Journal of the Nigerian Mathematical Society
Vol. 38, Issue 3, pp. 515-532, 2019
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Special Issue: The International Conference on Non-linear Analysis (organized to mark the 70th birthday anniversary of Prof. Charles Chidume)

APPROXIMATING SOLUTION OF GENERALIZED MIXED EQUILIBRIUM PROBLEM AND FIXED POINT OF MULTI VALUED STRICTLY PSEUDOCONTRACTIVE MAPPINGS

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ABSTRACT. A new iterative algorithm is constructed using the shrinking projection method, the sequence of the algorithm is used to approximate a common element of the set of solution of a finite family of generalized mixed equilibrium problem and the set of common fixed points of a finite family of multi-valued strictly pseudo-contractive mappings in real Hilbert space. The result obtained is a significant improvement on many recent important results.

Keywords and phrases: generalized mixed equilibrium problem; muti-valued mappings; pseudocontractive mapping; resolvent operator;
2010 Mathematical Subject Classification: 47H09; 47H10; 49J20; 49J40

1. INTRODUCTION

Let (X, d) be a metric space and $CB(X)$ be the family of all closed and bounded subsets of X. The Hausdorff metric induced by the metric d, is defined for all $A, B \in CB(X)$ by:

$$\mathcal{H}(A,B) = \max\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(b,A)\}, \tag{1}$$

where $d(a,B) := \inf_{b\in B} d(a,b)$.

Let C be a nonempty, closed and convex subset of a real Hilbert space H and $T : C \to 2^C$ be a multi-valued mapping. A point $x \in C$ is called a fixed point of T if $x \in Tx$. If $Tx = \{x\}$, then x is called a strict fixed point of T. We denote the set of fixed points

Received by the editors November 07, 2018; Revised December 01, 2019 ; Accepted: December 10, 2019

www.nigerianmathematicalsociety.org; Journal available online at https://ojs.ictp.it/jnms/

of T by $F(T)$.
The mapping T is said to be *L-Lipschitzian* if there exist $L > 0$ such that

$$\mathcal{H}(Tx, Ty) \leq L||x - y||,\ x, y \in C. \tag{2}$$

In (2), if $L \in (0, 1)$, then T is called contraction while T is called nonexapansive if $L = 1$.
T is said to be

(i) *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\mathcal{H}(Tx, Ty) \leq ||x - y||,\ \forall x \in C, y \in F(T),$$

(ii) *k-strictly pseudocontractive* in the sense of [9], if there exists $k \in [0, 1)$ such that $\forall x, y \in C$

$$\mathcal{H}^2(Tx, Ty) \leq ||x - y||^2 + k||x - u - (y - v)||^2,\ \forall u \in Tx,\ v \in Ty.$$

Remark: It is obvious that every multivalued nonexpansive mapping is multivalued k-strictly pseudocontractive mapping.
The metric projection P_C is a map defined on H onto C which assigns to each $x \in H$, the unique point in C, denoted by $P_C x$ such that

$$||x - P_C x|| = \inf\{||x - y|| :\ y \in C\}.$$

It is well known that $P_C x$ is characterized by the inequality $\langle x - P_C x, z - P_C x\rangle \leq 0,\ \forall z \in C$ and P_C is a firmly nonexpansive mapping, i.e., for any $x, y \in H$

$$||P_C x - P_C y||^2 \leq \langle P_C x - P_C y, x - y\rangle.$$

Approximation of fixed points for both single valued and multivalued mappings have been widely studied by numerous researchers using different iterative algorithms. One of such algorithms is the classical Mann iterative algorithm (see Mann [21]):

$$x_{n+1} := (1 - \alpha_n)x_n + \alpha_n T x_n,\ n \geq 1;$$

where T is nonexpansive. The Mann-scheme is known to yield weak convergence, even in Hilbert spaces for nonexpansive mappings. To overcome this challenge, Nakajo and Takahashi [25] modified the Mann's iteration process and introduced the so-called CQ method . Using their algorithm, they obtained strong convergence result for fixed point of nonexpansive mappings in Hilbert spaces. In 2008, Takahashi *et al* [29], introduced and studied yet another modification, "the shrinking projection method" of the Mann's iteration process and obtained strong convergence result for approximating common fixed point of family of nonexpansive mappings in Hilbert spaces. Precisely, they proved the following result: For $x_0 \in H,\ C_1 = C$ and $x_1 = P_{C_1}x_0$, define $\{x_n\}$ iteratively by

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T_n x_n, \\ C_{n+1} = \{z \in C_n : ||y_n - z|| \leq ||x_n - z||\}, \\ x_{n+1} = P_{C_{n+1}}x_0,\ n \geq 1, \end{cases} \tag{3}$$

where P_{C_n} is the metric projection on H onto C_n and $\{T_n\}$ is a family of nonexpansive mappings. Furthermore, they proved that the sequence $\{x_n\}$ converges strongly to $z = P_{F(T)}x_0$, where $F(T) = \cap_{n=1}^{\infty} F(T_n)$. Since then, the shrinking projection method has been widely studied by numerous researchers (see for example [1, 14, 13, 5, 18, 19] and the references therein).
For several years, the study of fixed point theory for multi-valued nonlinear mappings has continued to attract the interest of many l researchers (see, for example, Brouwer [3], Sastry and Babu [28], Chidume and Ezeora[10], Chang [7], Nadler [24], Isiogugu and Osilike [11], Hussain Khan [16], [15, 12] and the references therein). Interest in such studies stems, perhaps, mainly from the usefulness of such fixed point theory in real-world applications, such as in Game Theory and Market Economy and in other areas of mathematics, such as in Non-Smooth Differential Equations. Game theory is perhaps the most successful area of application of fixed point theory for multi-valued mappings.

