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NEW INERTIAL METHOD FOR NONEXPANSIVE MAPPINGS

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ABSTRACT. There have been increasing interests in studying inertial Krasnoselskii-Mann iterations due to the presence of inertial extrapolation step which improves the rate of convergence of Krasnoselskii-Mann iterations. These results analyzed the convergence properties of inertial Krasnoselskii-Mann iterations and demonstrated their performance numerically on some imaging and data analysis problems. It is discovered that these proposed inertial Krasnoselskii-Mann iterations assumed some stringent conditions on the inertial factor which make the implementations difficult in some numerical examples. In this present paper, we provide a new inertial Krasnoselskii-Mann iteration, prove its weak convergence and the corresponding rate of convergence under some suitable conditions.

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1. INTRODUCTION

Throughout this paper, we consider the real Hilbert space setting: H denotes a real Hilbert space with scalar product $\langle ., . \rangle$ and induced norm $\|\cdot\|$. We assume that $T : H \to H$ is a nonexpansive mapping, i.e., T satisfies

$$\|Tx - Ty\| \leq \|x - y\| \quad \forall x, y \in H.$$

We further denote the set of fixed points of T by

$$F(T) := \{x \in X \mid Tx = x\}.$$

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Prominent examples for nonexpansive mappings in Hilbert spaces are, the projection map, the proximal point map, and several composite maps which involve at least one of these two mappings, see, e.g., [6] for more details.

Many iterative schemes for approximating fixed points of nonexpansive mappings T are well-known from the literature, cf. [7, 9, 14, 17, 15] and references therein for some relevant results in this direction. One of the most famous fixed point methods is the Krasnoselskii-Mann iteration from [29, 34] that starts at some given point $x_1 \in H$ and uses the recursion

$$x_{n+1} = (1 - \beta_n)x_n + \beta_n T x_n \quad \forall n = 1, 2, \ldots \tag{1}$$

for some suitably chosen scalars $\beta_n \in [0, 1]$. The most general convergence result for this procedure is due to Reich [39] and assumes that if $F(T)$ is nonempty and $\{\beta_n\}$ satisfies the condition

$$\sum_{n=1}^{\infty} \beta_n(1 - \beta_n) = \infty, \tag{2}$$

then the iterates $\{x_n\}$ converge weakly to a fixed point of T. Many authors have studied approximation of fixed points of nonexpansive mappings using Krasnoselskii-Mann iteration (1) in both Hilbert spaces and Banach spaces (see, for example, [17, 18, 25, 26, 28, 30, 37, 39, 42] and the references contained therein).

In [20], Cominetti *et al.* showed that $\|x_n - Tx_n\|$ in (1) converges to zero at a rate of $O(1/\sqrt{\sigma_n})$ (big-O), where $\sigma_n := \sum_{k=1}^{n} \lambda_k(1 - \lambda_k), n \in \mathbb{N}$. Further convergence rate analysis for both exact and inexact Krasnoselski-Mann iterations built from nonexpansive mappings have been established recently in [22, 30, 35]. For example, it has been shown in [30, Theorem 1] that $\|x_n - Tx_n\|^2 = O(1/n)$.

We recall that the inertial extrapolation term is based upon a discrete version of a second order dissipative dynamical system [2, 3] and has been regarded as a procedure of speeding up the convergence properties (see, e.g., [1, 8, 32, 33, 38]). In the light of this observation, there have been increasing interests in studying inertial type algorithms. See, for example, inertial forward-backward splitting methods [4, 31, 36], inertial Douglas-Rachford splitting method [10], inertial ADMM [11, 16], and inertial forward-backward-forward method [12]. For example, it is known that acceleration scheme developed by Nesterov improves the theoretical rate of convergence of forward-backward method from the standard $O(1/(n+1))$ down to $O(1/(n+1)^2)$ and the inertial extrapolation scheme of Nesterov's accelerated forward-backward method is actually $o(1/(n+1)^2)$ rather than $O(1/(n+1)^2)$ (see [5]). These results and other related ones analyzed the convergence properties of inertial type algorithms and demonstrated their performance numerically on some problems arising from image reconstructions.

