5 = Number of trainees at the beginning of May
x_6 = Number of trainees at the beginning of June
In order to make the financial values too large, we divide 80,000 and 40,000 (i.e. the amount paid to a regular stewardess and trainee respectively) by 10^4 giving us 8 and 4.
To form the objective function z, we compute the following; since we started with 60 regular stewardess for the month of January we have $(60 * 8) + 4x_1$. Since we have that at the end of each month, 0.1 of the regular stewardess may quit leaving 0.9 then
For February we have $0.9\,(60 + x_1) * 8 + 4x_2$
March $\to 0.9\,[0.9\,(60 + x_1) + x_2] * 8 + 4x_3$
April $\to 0.9\,\{0.9\,[0.9\,(60 + x_1) + x_2] + x_3\} * 8 + 4x_4$
May $\to 0.9\,\{0.9\,[0.9\,[0.9\,(60 + x_1) + x_2] + x_3] + x_4\} * 8 + 4x_5$
June $\to 0.9\,\{0.9\,[0.9\,[0.9\,[0.9\,(60 + x_1) + x_2] + x_3] + x_4] + x_5\} * 8 + 4x_6$
The above computation can be written in another form as;

$$
\begin{aligned}
&60 * 8\left(0.9^0 + 0.9^1 + 0.9^2 + 0.9^3 + 0.9^4 + 0.9^5\right) + 4x_1 \\
&+8\left(0.9 + 0.9^2 + 0.9^3 + 0.9^4 + 0.9^5\right)x_1 + 4x_2 \\
&+8\left(0.9 + 0.9^2 + 0.9^3 + 0.9^4\right)x_2 + 4x_3 + 8\left(0.9 + 0.9^2 + 0.9^3\right)x_3 + 4x_4
\end{aligned}
$$

$$+8\left(0.9+0.9^2\right)x_4+4x_5+8\left(0.9\right)x_5+4x_6$$

Putting this together, the objective function z becomes

$$Z=2249.0832+33.48472x_1+28.7608x_2+23.512x_3+17.68x_4+11.2x_5+4x_6$$

Computing the Constraints. To do this, we look at the restrictions or conditions under which the stewardesses are subjected to. From the data we have that each experienced stewardess can work up to 150 hours in a month and we have 60 experienced stewardesses available at the beginning of January. And also from the data, we know that a trainee requires 100 hours of actual in-flight experience during the month of training. Finally we have to remember that at the end of each month, 10% of experienced stewardesses quit their job. Then the constraint for each is

For January $\rightarrow (150*60)+100x_1 \geq 8000$

For February $\rightarrow 150*0.9\left(60+x_1\right)+100x_2 \geq 9000$

For March $\rightarrow !50*0.9\left[0.9\left(60+x_1\right)+x_1\right]+100x_3 \geq 8000$

$100x_1 \geq -1000$

$135x_1+100x_2 \geq 900$

$121.5x_1+135x_2+100x_3 \geq 710$

$109.35x_1+121.5x_2+135x_3+100x_4 \geq 3439$

$98.415x_1+109.35x_2+121.5x_3+135x_4+100x_5 \geq 3095.1$

$88.5735x_1+98.415x_2+109.35x_3+121.5x_4+135x_5+100x_6 \geq 6685.59$

For all $x_i \geq 0$ and integers

Therefore, the linear programming problem becomes

Minimize

$$Z=2249.0832+33.48472x_1+28.76608x_2+23.512x_3+17.68x_4+11.2x_5+4x_6$$

Subject to

$100x_1+s_1 \geq -1000$

$135x_1+100x_2+s_2 \geq 900$

$121.5x_1+135x_2+100x_3+s_3 \geq 710$

$109.35x_1+121.5x_2+135x_3+100x_4+s_4 \geq 3439$

$98.415x_1+109.35x_2+121.5x_3+135x_4+100x_5+s_5 \geq 3095.1$

$88.5735x_1+98.415x_2+109.35x_3+121.5x_4+135x_5+100x_6+s_6 \geq 6685.59$

Then at this point we dualize the above L.P. problem to obtain the form below. Hence, we apply the Picard's simplex algorithm to the following tableau using a computer programming package called MATLAB so as to reduce the computational time, error and to enhance fast and accurate results.

Problem.

$$F=2249.0832x_1+28.76080x_2+23.51200x_3+1768000x_4+11.20000x_5+4.00000x_6$$

Subject to

$100x_1 \leq 1000$

$-135x_1-100x_2 \leq -900$

$-121.5x_1 - 135x_2 - 100x_3 \leq -710$
$-109.35x_1 - 121.5x_2 - 135x_3 - 100x_4 \leq -3439$
$-98.415x_1 - 109.35x_2 - 121.5x_3 - 135x_4 - 100x_5 \leq -3095.1$
$-88.5735x_1 - 98.415x_2 - 109.35x_3 - 121.5x_4 - 135x_5 - 100x_6 \leq -6685.59$

Solution.