For a nonempty, closed and convex subset C of a real Hilbert space H. Let $B : C \to H$ be a nonlinear mapping, $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a function and $F : C \times C \to \mathbb{R}$ be a bifunction, where $\mathbb{R}$ is the set of real numbers. The *generalized mixed equilibrium problem* is defined as follows:

$$\begin{aligned} \text{find } x \in C : F(x,y) + \varphi(y) - \varphi(x) \\ +\langle Bx, y - x\rangle \geq 0\ \forall\ y \in C. \end{aligned} \tag{4}$$

The set of solutions of (4) is denoted by $GMEP(F, \varphi, B)$.
If $B = 0$, problem (4) reduces to the following *mixed equilibrium* problem:

$$\text{find } x \in C : F(x,y) + \varphi(y) - \varphi(x) \geq 0\ \forall\ y \in C. \tag{5}$$

The set of solutions of (5) is denoted by $MEP(F, \varphi)$.
If $\varphi = 0$, problem (4) becomes the following *generalized equilibrium* problem:

$$\text{find } x \in C : F(x,y) + \langle Bx, y - x\rangle \geq 0\ \forall\ y \in C. \tag{6}$$

The set of solutions of (6) is denoted by $GEP(F, B)$.
If $\varphi = 0$ and $B = 0$, problem (4) becomes the following *equilibrium* problem:

$$\text{find } x \in C : F(x, y) \geq 0 \ \forall \ y \in C. \tag{7}$$

The set of solutions of (7) is denoted by $EP(F)$.
If $F(x, y) = 0$ for all $x, y \in C$, problem (4) becomes the following *generalized variational inequality* problem:

$$\text{find } x \in C : \varphi(y) - \varphi(x) + \langle Bx, y - x \rangle \geq 0 \ \forall \ y \in C. \tag{8}$$

If $\varphi = 0$ and $F(x, y) = 0$ for all $x, y \in C$, problem (4) becomes the following *variational inequality* problem:

$$\text{find } x \in C : \langle Bx, y - x \rangle \geq 0 \ \forall \ y \in C. \tag{9}$$

If $B = 0$ and $F(x, y) = 0$ for all $x, y \in C$, problem (4) becomes the following *convex minimization* problem:

$$\text{find } x \in C : \varphi(y) \geq \varphi(x) \ \forall \ y \in C. \tag{10}$$

These problems are of importance in the study of problems arising from economics, finance, network, optimization, image reconstruction, operation research, among others. Equilibrium Problem was first introduced by Blum and Oettli [2]. Since its introduction in the literature, it has been studied extensively by many authors under diffrent settings, (see for example [6, 17, 27, 23, 20]).

Recently, Bunyawat and Suantai [4] introdueced a new hybrid method for approximating a common solution of MEP and fixed points of a countable family of multivalued nonexpansive mappings in Hilbert spaces. They proved the following theorem.
Theorem:(Bunyawat and Suantai) Let H be a real Hilbert space and C be a nonempty, closed and convex subset of H. Let $F : C \times C \to \mathbb{R}$ be a bifunction which satisfies $(A1) - (A4)$ and $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be proper lower semi-continuous and convex function such that $C \cap \text{dom}\varphi \neq \emptyset$. Let $T_i : C \to P(C)$ be multivalued nonexpansive mappings for all $i \in \mathbb{N}$ with $\Omega := \cap_{i=1}^{\infty} F(T_i) \cap MEP(F, \varphi) \neq \emptyset$ such that all P_{T_i} are nonexpansive. Assume that either (B1) or (B2) holds and $\{\alpha_{n,i}\} \subset [0, 1)$ satisfies the condition $\liminf_{n\to\infty} \alpha_{n,i}\alpha_{n,0} > 0$ for all $i \in \mathbb{N}$. Define the sequence $\{x_n\}$ as follows: $x_1 \in C = C_1$,

$$\begin{cases} F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0,\ y \in C, \\ y_n = \alpha_{n,0}u_n + \sum_{i=1}^{n} \alpha_{n,i}x_{n,i}, \\ C_{n+1} = \{z \in C_n : ||y_n - z|| \leq ||x_n - z||\}, \\ x_{n+1} = P_{C_{n+1}}x_0,\ n \geq 0, \end{cases} \tag{11}$$

where $r_n \in (0, \infty)$ with $\liminf_{n\to\infty} r_n > 0$, $\{\alpha_{n,i}\} \in [0, 1)$ such that $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ and $x_{n,i} \in P_{T_i}u_n$ for $i \in \mathbb{N}$. Then the sequence $\{x_n\}$ converges strongly to $P_\Omega x_0$.

Chidume and Ezeora [10] introduced a new Krasnoselskii-type algorithm and proved strong convergence theorems for the sequence of the algorithm for approximating a common fixed point of a finite family of multivalued $k-$strictly-pseudocontractive mappings in a real Hilbert space. To achieve their result, they imposed some compactness-type condition on some of the mappings.

We remark that, in order to prove their results, Sastry and Babu [28], Chidume and Ezeora [10] imposed some compactness condition either on the domain of the mapping, or on the mapping itself. This restriction, no doubt is a very strong limitation to their results from the point of view of application. Furthermore, we observe that the result of Hussain Khan [16], though fascinating, is also very difficult to use in any possible application due to the condition on the operator P_T.

In this paper, motivated by the results of Bunyawat and Suantai [4], Sastry and Babu [28], Hussain Chan [16] and Chidume and Ezeora [10], we study the following Generalized Mixed Equilibrium Problem and Fixed Point Problem.