Recently, Bot *et al.* [10] proposed the following inertial version of the Krasnoselskii-Mann algorithm for approximating the set of fixed points of a nonexpansive operator: $x_0, x_1 \in H$,

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = w_n + \beta_n(Tw_n - w_n). \end{cases} \tag{3}$$

They gave weak convergence analysis in real Hilbert spaces under the conditions:
(a) $0 \leq \theta_n \leq \theta_{n+1} \leq \theta < 1, \forall n \geq 1$ and $\lambda, \sigma, \delta > 0$ such that
(b) $\delta > \frac{\theta^2(1+\theta)+\theta\sigma}{1-\theta^2}$; and
(c) $0 < \lambda \leq \beta_n \leq \rho := \frac{\delta - \theta[\theta(1+\theta)+\theta\delta+\sigma]}{\delta[1+\theta(1+\theta)+\theta\delta+\sigma]}$.
They also applied their results to inertial Douglas-Rachford splitting algorithm for finding common zeros of the sum of two maximally monotone operators in Hilbert spaces and illustrate their results through some numerical experiments in clustering and generalized Hebron problems. The convergence rate of (3) was also established in [40].

The results of Bot *et al.* [10] improved on the results of Maingé [33], where it was assumed, given the inertial Krasnoselskii-Mann (3), that
(i) $\theta_n \in [0, \theta)$ and $\theta \in [0, 1)$;
(ii) $\sum \theta_n \|x_n - x_{n-1}\|^2 < \infty$; and
(iii) $0 < \alpha \leq \beta_n \leq \beta < 1$.
However, one can see from the results of [10] that the conditions (b) and (c) imposed on the inertial factor θ_n and the iterative parameter β_n are stringent and not easy to implement during computations. The same conditions in [10] have been used in [23] to solve variational inequality problem.

Motivated by the results of Bot *et al.* [10], our aim in this paper is to introduce a new inertial Krasnoselskii-Mann iteration with weaker conditions on the inertial factor θ_n and the iterative parameters than the conditions (a)-(c) assumed in [10]. Using this new proposed inertial Krasnoselskii-Mann iteration, we establish both weak convergence and corresponding nonasymptotic $O(1/n)$ convergence rate result. It has been shown numerically (see, e.g., [10, 11, 12]) that inertial Krasnoselskii-Mann iteration gives faster convergence when the inertial factor θ_n approaches 1. One of our contributions in this paper is that the inertial factor $\theta_n \in [0, 1]$ ($\theta_n = 1$ is allowed) unlike the assumption in [10, 33], where $\theta_n \in [0, 1)$. This means that our results in this paper bring a major contribution to the state-of-the-art in the literature on the inertial Krasnoselskii-Mann iterations. Furthermore, the conditions on our iterative parameters (see conditions (b) and (c) of Theorem 1 below) seem simpler than the conditions (b) and (c) imposed in [10] and the assumptions of Maingé [33]. Our result complement the results given in [10, 13, 21, 24, 27, 41] and many other recent results in the literature. Our method of proof is of independent interest.

The paper is therefore organized as follows: We first recall some basic definitions and results in Section 2. We propose the new inertial Krasnoselskii-Mann iteration and give its weak convergence analysis in Section 3. In Section 4, we give the nonasymptotic $O(1/n)$ convergence rate analysis of the proposed method. We conclude with some final remarks in Section 5.

2. PRELIMINARIES

We give some basic properties which will be used in our convergence analysis in the next section. We start with the following lemma whose proof is elementary and therefore omitted.