Optimization terminated successfully
First order optimally measure less than option TolFun
Active inequalities (to within options. TolCon $= 1e^{-0.006}$

Iteration	x_1	x_2	x_3	x_4	x_5	x_6	$f(x)$
0	0	0	0	0	0	0	0
1	0	9	0	24	0	29	3048.2504
2	0	9	0	24	0	29	3048.2504

Table 1

$x_1 = 0$
$x_2 = 9$
$x_3 = 0$
$x_4 = 24$
$x_5 = 0$
$x_6 = 29$
$Z = 3048.2504 \times 10^4$

ANALYSIS OF RESULT AND SENSITIVITY ANALYSIS.

Analysis of result. The result obtained from optimal simplex tableau can be interpreted and presented in tabular form as follows

DECISION VARIABLE	OPTIMAL VALUE	MEANING
x_1	0	Hire no trainee stewardess in the month of January
x_2	9	Hire nine trainee stewardesses in February
x_3	0	Hire no trainee stewardess
x_4	24	Hire 24 in April
x_5	0	Hire none in May
x_6	29	Hire 29 stewardesses in June
Z	30482504	Maximum amount to be spent on hiring and training of stewardesses for the period of 6 months

Table 2

Also in a tabular form, we want to summarize the number of regular and trainee stewardesses we will have in each month according to the result obtained above and then finally compute or show how the total cost will be obtained. But before we continue, let us have the following in mind;

i.: According to the data collected, we have that at the end of each month, approximately 10% of the regular stewardesses quits their job.

ii.: It costs the airways #80,000 and #40,000 in payment and other benefits to maintain a regular and trainee stewardess respectively. With these things in mind, we now present the table below

Months	No of trainees hired	No of regulars at the beginning of the month	No of regulars at the end of the month	10% that left at the end of the month	No of regulars remaining which was carried over to the next month
January	0	60	60	6	54
February	9	54	63	6.3	56.7
March	0	56.7	56.7	5.67	51.03
April	24	51.03	75.03	7.503	67.527
May	0	67.527	67.527	6.7527	60.7743
June	29	60.7743	67.527	6.7527	60.7743
Total	62	350.0313			

Table 3

From the table above, the number of trainee stewardesses that received #40,000 for the period of six months is 62. Thus for six months, the trainees cost the airways $62 \times 40,000 = \#248,000$ in payment and other benefits.

For regular stewardesses we have that for the period of six months. 350,0313 received #80,000. This amounts to #28002504 because $350.0313 \times 80,000 = 28002504$.

Hence the total amount spent on trainees and regular stewardesses for the period is given by #2480000 + #28002504 = #30482504 which corresponds with the value of z in the optimal simplex tableau.

Sensitivity Analysis. Sensitivity analysis investigates the change in the optimum solution resulting from making changes in parameters of the LP model. It tries to find out how sensitive the optimum solution is to small change in a parameter. These changes often come from

i.: Changes in objective function coefficient

ii.: Changes in the right hand side of the constraints

iii.: Changes due to additional constraints or variables to the problem

Suppose from the problem, that the 10% of the experienced stewardesses does not quit their jobs at the end of each month, what will happen to the optimum solution? Will the value of the variables be affected? How many stewardesses should the company hire?

Hence the problem model becomes since no stewardess leaves the company at the end of each month to form the objective function we have that for the month of

January$-60 * 8 + 4x_1$

February-$(60 + x_1) * 8 + 4x_2$

March $-(60 + x_1 + x_2) * 8 + 4x_3$

April $-(60 + x_1 + x_2 + x_3) * 8 + 4x_4$
May $-(60 + x_1 + x_2 + x_3 + x_4) * 8 + 4x_5$
June $-(60 + x_1 + x_2 + x_3 + x_4 + x_5) * 8 + 4x_6$
putting this together we have
$480 + 4x_1$
$480 + 8x_1 + 4x_2$
$480 + 8x_1 + 8x_2 + 4x_3$
$480 + 8x_1 + 8x_2 + 8x_3 + 4x_4$
$480 + 8x_1 + 8x_2 + 8x_3 + 8x_4 + 4x_5$
$480 + 8x_1 + 8x_2 + 8x_3 + 8x_4 + 8x_5 + 4x_6$
$Z = 2880 + 44x_1 + 36x_2 + 28x_3 + 20x_4 + 12x_5 + 4x_6$
For constraints we have
January $-150 * 60 + 100x_1 \geq 8000$
February $-150 * (60 + x_1) + 100x_2 \geq 9000$
March $-150 * (60 + x_1 + x_2) + 100x_3 \geq 8000$
April $-150 * (60 + x_1 + x_2 + x_3) + 100x_4 \geq 10,000$
May $-150 * (60 + x_1 + x_2 + x_3 + x_4) + 100x_5 \geq 9,000$
June $-150 * (60 + x_1 + x_2 + x_3 + x_4 + x_5) + 100x_6 \geq 12,000$
Putting together the constraints becomes
$100x_1 \geq -1000$
$150x_1 + 100x_2 \geq 0$
$150x_1 + 150x_2 + 100x_3 \geq -1000$
$150x_1 + 150x_2 + 150x_3 + 100x_4 \geq 1000$
$150x_1 + 150x_2 + 150x_3 + 150x_4 + 100x_5 \geq 0$
$150x_1 + 150x_2 + 150x_3 + 150x_4 + 150x_5 + 100x_6 \geq 3000$
After dualising by multiplying with -1, the problem becomes
Maximize