Let H be a real Hilbert space and C be a nonempty, closed and convex subset of H. Let $T_i : C \to CB(C)$ be finite family of multivalued k_i-strictly pseudo-contractive mappings, $k_i \in (0, 1)$, $i = 1, 2, \ldots, m$. For $j = 1, 2, \ldots, N$, let $F_j : C \times C \to \mathbb{R}$ be nonlinear bifunctions, let $f_j : C \to C$ be nonlinear mappings, and $\varphi_j : C \to \mathbb{R} \cup \{+\infty\}$ be proper lower semi-continuous and convex functions such that $C \cap \text{dom}\varphi_j \neq \emptyset$: Find $x \in \cap_{i=1}^{m} F(T_i)$ such that

$$F_j(x, y) + \langle f_j x, y - x\rangle + \varphi_j(y) - \varphi_j(x) \geq 0$$
$$\forall y \in C, j = 1, 2, \ldots, N. \tag{12}$$

We denote the solution set of problem (12) by Γ defined by

$$\Gamma := (\cap_{i=1}^{m} F(T_i)) \cap (\cap_{j=1}^{N} GMEP(F_j, f_j, \varphi_j)).$$

Observe that if $j = 1$, $f_j \equiv 0$ and T_i is reduced to a multivalued nonexpansive mapping (see Remark above), then problem (12) reduces to the problem studied by Bunyawat and Suantai [4]. Hence, our result extends and improves the result in [4] and host of other important results in literature. Furthermore, the result obtained is devoid of the above named restrictions.

2. PRELIMINARY

In this section, we state some known and useful results which will be needed in the proof of our main result.
Lemma 1: Let H be a real Hilbert space, then

$$2\langle x, y\rangle = ||x||^2+||y||^2-||x-y||^2 = ||x+y||^2-||x||^2-||y||^2,\ \forall\, x, y \in H.$$

Lemma 2:[25]. Let H be a real Hilbert space, then $\forall\, x, y \in H$ and $\alpha \in (0, 1)$, we have

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2.$$

Lemma 3:[4]. Let C be a nonempty closed and convex subset of a real Hilbert space H. Given $x, y, z \in H$ and also given $a \in \mathbb{R}$, the set

$$\{v \in C : ||y-v||^2 \leq ||x-v||^2 + \langle z, v\rangle + a\}$$

is convex and closed.
For solving the equilibrium problem, we assume that the bifunction $F : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$,
(A2) F is monotone, that is $F(x, y)+F(y, x) \leq 0$ for all $x, y \in C$,
(A3) for each $x, y, z \in C$, $\limsup_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x, y)$,
(A4) $F(x, .)$ is convex and lower semicontinuous for each $x \in C$.

Lemma 4:[26]. Let C be a nonempty closed and convex subset of a real Hilbert spaces H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions (A1)-(A4), $f : C \to C$ be a continuous and monotone mapping, and $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex function such that $C \cap \mathrm{dom}\varphi \neq \emptyset$. For $r > 0$ and $x \in C$, we have the following:

(B1) there exist $z \in C$ such that $\forall y \in C$,

$$F(z, y) + \langle fz, y-z\rangle + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \quad (13)$$

(B2) if we define a resolvent mapping $T_r : H \to C$ by

$$T_r(x) = \{z \in C : F(z,y) + \langle fz, y-z\rangle + \varphi(y) - \varphi(z) \\ + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \forall y \in C\} \tag{14}$$

for all $x \in H$. Then the following conclusions hold:

(i) for each $x \in H$, $T_r(x) \neq \emptyset$,
(ii) T_r is single-valued,
(iii) T_r is firmly nonexpansive,
(iv) $F(T_r) = GMEP(F, f, \varphi)$,
(v) $GMEP(F, f, \varphi)$ is closed and convex.

Lemma 5:[25].

Let C be a nonempty closed and convex subset of a real Hilbert space H and be the metric projection from H onto C. Then the following inequity holds:

$$||y - P_C x||^2 + ||x - P_C x||^2 \leq ||x-y||^2 \ \forall x \in H, y \in C. \tag{15}$$

Lemma 6:[9]. Let E be a reflexive real Banach space and let $A, B \in CB(E)$, assume that B is weakly closed. Then for every $a \in A$, there exists $b \in B$ such that

$$||a-b|| \leq \mathcal{H}(A, B). \tag{16}$$

Lemma 7:[10]. Let H be a real Hilbert space and $\{x_i, i = 1, 2, \dots, m\} \subset H$. For $\alpha_i \in (0,1), i = 1, 2, \dots m$, such that $\sum_{i=1}^{m} \alpha_i = 1$, the following identity holds:

$$||\sum_{i=1}^{m} \alpha_i x_i||^2 = \sum_{i=1}^{m} \alpha_i ||x_i||^2 - \sum_{i,j=1, i\neq j} \alpha_i \alpha_j ||x_i - x_j||^2.$$

3. MAIN RESULTS

In this section, we the main results of our work. Let C be a nonempty closed and convex subset of a real Hilbert space H. In this section, we denote by $CB(C)$, the family of nonempty closed, convex and bounded subsets of C.