Lemma 1: The following statement holds in H:

$$\|tx+sy\|^2 = t(t+s)\|x\|^2+s(t+s)\|y\|^2-st\|x-y\|^2, \quad \forall x, y \in H, \forall s, t \in \mathbb{R}.$$

Lemma 2: (Maingé [33]) Let $\{\varphi_n\}, \{\delta_n\}$ and $\{\theta_n\}$ be sequences in $[0, +\infty)$ such that

$$\varphi_{n+1} \leq \varphi_n + \theta_n(\varphi_n - \varphi_{n-1}) + \delta_n, \ \forall n \geq 1, \ \sum_{n=1}^{+\infty} \delta_n < +\infty,$$

and there exists a real number θ with $0 \leq \theta_n \leq \theta < 1$ for all $n \in \mathbb{N}$. Then the following hold:

(i) $\sum_{n=1}^{+\infty}[\varphi_n - \varphi_{n-1}]_+ < +\infty$, where $[t]_+ := \max\{t, 0\}$;
(ii) there exists $\varphi^* \in [0, +\infty)$ such that $\lim_{n\to\infty} \varphi_n = \varphi^*$.

Lemma 3: (Opial [37]) Let C be a nonempty set of H and $\{x_n\}$ be a sequence in H such that the following two conditions hold:
(i) for any $x \in C$, $\lim_{n\to\infty} \|x_n - x\|$ exists;
(ii) every sequential weak cluster point of $\{x_n\}$ is in C.
Then $\{x_n\}$ converges weakly to a point in C.

Definition: A mapping S is said to be demiclosed if for any sequence $\{x_n\}$ which weakly converges to y, and if the sequence $\{Sx_n\}$ strongly converges to z, then $S(y) = z$.

Lemma 4: (Goebel and Reich [26]) Let C be a nonempty, closed and convex subset of H. Let $T : C \mapsto C$ be a nonexpansive mapping. Then $I - T$ is demiclosed at 0.

3. WEAK CONVERGENCE ANALYSIS

This section investigates the weak convergence analysis of the sequence of iterates $\{x_n\}$ generated by our proposed inertial Krasnoselskii-Mann iteration.

Theorem 1: Suppose that $T : H \to H$ is a nonexpansive mapping such that its set of fixed points $F(T)$ is nonempty. Let the sequence $\{x_n\}$ in H be generated by choosing $x_0 = x_1 \in H$ and using the recursion

$$\left\{ \begin{array}{l} w_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n((1-\beta_n)w_n + \beta_n T w_n), \end{array} \right. \tag{4}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\theta_n\}$ are sequences such that
(a) $0 \le \theta_n \le \theta_{n+1} \le 1$;
(b) $0 < \alpha \le \alpha_n \le \alpha_{n+1} \le \frac{1}{2+\delta} := \epsilon,\ \delta > 0$; and
(c) $0 < \beta \le \beta_n \le \gamma < 1$.
Then the sequence $\{x_n\}$ generated by (4) converges weakly to a fixed point of T.

Proof: Let $z \in F(T)$ and define $u_n := (1-\beta_n)w_n + \beta_n T w_n$. Then by Lemma 1,

$$\begin{aligned} \|u_n - z\|^2 &= \|(1-\beta_n)(w_n - z) + \beta_n(Tw_n - z)\|^2 \\ &= (1-\beta_n)\|w_n - z\|^2 + \beta_n\|Tw_n - z\|^2 \\ &\quad -\beta_n(1-\beta_n)\|Tw_n - w_n\|^2 \\ &\le \|w_n - z\|^2 - \beta_n(1-\beta_n)\|Tw_n - w_n\|^2. \end{aligned} \tag{5}$$

Therefore,

$$\|u_n - z\| \leq \|w_n - z\|. \tag{6}$$

Now, by (4), we get

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|(1-\alpha_n)(x_n - z) + \alpha_n(u_n - z)\|^2 \\ &= (1-\alpha_n)\|x_n - z\|^2 + \alpha_n\|u_n - z\|^2 \\ &\quad -\alpha_n(1-\alpha_n)\|x_n - u_n\|^2, \end{aligned}\tag{7}$$

which in turn implies that (noting (6))

$$\begin{aligned}\|x_{n+1} - z\|^2 &\leq (1-\alpha_n)\|x_n - z\|^2 + \alpha_n\|w_n - z\|^2 \\ &\quad -\alpha_n(1-\alpha_n)\|x_n - u_n\|^2. \end{aligned}\tag{8}$$