$$Z = 2880 + 44x_1 + 36x_2 + 28x_3 + 20x_4 + 12x_5 + 4x_6$$

Subject to
$-100x_1 + S_1 \leq -1000$
$-150x_1 - 100x_2 + S_2 \leq 0$
-$150x_1 - 150x_2 - 100x_3 + S_3 \leq -1000$
$-150x_1 - 150x_2 - 150x_3 - 100x_4 + S_4 \leq 1000$
$-150x_1 - 150x_2 - 150x_3 - 150x_4 - 100x_5 + S_5 \leq 0$
$-150x_1 - 150x_2 - 150x_3 - 150x_4 - 150x_5 - 100x_6 + S_6 \leq 3000$

Solution. Using the simplex algorithm presented earlier in this work, the following iterations was done and the last table gives the optimal and feasible solution for the sensitivity analysis problem.

	x_1	x_2	x_3	x_4	x_5	x_6	S_1	S_2	S_3	S_4	S_5	S_6	**Solution**
Z	-44	-36	-28	-20	-12	-4	0	0	0	0	0	0	**288010**
S_1	100	0	0	0	0	0	1	0	0	0	0	0	**1000**
S_2	-150	100	0	0	0	0	0	1	0	0	0	0	**0**
S_3	-150	-150	-100	0	0	0	0	0	1	0	0	0	**1000**
S_4	-150	-150	-150	-100	0	0	0	0	0	1	0	0	**-1000**
S_5	-150	-150	-150	-150	-100	0	0	0	0	0	1	0	**0**
S_6	-150	-150	-150	-150	-150	-100	0	0	0	0	0	1	**-3000**

Table 4

	x_1	x_2	x_3	x_4	x_5	x_6	S_1	S_2	S_3	S_4	S_5	S_6	**Solution**
Z	0	-6.52	-28	-20	-12	-4	0	0	0	0	0	0	**288010**
S_1	0	67	0	0	0	0	1	0	0	0	0	0	**1000**
x_1	1	0.67	0	0	0	0	0	1	0	0	0	0	**0**
S_3	0	-49.5	-100	0	0	0	0	0	1	0	0	0	**1000**
S_4	0	-49.5	-150	-100	0	0	0	0	0	1	0	0	**-1000**
S_5	0	-49.5	-150	-150	-100	0	0	0	0	0	1	0	**0**
S_6	0	-49.5	-150	-150	-150	-100	0	0	0	0	0	1	**-3000**

Table 5

	x_1	x_2	x_3	x_4	x_5	x_6	S_1	S_2	S_3	S_4	S_5	S_6	**Solution**
Z	0	2.72	0	8	6.76	-4	0	0	0	0	0	0	**288010**
S_1	0	67	0	0	0	0	1	0	0	0	0	0	**1000**
x_1	1	0.67	0	0	0	0	0	1	0	0	0	0	**0**
S_3	0	-16.5	0	100	67	0	0	0	1	0	0	0	**1000**
S_4	0	0	0	50	100.5	0	0	0	0	1	0	0	**-1000**
x_3	0	0.33	1	1	0.67	0	0	0	0	0	1	0	**0**
S_6	0	0	0	0	-49.5	-100	0	0	0	0	0	1	**-3000**

Table 6

	x_1	x_2	x_3	x_4	x_5	x_6	S_1	S_2	S_3	S_4	S_5	S_6	**Solution**
Z	0	2.72	0	8	8.76	-4	0	0	0	0	0	0	**3000**
S_1	0	67	0	0	0	0	1	0	0	0	0	0	**1000**
x_1	1	0.67	0	0	0	0	0	1	0	0	0	0	**0**
S_3	0	-16.5	0	100	67	0	0	0	1	0	0	0	**1000**
S_4	0	0	0	50	100.5	0	0	0	0	1	0	0	**-1000**
x_3	0	0.33	1	1	0.67	0	0	0	0	0	1	0	**0**
x_6	0	0	0	0	0.50	1	0	0	0	0	0	1	**30**

Table 7

4 CONCLUSION AND SUGGESTION

The topological conclusion of this research shows that the traditional simplex method in optimization convincingly is the Picard's fixed point iterative method. Secondly, from the solution result of the sensitivity analysis problem as seen in the computation, we can deduce that only 30 trainee stewardesses will be hired within the period in question and that will be in the month of June. And from our data, it costs the airline #40,000 to maintain one trainee, so that the total amount spent on maintaining the 30 trainees is 30*#40,000 = #1,200,000.
But #30,000,000 - #1,200,000 = #28,800,000
This means that the difference (i.e. #28,800,000) will then be the total amount spent on maintaining the regular stewardesses which is greater than that spent for the trainees.