Theorem 1: Let H be a real Hilbert space and C be a nonempty, closed and convex subset of H. Let $T_i : C \to CB(C)$ be finite family of multi-valued k_i-strictly pseudo-contractive mappings, $k_i \in [0,1)$, $i = 1, 2, \dots, m$. Assume that for $x^* \in \cap_{i=1}^{m} F(T_i)$, $T_i x^* = \{x^*\}$. For $j = 1, 2, \dots, N$, let $F_j : C \times C \to \mathbb{R}$ be bifunctions which satisfy $(A1) - (A4)$, let $f_j : C \to C$ be continuous and monotone mappings, and $\varphi_j : C \to \mathbb{R} \cup \{+\infty\}$ be proper lower semi-continuous and convex functions such that $C \cap \mathrm{dom}\varphi_j \neq \emptyset$. Assume that $\Gamma := (\cap_{i=1}^m F(T_i)) \cap (\cap_{j=1}^N GMEP(F_j, f_j, \varphi_j)) \neq \emptyset$. Let the sequence $\{x_n\}$ be generated by

$$\begin{cases} x_0 \in C = C_0, \\ u_n = T_{r_{N,n}} \circ T_{r_{N-1,n}} \circ \cdots \circ T_{r_{2,n}} \circ T_{r_{1,n}} x_n, \\ y_n = \alpha_0 u_n + \sum_{i=1}^m \alpha_i w_n^i, \\ C_{n+1} = \{z \in C_n : ||y_n - z|| \leq ||x_n - z||\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \ n \geq 0, \end{cases} \tag{17}$$

where $r_{j,n} \in (0, \infty)$ with $\liminf_{n\to\infty} r_{j,n} > 0, \ j = 1, 2, \ldots, N, \ w_n^i \in T_i u_n, n \geq 1, \ \alpha_i \in (k, 1), \ i = 0, 1, 2, \ldots, m$ such that $\sum_{i=0}^m \alpha_i = 1$ and $k := \max\{k_i, i = 1, 2, \ldots, m\}$. Then the sequence $\{x_n\}$ converges strongly to $P_\Gamma x_0$.

Proof: .We divide our proof into 7 steps.

Step 1: We show that $P_{C_{n+1}} x_0$ is well defined for $x_0 \in C$.
It follows from Lemma [4] and the condition on $CB(C)$ that $\cap_{i=1}^m F(T_i)$ and $\cap_{j=1}^N GMEP(F_j, f_j, \varphi_j)$ are closed and convex subsets of C. Hence, Γ is a closed and convex subset of C. Also, by Lemma[3] , we obtain that C_{n+1} is closed and convex for $n \geq 0$.
Let $x^* \in \Gamma$ and $\Phi_n^N = T_{r_{N,n}} \circ T_{r_{N-1,n}} \circ \cdots \circ T_{r_{2,n}} \circ T_{r_{1,n}}$, where $\Phi_n^0 = I$, then it follows from Lemma [4] (iv) that $\Phi_n^N x^* = x^*$. Also from Algorithm 17, we have

$$\begin{aligned} ||u_n - x^*|| &= ||\Phi_n^N x_n - x^*|| \\ &= ||T_{r_{N,n}} \Phi_n^{N-1} x_n - x^*|| \\ &\leq ||\Phi_n^{N-1} x_n - x^*|| \\ &\vdots \\ &\leq ||x_n - x^*||. \end{aligned} \tag{18}$$

From Lemma [7] , Lemma [1] , Lemma [6] and conclusion (18), we obtain the following:

$$\begin{aligned} ||y_n - x^*||^2 &= ||\alpha_0 u_n + \sum_{i=1}^m \alpha_i w_n^i - x^*||^2 \\ &= ||\alpha_0 (u_n - x^*) + \sum_{i=1}^m \alpha_i (w_n^i - x^*)||^2 \end{aligned}$$

$$
\begin{aligned}
&= \alpha_0||u_n - x^*||^2 + \sum_{i=1}^{m} \alpha_i ||w_n^i - x^*||^2 \\
&- \sum_{i=1}^{m} \alpha_0\alpha_i ||u_n - w_n^i||^2 - \sum_{i,j=1, i\neq j} \alpha_i\alpha_j ||w_n^i - w_n^j||^2 \\
&\leq \alpha_0||u_n - x^*||^2 + \sum_{i=1}^{m} \alpha_i \mathcal{H}^2(T_i u_n, T_i x^*) \\
&- \sum_{i=1}^{m} \alpha_0\alpha_i ||u_n - w_n^i||^2 \\
&\leq \alpha_0||u_n - x^*||^2 + \sum_{i=1}^{m} \alpha_i \left[||u_n - x^*||^2 + k_i||u_n - w_n^i||^2 \right] \\
&- \sum_{i=1}^{m} \alpha_0\alpha_i ||u_n - w_n^i||^2 \\
&= ||u_n - x^*||^2 + (k - \alpha_0)\sum_{i=1}^{m} \alpha_i ||u_n - w_n^i||^2 \qquad (19) \\
&\leq ||u_n - x^*||^2 \\
&\leq ||x_n - x^*||^2,
\end{aligned}
$$

which implies that $x^* \in C_{n+1}$. Hence, $\Gamma \subset C_{n+1}$, thus $P_{C_{n+1}}x_0$ is well defined.

Step 2: We show that $\lim_{n\to\infty} ||x_n - x^*||$ exists.

Since $x_n = P_{C_n}x_0$ and $x_{n+1} \in C_{n+1} \subset C_n \ \forall n \geq 0$. We have

$$||x_n - x_0|| \leq ||x_{n+1} - x_0||. \qquad (20)$$

For $x^* \in \Gamma \subset C_{n+1} \subset C_n$, we have

$$||x_n - x_0|| \leq ||x^* - x_0||. \qquad (21)$$

From (20) and (21), we have that $\{x_n\}$ is a non-decreasing and bounded sequence. Therefore, $\lim_{n\to\infty} ||x_n - x_0||$ exists.

Step 3: We show that the sequence $\{x_n\}$ converges strongly to $\bar{x} \in C$.