Note that

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n u_n$$

and this implies

$$u_n - x_n = \frac{1}{\alpha_n}(x_{n+1} - x_n), \ \forall n. \tag{9}$$

Using (9) in (8), we get

$$\begin{aligned}\|x_{n+1} - z\|^2 &\leq (1-\alpha_n)\|x_n - z\|^2 + \alpha_n\|w_n - z\|^2 \\ &\quad -\frac{(1-\alpha_n)}{\alpha_n}\|x_{n+1} - x_n\|^2. \end{aligned}\tag{10}$$

Also, by Lemma 1,

$$\begin{aligned}\|w_n - z\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - z\|^2 \\ &= \|(1+\theta_n)(x_n - z) - \theta_n(x_{n-1} - z)\|^2 \\ &= (1+\theta_n)\|x_n - z\|^2 - \theta_n\|x_{n-1} - z\|^2 \\ &\quad +\theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2. \end{aligned}\tag{11}$$

Using (11) in (10):

$$\begin{aligned}\|x_{n+1} - z\|^2 &\leq (1-\alpha_n)\|x_n - z\|^2 + \alpha_n(1+\theta_n)\|x_n - z\|^2 \\ &\quad -\alpha_n\theta_n\|x_{n-1} - z\|^2 + \alpha_n\theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2 \\ &\quad -\frac{1-\alpha_n}{\alpha_n}\|x_{n+1} - x_n\|^2 \\ &= (1+\alpha_n\theta_n)\|x_n - z\|^2 - \alpha_n\theta_n\|x_{n-1} - z\|^2 \\ &\quad +\alpha_n\theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2 \\ &\quad -\frac{1-\alpha_n}{\alpha_n}\|x_{n+1} - x_n\|^2. \end{aligned}\tag{12}$$

Define

$$\Gamma_n := \|x_n - z\|^2 - \alpha_n\theta_n\|x_{n-1} - z\|^2 + \alpha_n\theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2, \ n \geq 1.$$

Since $\alpha_n \leq \alpha_{n+1}$ and $\theta_n \leq \theta_{n+1}$, then $\alpha_n\theta_n \leq \alpha_{n+1}\theta_{n+1}$. So,

$$\begin{aligned}
\Gamma_{n+1} - \Gamma_n &= \|x_{n+1} - z\|^2 - (1 + \alpha_{n+1}\theta_{n+1})\|x_n - z\|^2 \\
&\quad + \alpha_n\theta_n\|x_{n-1} - z\|^2 + \alpha_{n+1}\theta_{n+1}(1 + \theta_{n+1})\|x_{n+1} - x_n\|^2 \\
&\quad - \alpha_n\theta_n(1 + \theta_n)\|x_n - x_{n-1}\|^2 \\
&\leq \|x_{n+1} - z\|^2 - (1 + \alpha_n\theta_n)\|x_n - z\|^2 + \alpha_n\theta_n\|x_{n-1} - z\|^2 \\
&\quad + \alpha_{n+1}\theta_{n+1}(1 + \theta_{n+1})\|x_{n+1} - x_n\|^2 \\
&\quad - \alpha_n\theta_n(1 + \theta_n)\|x_n - x_{n-1}\|^2. \qquad (13)
\end{aligned}$$