Also from the analysis of the result obtained earlier from our main problem, we can see that the amount spent on maintaining the regular stewardesses is far more than that spent on trainees.
Since from the analysis of our result we have seen that it will cost the airline more to maintain regular stewardesses than trainees, I suggest that they depends on hiring more trainees than regulars in other to minimize cost.
Finally, I suggest that hiring of trainee stewardesses should be done at 2 months interval as seen from the solution result of our main problem.

References

[1] Siddiqi A.H. (2004) "Applied Functional Analysis (Numerical Methods, Wavelet Methods and Image Processing)"; Marcel Dekker Inc. New York, Basel USA

[2] Bender, E.A. (2000)"An introduction to Mathematical Modelling"; John Wiley, New York

[3] Beveridge, G.S.G. and R. Schechter, (1970) "Optimization theory and Practice"; McGraw-Hill, New York

[4] Boggs P.T.R, H Byrds and R.B. Schnabel, (1985)"Numerical Optimization"; SIAM Piladelphia

[5] Brent, R.P, (1973)"Algorithms for Mi nimization without Derivatives"; Prentice-Hall, Eaglewood Cliffs, New York

[6] Dantzig, G.B, (1998) "Linear Programming and Extensions"; Princeton NJ

[7] Fourier, R. (1997) "Linear Programming"; ORMS Taday 24: 54-67

[8] Fourier, R. (1999) "Software for Optimization"; ORMS Today 26: 40-44

[9] Hofmann K., (2011) "Ethopian Increases Star Alliance pressure in Africa" Airtransport world, 14 December, 2011

[10] Gass, S.I (1990) "An Illustrated Guide to Linear Programming" Dover Mineola NY

[11] Taha H.A. (2005) "Operation Research, an Introduction" 7th ed.

[12] Jeter, M.W. (1986)"Mathematical Programming"; Mercel Dekker New York

[13] Ladan, Suleiman (2002) "An analysis of air transportation in Nigeria", Journal of Research in National Development

[14] Lueaberger, D.G. (1984) "Linear and Nonlinear Programming"; 2nd Ed. Addison-Wisley, Menlo Park, CA

[15] Martin R.K, (1999)"Large Scale Linear and Integer Optimization"; Kluwer Academic Publishers, Norwell MA

[16] Mikairu, L and Eteghe, D. (2012)"Why Nigerian Airlines do not survive" Vanguard Newspaper, December 24, 2012. http/www.vanguardngr.com/2012/why Nigerian Airline do not survive. Accessed 20/2/14

[17] Murray, K.G. (1983)"Linear Programming"; Wiley New York

[18] Nash, S.G and A Sofar (1996) "Linear and Nonlinear Programming"; McGraw-Hill, New York

[19] Nocedal, J. and Wright, S.J (1999) "Numerical Optimization"; Springer series in Operations Research New York

[20] Ogbeidi, M.M. (2006) The aviation industry in Nigeria. A historical overview.Lagos historical review, Vol.6 pp133-147

[21] Schittkawski, K. (1987) "Computational Mathematical Programming"; Springer-Verlag, Berlin

[22] Schrijver, A. (1990) "Theory of Linear and Integer Programming"; Wiley, New York

[23] Vanderbei, R.J, (1999) "Linear Programming Foundations and Economics"; Kluwer Academic Publishers Norwell MA

[24] Wesley, L.A (1983) "Integer Programming"; McGraw-Hill New York

[25] Charbonnier, P., Blanc-Féraud, Aubert, G., and Barlaud, M., (1994) "Two deterministic half quadratic regularization algorithm for computed imaging" in proceedings of IEEE ICIP, vol.2 pp 162-178

[26] Aubert, G., and Vese, L., (1997) "A varitional method in image recorvery" *SIAM Journal on Numerical Analysis*, vol.34 no.5 pp 1948-1979

[27] Nikolova, M and Ng, M., (2004) "Analysis of half quadratic criteria and em-algorithm", IEEE Signal Processing Letters, vol II, no.9, pp709-712

[28] Wikipedia (2014) Air Nigeria. http:/enwikipedia.org/wiki/AirNigeria.Accessed 5/2/2014

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