Since $x_m = P_{C_m}x_0 \in C_m \subset C_n$ for $m > n$, we obtain from Lemma [5] that

$$||x_m - x_n||^2 \leq ||x_m - x_0||^2 - ||x_n - x_0||^2. \qquad (22)$$

Since $\lim_{n\to\infty} ||x_n - x_0||$ exists, we have from (22) that

$$\lim_{n\to\infty} ||x_m - x_n|| = 0.$$

Thus, $\{x_n\}$ is a Cauchy sequence. By the completeness of H and the closeness of C there exists $\bar{x} \in C$ such that $\{x_n\}$ converges to $\bar{x}$.

Step 4: We show that $\lim_{n\to\infty} ||w_n^i - x_n|| = 0,\ i = 1, 2, \ldots, N$.
From $x_{n+1} \in C_{n+1}$, we have that $||y_n - x_{n+1}|| \le ||x_n - x_{n+1}||$. From which we obtain that

$$\begin{aligned} ||x_n - y_n|| &\le ||x_n - x_{n+1}|| + ||x_{n+1} - y_n|| \\ &\le 2||x_n - x_{n+1}|| \to 0,\ \text{as } n \to \infty. \end{aligned} \tag{23}$$

From (19), we have

$$\begin{aligned} (\alpha_0 - k)\sum_{i=1}^{m} \alpha_i ||u_n - w_n^i||^2 &\le ||u_n - x^*||^2 - ||y_n - x^*||^2 \\ &\le ||x_n - x^*||^2 - ||y_n - x^*||^2 \end{aligned} \tag{24}$$

and for each $i = 1, 2, \ldots, m$, we have

$$\begin{aligned} (\alpha_0 - k)\alpha_i ||u_n - w_n^i||^2 &\le ||x_n - x^*||^2 - ||y_n - x^*||^2 \\ &\le M||x_n - y_n|| \to 0,\ \text{as } n \to \infty, \end{aligned} \tag{25}$$

where $M = \sup_{n\ge 0}\{||x_n - x^*|| + ||y_n - x^*||\}$.
Using the condition on α_i in (25), we have

$$\lim_{n\to\infty} ||u_n - w_n^i||^2 = 0,\ i = 1, 2, \ldots, m. \tag{26}$$

From Lemma [4] (iii), we have

$$\begin{aligned} ||u_n - x^*||^2 &= ||T_{r_{N,n}} \Phi_n^{N-1} x_n - x^*||^2 \\ &\le \langle u_n - x^*, \Phi_n^{N-1} x_n - x^* \rangle \\ &= \frac{1}{2}\big[||u_n - x^*||^2 + ||\Phi_n^{N-1} x_n - x^*|| \\ &- ||u_n - \Phi_n^{N-1} x_n||^2\big], \end{aligned} \tag{27}$$

which implies

$$\begin{aligned} ||u_n - \Phi_n^{N-1} x_n||^2 &\le ||\Phi_n^{N-1} x_n - x^*||^2 - ||u_n - x^*||^2 \\ &\vdots \\ &\le ||x_n - x^*||^2 - ||u_n - x^*||^2 \\ &\le ||x_n - x^*||^2 - ||y_n - x^*||^2 \\ &\le M||x_n - y_n|| \to 0,\ \text{as } n \to \infty, \end{aligned}$$

which implies

$$\lim_{n\to\infty} ||\Phi_n^N x_n - \Phi_n^{N-1} x_n|| = 0. \tag{28}$$

By similar argument as in (27), we obtain

$$\begin{aligned} ||\Phi_n^{N-1}x_n - \Phi_n^{N-2}x_n||^2 &\le ||\Phi_n^{N-2}x_n - x^*||^2 - ||\Phi_n^{N-1}x_n - x^*||^2 \\ &\vdots \\ &\le ||x_n - x^*||^2 - ||\Phi_n^{N-1}x_n - x^*||^2 \\ &\le ||x_n - x^*||^2 - ||u_n - x^*||^2 \\ &\le ||x_n - x^*||^2 - ||y_n - x^*||^2 \\ &\le M||x_n - y_n|| \to 0, \text{ as } n \to \infty, \end{aligned}$$

which implies

$$\lim_{n\to\infty} ||\Phi_n^{N-1}x_n - \Phi_n^{N-2}x_n|| = 0. \tag{29}$$

Continuing in the same manner, we have that

$$\begin{gathered} \lim_{n\to\infty} ||\Phi_n^{N-2}x_n - \Phi_n^{N-3}x_n|| = \cdots = \\ \lim_{n\to\infty} ||\Phi_n^{2}x_n - \Phi_n^{1}x_n|| = \lim_{n\to\infty} ||\Phi_n^{1}x_n - x_n|| = 0. \end{gathered} \tag{30}$$

From (28), (29) and (30), we conclude that

$$\lim_{n\to\infty} ||\Phi_n^{j}x_n - \Phi_n^{j-1}x_n|| = 0, \ j = 1, 2, \ldots, N. \tag{31}$$

Thus,

$$\begin{gathered} ||u_n - x_n|| \le ||\Phi_n^{N}x_n - \Phi_n^{N-1}x_n|| + \\ ||\Phi_n^{N-1}x_n - \Phi_n^{N-2}x_n|| + \cdots + ||\Phi_n^{1}x_n - x_n||, \end{gathered}$$

which implies

$$\lim_{n\to\infty} ||u_n - x_n|| = 0. \tag{32}$$

From (26) and (32), we have

$$\lim_{n\to\infty} ||w_n^i - x_n|| = 0, \ i = 1, 2, \ldots, m. \tag{33}$$

Step 5: We show that $\bar{x} \in \cap_{j=1}^{N} GMEP(F_j, f_j\varphi_j)$.
From (31) and by the condition that $\liminf_{n\to\infty} r_{j,n} > 0, \ j = 1, 2, \ldots, N$, we have

$$\lim_{n\to\infty} ||\frac{\Phi_n^{j}x_n - \Phi_n^{j-1}x_n}{r_{j,n}}|| = 0, j = 1, 2, \ldots, N. \tag{34}$$