Now, using (12) in (13):

$$\begin{aligned}
\Gamma_{n+1} - \Gamma_n &\leq -\frac{1 - \alpha_n}{\alpha_n}\|x_{n+1} - x_n\|^2 + \alpha_{n+1}\theta_{n+1}(1 + \theta_{n+1})\|x_{n+1} - x_n\|^2 \\
&= -\Big(\frac{1 - \alpha_n}{\alpha_n} - \alpha_{n+1}\theta_{n+1}(1 + \theta_{n+1})\Big)\|x_{n+1} - x_n\|^2. \qquad (14)
\end{aligned}$$

By conditions (a) and (b), one gets

$$\begin{aligned}
\frac{1 - \alpha_n}{\alpha_n} - \alpha_{n+1}\theta_{n+1}(1 + \theta_{n+1}) &= \frac{1}{\alpha_n} - 1 - \alpha_{n+1}\theta_{n+1}(1 + \theta_{n+1}) \\
&\geq 2 + \delta - 1 - \frac{2}{2 + \delta} \\
&= \delta + \frac{\delta}{2 + \delta} \geq \delta. \qquad (15)
\end{aligned}$$

Using (15) in (14), we have

$$\Gamma_{n+1} - \Gamma_n \leq -\delta\|x_{n+1} - x_n\|^2. \qquad (16)$$

Therefore, $\{\Gamma_n\}$ is non-increasing. Similarly,

$$\begin{aligned}
\Gamma_n &= \|x_n - z\|^2 - \alpha_n\theta_n\|x_{n-1} - z\|^2 + \alpha_n\theta_n(1 + \theta_n)\|x_n - x_{n-1}\|^2 \\
&\geq \|x_n - z\|^2 - \alpha_n\theta_n\|x_{n-1} - z\|^2. \qquad (17)
\end{aligned}$$

Note that

$$\alpha_n\theta_n \leq \frac{1}{2 + \delta} = \epsilon < 1.$$

From (17), we have

$$\begin{aligned}
\|x_n - z\|^2 &\leq \alpha_n\theta_n\|x_{n-1} - z\|^2 + \Gamma_n \\
&\leq \epsilon\|x_{n-1} - z\|^2 + \Gamma_1 \\
&\vdots \\
&\leq \epsilon^n\|x_0 - z\|^2 + (1 + \cdots + \epsilon^{n-1})\Gamma_1 \\
&\leq \epsilon^n\|x_0 - z\|^2 + \frac{\Gamma_1}{1 - \epsilon}. \qquad (18)
\end{aligned}$$

Consequently,

$$\begin{aligned}\Gamma_{n+1} &= \|x_{n+1}-z\|^2 - \alpha_{n+1}\theta_{n+1}\|x_n - z\|^2 \\ &\quad + \alpha_{n+1}\theta_{n+1}(1+\theta_{n+1})\|x_{n+1}-x_n\|^2 \\ &\geq -\alpha_{n+1}\theta_{n+1}\|x_n - z\|^2\end{aligned}$$

and this means from (18) that

$$\begin{aligned}-\Gamma_{n+1} &\leq \alpha_{n+1}\theta_{n+1}\|x_n - z\|^2 \\ &\leq \epsilon\|x_n - z\|^2 \\ &\vdots \\ &\leq \epsilon^{n+1}\|x_0 - z\|^2 + \frac{\epsilon\Gamma_1}{1-\epsilon}. \qquad (19)\end{aligned}$$

By (16) and (19), we get

$$\begin{aligned}\delta\sum_{n=1}^{k}\|x_{n+1}-x_n\|^2 &\leq \Gamma_1 - \Gamma_{k+1} \\ &\leq \epsilon^{k+1}\|x_0 - z\|^2 + \frac{\Gamma_1}{1-\epsilon}. \qquad (20)\end{aligned}$$

This implies

$$\sum_{n=1}^{\infty}\|x_{n+1}-x_n\|^2 \leq \frac{\Gamma_1}{\delta(1-\epsilon)} < +\infty. \qquad (21)$$

Therefore, $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$. Also, from (4), we get

$$\begin{aligned}\|w_n - x_n\| &= \theta_n\|x_n - x_{n-1}\| \\ &\leq \|x_n - x_{n-1}\| \to 0, n \to \infty. \qquad (22)\end{aligned}$$