From Lemma[4] , we have for each $j = 1, 2, \ldots, N$,

$$\begin{aligned} F_j(\Phi_n^{j}x_n, y) + \langle f_j\Phi_n^{j}x_n, y - \Phi_n^{j}x_n\rangle + \varphi_j(y) - \varphi_j(\Phi_n^{j}x_n) \\ + \frac{1}{r_{j,n}}\langle y - \Phi_n^{j}x_n, \Phi_n^{j}x_n - \Phi_n^{j-1}x_n\rangle \ge 0, \ \forall y \in C. \end{aligned} \tag{35}$$

By (A2) and since f_j is monotone for each $j = 1, 2, \dots, N$, we obtain

$$\varphi_j(y) - \varphi_j(\Phi_n^j x_n) + \langle y - \Phi_n^j x_n, \frac{\Phi_n^j x_n - \Phi_n^{j-1} x_n}{r_{j,n}}\rangle \geq F_j(y, \Phi_n^j x_n) + \langle f_j y, \Phi_n^j x_n - y\rangle, \ \forall y \in C. \quad (36)$$

From (32), we have that $\{u_n\} = \{\Phi_n^j x_n\}$ converges to $\bar{x}$. Thus, taking limit as $n \to \infty$ in (36), then we have from (34), (A4) and the lower semicontinuity of φ that

$$F_j(y, \bar{x}) + \langle f_j y, \bar{x} - y\rangle + \varphi_j(\bar{x}) - \varphi_j(y) \leq 0, \ \forall y \in C.$$

Let $z_t = ty + (1-t)\bar{x}$ for $t \in (0, 1]$, $y \in C$. Then $z_t \in C$ for each $j = 1, 2, \dots, N$. Hence,

$$F_j(z_t, \bar{x}) + \langle f_j z_t, \bar{x} - z_t\rangle + \varphi_j(\bar{x}) - \varphi_j(z_t) \leq 0.$$

From $(A1), (A4)$ and the convexity of φ, we have

$$\begin{aligned}
0 &= F_j(z_t, z_t) + \langle f_j z_t, z_t - z_t\rangle + \varphi_j(z_t) - \varphi(z_t) \\
&\leq tF_j(z_t, y) + (1-t)F_j(z_t, \bar{x}) + t\langle f_j z_t, y - z_t\rangle + (1-t)\langle f_j z_t, \bar{x} - z_t\rangle \\
&\quad + t\varphi_j(y) + (1-t)\varphi_j(\bar{x}) - (t\varphi_j(z_t) + (1-t)\varphi_j(z_t)) \\
&= t\left[F_j(z_t, y) + \langle f_j z_t, y - z_t\rangle + \varphi_j(y) - \varphi_j(z_t)\right] \\
&\quad + (1-t)\left[F_j(z_t, \bar{x}) + \langle f_j z_t, \bar{x} - z_t\rangle + \varphi_j(\bar{x}) - \varphi_j(z_t)\right] \\
&\leq t\left[F_j(z_t, y) + \langle f_j z_t, y - z_t\rangle + \varphi_j(y) - \varphi_j(z_t)\right], \quad (37)
\end{aligned}$$

which implies for each $j = 1, 2, \dots, N$ that

$$F_j(z_t, y) + \langle f_j z_t, y - z_t\rangle + \varphi_j(y) - \varphi_j(z_t) \geq 0, \ y \in C. \quad (38)$$

From the definition of z_t, we have that $z_t \to \bar{x}$ if $t \to 0$. Therefore, letting $t \to 0$ in (38), we have

$$F_j(\bar{x}, y) + \langle f_j \bar{x}, y - \bar{x}\rangle + \varphi_j(y) - \varphi_j(\bar{x}) \geq 0, \ y \in C,$$

which implies that $\bar{x} \in GMEP(F_j, f_j, \varphi_j)$, for each $j = 1, 2, \dots, N$. Hence,

$$\bar{x} \in \cap_{j=1}^N GMEP((F_j, f_j, \varphi_j).$$

Step 6: We show that $\bar{x} \in \cap_{i=1}^m F(T_i)$.
Since $w_n^i \in T_i u_n$, $i = 1, \dots, m$, we have that $d(u_n, T_i u_n) \leq ||u_n - w_n^i||$, which implies from (26) that

$$\lim_{n\to\infty} d(u_n, T_i u_n) = 0, i = 1, 2, \dots, m. \quad (39)$$

From (39) and the demiclosedness of T_i, we have that $\bar{x} \in \cap_{n=1}^m F(T_i)$.
Step 7: We finally show that $\bar{x} = P_\Gamma x_0$.
Since $x_n = P_{C_n} x_0$, we have

$$\langle z - x_n, x_0 - x_n\rangle \leq 0 \ \forall z \in C_n.$$

Therefore,

$$\langle x^* - x_n, x_0 - x_n \rangle \leq 0 \ \forall x^* \in \Gamma \subset C_n. \tag{40}$$

Letting $n \to \infty$ in (40), we obtain

$$\langle x^* - \bar{x}, x_0 - \bar{x} \rangle \leq 0 \ \forall x^* \in \Gamma,$$

which implies that $\bar{x} = P_\Gamma x_0$, this completes the proof.

If for each $i = 1, 2, \ldots, m$, $T_i : C \to CB(C)$ is a multivalued nonexpansive mappings, then we obtain the following result.
Corollary 1: Let H be a real Hilbert space and C be a nonempty, closed and convex subset of H. Let $T_i : C \to CB(C)$ be finite family of multi-valued nonexpansive mappings. For $j = 1, 2, \ldots, N$, let $F_j : C \times C \to \mathbb{R}$ be bifunctions which satisfy $(A1) - (A4)$, let $f_j : C \to C$ be continuous and monotone mappings, and $\varphi_j : C \to \mathbb{R} \cup \{+\infty\}$ be proper lower semi-continuous and convex functions such that $C \cap \mathrm{dom}\varphi_j \neq \emptyset$. Assume that $\Gamma := (\cap_{i=1}^m F(T_i)) \cap (\cap_{j=1}^N GMEP(F_j, f_j, \varphi_j)) \neq \emptyset$. Let the sequence $\{x_n\}$ be generated by

$$\begin{cases} x_0 \in C = C_0, \\ u_n = T_{r_{N,n}} \circ T_{r_{N-1,n}} \circ \cdots \circ T_{r_{2,n}} \circ T_{r_{1,n}} x_n, \\ y_n = \alpha_0 u_n + \sum_{i=1}^m \alpha_i w_n^i, \\ C_{n+1} = \{z \in C_n : ||y_n - z|| \leq ||x_n - z||\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \ n \geq 0, \end{cases} \tag{41}$$

where $r_{j,n} \in (0, \infty)$ with $\liminf_{n\to\infty} r_{j,n} > 0, \ j = 1, 2, \ldots, N, \ w_n^i \in T_i u_n, n \geq 1, \ \alpha_i \in (0,1)$ such that $\liminf\limits_{n\to\infty} \alpha_i \alpha_0 > 0, \ i = 1, 2, \ldots, m$ and $\sum_{i=0}^m \alpha_i = 1$. Then the sequence $\{x_n\}$ converges strongly to $P_\Gamma x_0$.

Setting $f \equiv 0$ and $N = 1$ in Theorem , we obtain the following result.
Corollary 2: Let H be a real Hilbert space and C be a nonempty, closed and convex subset of H. Let $T_i : C \to CB(C)$ be finite family of multi-valued k_i-strictly pseudo-contractive mappings, $k_i \in (0,1), \ i = 1, 2, \ldots, m$. Assume that for $x^* \in \cap_{i=1}^m F(T_i), \ T_i x^* = \{x^*\}$. Let $F_1 : C \times C \to \mathbb{R}$ be bifunctions which satisfy $(A1) - (A4)$ and $\varphi_1 : C \to \mathbb{R} \cup \{+\infty\}$ be proper lower semi-continuous and convex functions such that $C \cap \mathrm{dom}\varphi_1 \neq \emptyset$. Assume that $\Gamma := \cap_{i=1}^m F(T_i) \cap GMEP(F_1, \varphi_1) \neq \emptyset$. Let the sequence $\{x_n\}$ be

generated by

$$\begin{cases} x_0 \in C = C_0, \\ F_1(u_n, y) + \varphi_1(y) - \varphi_1(u_n) + \frac{1}{r_{1,n}}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ y_n = \alpha_0 u_n + \sum_{i=1}^{m} \alpha_i w_n^i, \\ C_{n+1} = \{z \in C_n : ||y_n - z|| \leq ||x_n - z||\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \ n \geq 0, \end{cases} \tag{42}$$

where $r_{1,n} \in (0, \infty)$ with $\liminf_{n\to\infty} r_{1,n} > 0$, $w_n^i \in T_i u_n, n \geq 1$, $\alpha_i \in (k, 1)$, $i = 0, 1, 2, \dots, m$ such that $\sum_{i=0}^{m} \alpha_i = 1$ and $k := \max\{k_i, i = 1, 2, \dots, m\}$. Then the sequence $\{x_n\}$ converges strongly to $P_\Gamma x_0$.

Setting $\varphi \equiv 0$ and $j = 1$ in Theorem , we obtain the following result.
Corollary 3: Let H be a real Hilbert space and C be a nonempty, closed and convex subset of H. Let $T_i : C \to CB(C)$ be finite family of multi-valued k_i-strictly pseudo-contractive mappings, $k_i \in (0, 1)$, $i = 1, 2, \dots, m$. Assume that for $x^* \in \cap_{i=1}^{m} F(T_i)$, $T_i x^* = \{x^*\}$. Let $F_1 : C \times C \to \mathbb{R}$ be bifunctions which satisfy $(A1) - (A4)$ and let $f_1 : C \to C$ be continuous and monotone mappings. Assume that $\Gamma := \cap_{i=1}^{m} F(T_i) \cap GMEP(F_1, f_1) \neq \emptyset$. Let the sequence $\{x_n\}$ be generated by

$$\begin{cases} x_0 \in C = C_0, \\ F_1(u_n, y) + \langle f_1 u_n, y - u_n\rangle + \frac{1}{r_{1,n}}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ y_n = \alpha_0 u_n + \sum_{i=1}^{m} \alpha_i w_n^i, \\ C_{n+1} = \{z \in C_n : ||y_n - z|| \leq ||x_n - z||\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \ n \geq 0, \end{cases} \tag{43}$$

where $r_{1,n} \in (0, \infty)$ with $\liminf_{n\to\infty} r_{1,n} > 0$, $w_n^i \in T_i u_n, n \geq 1$, $\alpha_i \in (k, 1)$, $i = 0, 1, 2, \dots, m$ such that $\sum_{i=0}^{m} \alpha_i = 1$ and $k := \max\{k_i, i = 1, 2, \dots, m\}$. Then the sequence $\{x_n\}$ converges strongly to $P_\Gamma x_0$.