From (12):

$$\begin{aligned}\|x_{n+1}-z\|^2 &\leq (1+\alpha_n\theta_n)\|x_n - z\|^2 - \alpha_n\theta_n\|x_{n-1}-z\|^2 \\ &\quad + 2\|x_n - x_{n-1}\|^2.\end{aligned}$$

Using Lemma 2 in the last inequality (noting (21)), we get

$$\lim_{n\to\infty}\|x_n - z\|^2 = l < \infty. \qquad (23)$$

By condition (a), we have that $\lim_{n\to\infty}\theta_n$ exists. Suppose $\lim_{n\to\infty}\theta_n = \theta \in [0,1]$. Then $\lim_{n\to\infty}\theta_n\|x_n - z\|^2 = \theta l$. Similarly, $\lim_{n\to\infty}\theta_n\|x_{n-1}-z\|^2 = \theta l$. Now, from (11)

$$\begin{aligned}\lim_{n\to\infty}\|w_n - z\|^2 &= \lim_{n\to\infty}\Big[(1+\theta_n)\|x_n - z\|^2 - \theta_n\|x_{n-1}-z\|^2 \\ &\quad + \theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2\Big] = l.\end{aligned}$$

Observe that $\lim_{n\to\infty}\Big[(1+\theta_n)\|x_n-z\|^2-\theta_n\|x_{n-1}-z\|^2$ $+\theta_n(1+\theta_n)\|x_n-x_{n-1}\|^2\Big]$ exists. Combining (5), (6) and (7), we have

$$\begin{aligned}\|x_{n+1}-z\|^2 &\leq (1-\alpha_n)\|x_n-z\|^2+\alpha_n\|w_n-z\|^2\\ &\quad -\alpha_n\beta_n(1-\beta_n)\|Tw_n-w_n\|^2\end{aligned}$$

and this implies that

$$\begin{aligned}\alpha\beta(1-\gamma)\|Tw_n-w_n\|^2 &\leq \alpha_n\beta_n(1-\beta_n)\|Tw_n-w_n\|^2\\ &\leq \|x_n-z\|^2-\|x_{n+1}-z\|^2-\alpha_n\|x_n-z\|^2\\ &\quad +\alpha_n\|w_n-z\|^2. \end{aligned} \tag{24}$$

By condition (b), we have that $\lim\limits_{n\to\infty}\alpha_n$ exists and this implies $\lim\limits_{n\to\infty}\alpha_n\|x_n-z\|^2=\lim\limits_{n\to\infty}\alpha_n\|w_n-z\|^2$. From (24), we have $\limsup\limits_{n\to\infty}\|Tw_n-w_n\|^2\leq 0$. Therefore,

$$\lim_{n\to\infty}\|Tw_n-w_n\|=0.$$

Since $\{x_n\}$ is bounded by (23), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges weakly to some element $p\in H$. By Lemma 4, we have that $p\in F(T)$. Invoking Lemma 3, one has that the entire sequence $\{x_n\}$ converges weakly to p. This completes the proof.

4. RATE OF CONVERGENCE ANALYSIS

In this section, we provide a nonasymptotic $O(1/n)$ convergence rate result for our proposed inertial Krasnoselskii-Mann iteration (4) under the same conditions assumed in Section 3.

Theorem 2: Suppose that $T:H\to H$ is a nonexpansive mapping such that its set of fixed points $F(T)$ is nonempty. Let the sequence $\{x_n\}$ in H be generated by choosing $x_0=x_1\in H$ and iteration (4). Assume that the conditions (a), (b) and (c) in Theorem 1 hold. Then, for any $z\in F(T)$, it holds that

$$\min_{1\leq i\leq n}\|w_i-Tw_i\|^2\leq\frac{M}{\alpha\beta(1-\gamma)}\frac{\|x_0-z\|^2}{n},\ n\geq 1,$$

where $M:=1+\frac{2\epsilon}{\delta(1-\epsilon)^2}$ and $\epsilon=\frac{1}{2+\delta},\ \delta>0.$

Proof: From (21), we get

$$\sum_{n=1}^{\infty}\|x_{n+1}-x_n\|^2 \leq \frac{\Gamma_1}{\delta(1-\epsilon)} = \frac{(1-\alpha_1\theta_1)}{\delta(1-\epsilon)}\|x_0-z\|^2. \quad (25)$$

Combining (5) and (12), one can obtain

$$\begin{aligned}\|x_{n+1}-z\|^2 &\leq (1+\alpha_n\theta_n)\|x_n-z\|^2-\alpha_n\theta_n\|x_{n-1}-z\|^2\\ &\quad +\alpha_n\theta_n(1+\theta_n)\|x_n-x_{n-1}\|^2-\frac{1-\alpha_n}{\alpha_n}\|x_{n+1}-x_n\|^2\\ &\quad -\alpha_n\beta_n(1-\beta_n)\|Tw_n-w_n\|^2.\end{aligned}$$

Thus,

$$\begin{aligned}\alpha_n\beta_n(1-\beta_n)\|Tw_n-w_n\|^2 &\leq \|x_n-z\|^2-\|x_{n+1}-z\|^2\\ &\quad +\alpha_n\theta_n(\|x_n-z\|^2-\|x_{n-1}-z\|^2)\\ &\quad +\alpha_n\theta_n(1+\theta_n)\|x_n-x_{n-1}\|^2. \quad (26)\end{aligned}$$

Define

$$\begin{aligned}\delta_n &:= \alpha_n\theta_n(1+\theta_n)\|x_n-x_{n-1}\|^2,\ \forall n\geq 1;\\ \varphi_n &:= \|x_n-x^*\|^2,\ \forall n\geq 1;\\ V_n &:= \varphi_n-\varphi_{n-1},\ \forall n\geq 1\end{aligned}$$

and

$$[V_n]_+ := \max\{V_n,0\},\ \forall n\geq 1.$$

Then, we obtain from (26) that

$$\begin{aligned}\alpha_n\beta_n(1-\beta_n)\|Tw_n-w_n\|^2 &\leq \varphi_n-\varphi_{n+1}+\alpha_n\theta_n(\varphi_n-\varphi_{n-1})+\delta_n\\ &\leq \varphi_n-\varphi_{n+1}+\epsilon[V_n]_++\delta_n. \quad (27)\end{aligned}$$

By (25), we get

$$\begin{aligned}\sum_{n=1}^{\infty}\delta_n &= \sum_{n=1}^{\infty}\alpha_n\theta_n(1+\theta_n)\|x_n-x_{n-1}\|^2\\ &\leq \sum_{n=1}^{\infty}2\epsilon\|x_n-x_{n-1}\|^2\\ &= 2\epsilon\sum_{n=1}^{\infty}\|x_n-x_{n-1}\|^2\\ &\leq \frac{2\epsilon\|x_0-z\|^2}{\delta(1-\epsilon)} := C_1.\end{aligned}$$

Then from (26), we obtain

$$V_{n+1} \leq \alpha_n\theta_nV_n+\delta_n \leq \epsilon[V_n]_++\delta_n.$$

Therefore,

$$[V_{n+1}]_+ \leq \epsilon[V_n]_+ + \delta_n \leq \epsilon^n[V_1]_+ + \sum_{j=1}^{n} \epsilon^{j-1}\delta_{n+1-j}. \tag{28}$$

Since by the assumption that $x_0 = x_1$, we get

$$V_1 = [V_1]_+ = 0, \delta_1 = 0.$$

From (28), one has

$$\sum_{n=2}^{\infty}[V_n]_+ \leq \frac{1}{1-\epsilon}\sum_{n=1}^{\infty}\delta_n = \frac{1}{1-\epsilon}\sum_{n=2}^{\infty}\delta_n. \tag{29}$$