4. NUMERICAL EXAMPLE

In this section, we give a numerical example of Theorem to illustrate the applicability of our result.
Let $H = \mathbb{R}$ be endowed with the usual norm and $C = [0, 1]$. Let $N = 2$, then for $j = 1$, we define $F_1 : C \times C \to \mathbb{R}$ by $F_1(x, y) = y^2 - x^2$,
$f_1 : C \to C$ by $f_1(x) = 2x$ and

$\varphi_1 : C \to \mathbb{R} \cup \{+\infty\}$ by $\varphi_1(x) = x^2$. For any $r_{1,n} > 0$ and $x \in C$, Lemma [26] ensures that there exists $z \in C$ such that

$$F_1(z,y)+\langle f_1 z, y-z\rangle+\varphi_1(y)-\varphi_1(z)+\frac{1}{r_{1,n}}\langle y-z, z-x\rangle \geq 0,\ \forall y \in C.$$

$$\begin{aligned}
&\iff\ y^2 - z^2 + 2z(y-z) + y^2 - z^2 + \frac{1}{r_{1,n}}(y-z)(z-x) \geq 0\\
&\iff\ 2y^2 - 4z^2 + 2yz + \frac{1}{r_{1,n}}(yz - xy - z^2 + xz) \geq 0\\
&\iff\ 2r_{1,n}y^2 + (2r_{1,n}z + z - x)y - (4r_{1,n}z^2 + z^2 - xz) \geq 0.
\end{aligned}$$

Letting $h(y) = 2r_{1,n}y^2 + (2r_{1,n}z + z - x)y - (4r_{1,n}z^2 + z^2 - xz)$, we have that $f(y)$ is a quadratic function of y. Thus, we compute the discriminant Δ of h as follows:

$$\begin{aligned}
\Delta &= (2r_{1,n}z + z - x)^2 - 4(2r_{1,n})(4r_{1,n}z^2 + z^2 - xz)\\
&= x^2 - 2(6r_{1,n}z + z)x + (6r_{1,n}z + z)^2\\
&= \left(x - (6r_{1,n}z + z)\right)^2,
\end{aligned}$$

thus $\Delta \geq 0$ for all $y \in C$. If it has at most one solution in C, $\Delta \leq 0$. Therefore,

$$\Delta = \left(x - (6r_{1,n}z + z)\right)^2 = 0.$$

Hence, $z = \frac{x}{6r_{1,n}+1}$. So that the resolvent function

$$T_{r_{1,n}}(x) = \frac{x}{6r_{1,n}+1}.$$

Now, for $j = 2$, we define $F_2 : C \times C \to \mathbb{R}$ by $F_2(x,y) = 2y^2 - 2x^2$, $f_2 : C \to C$ by $f_2(x) = 4x$ and
$\varphi_2 : C \to \mathbb{R} \cup \{+\infty\}$ by $\varphi_2(x) = 2x^2$.
Thus, using similar arguments as above, we obtain the resolvent function

$$T_{r_{2,n}}(x) = \frac{x}{12r_{2,n}+1}.$$

If we take $r_{1,n} = \frac{1}{1+\frac{1}{n}}$ and $r_{2,n} = \frac{2}{1+\frac{1}{n}}$, then

$$u_n = T_{r_{2,n}}\left(T_{r_{1,n}}x_n\right) = \frac{x_n}{(\frac{6}{1+\frac{1}{n}}+1)(\frac{24}{1+\frac{1}{n}}+1)}. \tag{44}$$

Using example 3.1 of [9], we have the following examples;
Let $H = \mathbb{R}$, with the usual metric and $S_i : \mathbb{R} \to CB(R),\ i = 1,2$

be the multivalued mapping defined by

$$S_1(x) = \begin{cases} [0, \frac{x}{2}] \text{ if } x \in (0, \infty), \\ [\frac{x}{2}, 0] \text{ if } x \in (-\infty, 0] \end{cases} \tag{45}$$

$$S_2(x) = \begin{cases} [0, \frac{x}{3}] \text{ if } x \in (0, \infty), \\ [\frac{x}{3}, 0] \text{ if } x \in (-\infty, 0] \end{cases} \tag{46}$$

Then P_{S_i} is strictly pseudocontractive, $i = 1, 2$. In Theorem , let $T_i = P_{S_i}$, $i = 1, 2$, then with u_n given by (44), algorithm (43) becomes

$$\begin{cases} x_0 \in C = C_0, \\ u_n = \frac{x_n}{(\frac{6}{1+\frac{1}{n}}+1)(\frac{24}{1+\frac{1}{n}}+1)}. \\ y_n = \alpha_0 u_n + \sum_{i=1}^{m} \alpha_i w_n^i, \\ C_{n+1} = \{z \in C_n : ||y_n - z|| \leq ||x_n - z||\}, \\ x_{n+1} = P_{C_{n+1}} x_0,\ n \geq 0, \end{cases} \tag{47}$$

where $w_n^i \in T_i u_n$.
We note that $\Gamma := (\cap_{i=1}^2 F(T_i)) \cap (\cap_{j=1}^2 GMEP(F_j, f_j, \varphi_j)) = \{0\}$.

5. CONCLUDING REMARKS

The main Theorem of this work, Theorem 3.1 is a significant improvement on the results of Sastry and Babu [28], Chidume and Ezeora [10] and a lot of other important recent results in the following ways;
(i) compactness assumption imposed on the domain of the multivalued operator, studied in Sastry and Babu [28] is dispensed with in Theorem of this work
(ii) To obtain convergence of the sequence defined in the work of Chidume and Ezeora [10], the authors required that at least one of the operators be Hemicompact, this is dispensed with here.
(ii) The sequence constructed in Theorem of this work is applicable in solving multiple problems jointly as against those of Sastry and Babu [28], Chidume and Ezeora [10] and a host of others in this direction of research.

ACKNOWLEDGEMENTS

The author would like to thank the anonymous referee whose comments improved the original version of this manuscript.

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