By (26), we get

$$\begin{aligned}\alpha\beta(1-\gamma)\sum_{i=1}^{n}\|w_i - Tw_i\|^2 &\leq \varphi_1 - \varphi_n + \epsilon\sum_{i=1}^{n}[V_i]_+ + \sum_{i=2}^{n}\delta_i \\ &\leq \varphi_1 + \epsilon C_2 + C_1, \end{aligned} \tag{30}$$

where

$$C_2 := \frac{C_1}{1-\epsilon} \geq \frac{1}{1-\epsilon}\sum_{i=2}^{\infty}\delta_i \geq \sum_{i=1}^{\infty}[V_i]_+$$

by (29). Now, since $\varphi_1 = \varphi_0$, we have

$$\begin{aligned}\varphi_1 + \epsilon C_2 + C_1 &= \varphi_1 + \frac{\epsilon C_1}{1-\epsilon} + C_1 \\ &= \varphi_0 + \frac{1}{1-\epsilon}C_1 \\ &= \left[1 + \frac{2\epsilon}{\delta(1-\epsilon)^2}\right]\|x_0 - z\|^2 \\ &= M\|x_0 - z\|^2. \end{aligned} \tag{31}$$

Hence by (30) and (31), we get

$$\sum_{i=1}^{n}\|w_i - Tw_i\|^2 \leq \frac{M}{\alpha\beta(1-\gamma)}\|x_0 - z\|^2.$$

Thus,

$$\min_{i\in\{1,2,\ldots,n\}}\|w_i - Tw_i\|^2 \leq \frac{M}{\alpha\beta(1-\gamma)}\frac{\|x_0 - z\|^2}{n}. \tag{32}$$

In other words,

$$\min_{i\in\{1,2,\ldots,n\}}\|w_i - Tw_i\|^2 = O(1/n).$$

This completes the proof.

We give the following remarks on our results.

Remark:

- It is known that $\|Tx - x\| = 0$ if and only if $Tx = x$ and by Theorem 1, we get that $\|Tw_n - w_n\| \to 0$ holds when $F(T) \neq \emptyset$. Therefore, Theorem 2 gives the convergence rate of our proposed inertial Krasnoselskii-Mann iteration (4) using the quantity $\|Tw_n - w_n\|$ as a measure of its convergence rate.
- We know that for the Krasnoselskii-Mann (1), the quantity $\|Tx_n - x_n\|$ is monotonically nonincreasing with n. However, this property does not hold for inertial Krasnoselskii-Mann (4). This implies that " $\min\limits_{i \in \{1,2,\dots,n\}}$ " cannot be removed in Theorem 2. Nonetheless, with or without the " $\min\limits_{i \in \{1,2,\dots,n\}}$ " a nonasymptotic $O(1/n)$ convergence rate would imply that an ϵ-accuracy solution, in the sense that $\|w_n - Tw_n\| \leq \epsilon$, is obtainable within no more than $O(1/\epsilon)$ iterations.
- If we assume that $\theta_n = 0$ for all n in (4), then the " $\min\limits_{i \in \{1,2,\dots,n\}}$ " in Theorem 2 can be removed by setting $i = n$. In this case, the nonasymptotic $O(1/n)$ convergence rate result given in Theorem 2 reduces to the $O(1/n)$ convergence rate for nonexpansive mappings in [30, Theorem 1] and [27, Theorem 3.1].

5. CONCLUDING REMARKS

We presents weak convergence analysis and nonasymptotic convergence rate results for a new inertial Krasnoselskii-Mann iteration. Our results are obtained under some weaker conditions than other previously obtained results on inertial Krasnoselskii-Mann iteration already in the literature. Part of our future research is devoted to application of our results to Douglas-Rachford splitting method and numerical comparisons of inertial Krasnoselskii-Mann iteration (3) and (4).

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