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NEW FIXED POINT THEOREMS IN DISLOCATED METRIC SPACES

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ABSTRACT. In this paper, the notion of dislocated metric space, which is a proper generalization of the notion of metric space is discussed. A new class of contractive mappings on dislocated metric space, called the class of 3α-contractive mappings is introduced. Using some classical results, new fixed point theorems are established in dislocated metric spaces for the class of 3α-contractive mappings. Concrete examples and an application of our theorems to establishment of existence of solution of certain two-point boundary value problem are given. As further findings of this research, certain anomalies detected in the existing literature are corrected.

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1. INTRODUCTION

The study of fixed points of mappings satisfying certain contractive type conditions has been an active field of research (see for example, [2, 4, 5, 8, 12, 15, 18, 20, 21]). Matthews [13, 14] introduced the concept of dislocated metric spaces under what was tagged metric domains in domain theory in which self distance of a point need not be equal to zero. A generalization of the celebrated

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Banach Contraction Mapping Principle in dislocated metric space was given by Hitzler and Seda [7]. The study of dislocated quasi metric space plays very important role in topology, logic programming and in electronics engineering (see for example, [6, 7]).

This work ia motivated by the work of Karapinar and Erhan [9] and Aydi and Karapinar [3]. In [9], the authors studies the classes of cyclic contractions and Kannan-type cyclic contractions. In trying to justify that a map can be a cyclic contraction but not a Kannan-type cyclic contraction, an example (see Example 8) was given which, however, ended up not justifying their claim. In [3], the authors introduced the so-called classes of type A and type B generalized $\alpha-\psi$ contraction (see Definitions 8 and 8). It is part of our aim in this paper to show that type A and type B generalized $\alpha-\psi$ contraction are the same.

In addition to putting right the anomalies observed in [3] and [9], it is also our aim in this paper to introduce the classes of 3α-contractive mappings. This class of mappings is observed to be independent of the class of mappings studied by Aydi and Karapinar [3]. Our motivation in bringing to view the class of 3α-contractive mappings (as we shall see in the sequel) stems mainly from the fact that the class is not vacuous and that fixed point theory of this class of mappings is applicable to establishment of solutions of certain class of two-point boundary value problem for second order ordinary differential equations. The results obtained in this paper complements the results obtained by several authors in this direction.

2. PRELIMINARY

Let X be a nonempty set. A function $d : X \times X \mapsto [0, \infty)$ is said to be a dislocated metric on X if for all $x, y, z \in X$, the following conditions hold:

$$\begin{aligned}
&\text{(D1)} \qquad d(x, y) = 0 = d(y, x) \implies x = y \\
&\text{(D2)} \qquad d(x, y) = d(y, x) \\
&\text{(D3)} \qquad d(x, y) \leq d(x, z) + d(z, y).
\end{aligned}$$

The pair (X, d) is then called a dislocated metric space.

Example 1. *Let $X = [0, \infty)$ and define the function $d : X \times X \mapsto [0, \infty)$ by*

$$d(x, y) = \max\{x, y\} \qquad \forall\, x, y \in X.$$

Then, d is a dislocated metric and the pair (X, d) is a dislocated metric space.

Example 2. *Let $X = \mathbb{R}$ and define $d : X \times X \mapsto [0, +\infty)$ by*

$$d(x, y) = \tfrac{1}{2}(|x - y| + |x| + |y|), \qquad \forall\ x, y \in X.$$

Then, d is not a metric on X since for $y = x \neq 0$,

$$d(x, x) = \tfrac{1}{2}(|x - x| + |x| + |x|) = |x| \neq 0.$$

However, d is a dislocated metric on X.

Remark 1. *Observe that in Example 2, if the action of d is restricted to $[0, +\infty) \times [0, +\infty)$, then we obtain Example 1. But $\forall$ $x, y \in \mathbb{R}$,*

$$d(x, y) = \tfrac{1}{2}(|x - y| + |x| + |y|) \neq \max\{x, y\}$$

in general. Thus, the function σ in Example 2 is an extension of the function d in Example 1 from $[0, +\infty) \times [0, +\infty)$ to $\mathbb{R} \times \mathbb{R}$.

Analogous to the concept of continuity in metric space, d-continuity in dislocated metric space is defined in the following way:

Definition 1. [1] *Let (X, σ) be a dislocated metric space. A mapping $T : (X, \sigma) \to (X, \sigma)$ is said to be d-continuous if for any sequence $\{x_n\}_{n\geq 1}$ in X such that*

$$\sigma(x_n, x) \to \sigma(x, x) \quad \text{as } n \to \infty,$$

we have that

$$\sigma(Tx_n, Tx) \to \sigma(Tx, Tx) \quad \text{as } n \to \infty.$$

Definition 2. [19] *For a nonempty set X, let $T : X \mapsto X$ and $\alpha : X \times X \mapsto [0, \infty)$ be given mappings. We say that T is α-admissible if for any $x, y \in X$ such that $\alpha(x, y) \geq 1$, then $\alpha(Tx, Ty) \geq 1$.*

Closely related to the concept of metric is the concept of partial metric.

Definition 3. [14] *Let X be a nonempty set. If for any $x,\, y,\, z \in X$, a mapping $\rho : X \times X \mapsto [0,\, \infty)$ satisfies*

(P1) $\rho(x,\, x) = \rho(y,\, y) = \rho(x,\, y)$ *if and only if* $x = y$

(P2) $\rho(x,\, x) \leq \rho(x,\, y)$

(P3) $\rho(x,\, y) = \rho(y,\, x)$

(P4) $\rho(x,\, z) \leq \rho(x,\, y) + \rho(y,\, z) - \rho(y,\, y)$,

then ρ is said to be a partial metric (or briefly, ρ-metric) on X and the pair $(X,\, \rho)$ is called a partial metric space.

Every partial metric space is not a metric space. It is well known that every partial metric space[14] is a dislocated metric space. The converse, however, is not necessarily true. The following example justifies this statement.

Example 3. *Let $X = \{1,\, 2,\, 3\}$ and define the function $\sigma : X \times X \mapsto [0,\, \infty)$ by*

$$\sigma(1,\, 1) = 0, \qquad \sigma(1,\, 2) = \sigma(2,\, 1) = \tfrac{9}{11}, \qquad \sigma(1,\, 3) = \sigma(3,\, 1) = \tfrac{7}{11}$$
$$\sigma(2,\, 2) = \tfrac{10}{11}, \qquad \sigma(2,\, 3) = \sigma(3,\, 2) = \tfrac{8}{11}, \qquad \sigma(3,\, 3) = 1.$$

Then, σ is a dislocated metric. Since, for example, $\sigma(2,\, 2) > \sigma(1,\, 2)$, we observe that σ is not a partial metric.

It is then natural to ask whether given any partial metric, there is an associated metric. Proposition 1 which can be readily proved answers this question in affirmative.

Proposition 1. *If $(X,\, \rho)$ is a partial metric space, then*

$$\sigma_\rho(x,\, y) = 2\rho(x,\, y) - \rho(x,\, x) - \rho(y,\, y), \qquad x,\, y \in X, \tag{1}$$

is a metric on X.

Lemma 1 (Compare with Lemma 2.2 of [16]). *Let $(X,\, \rho)$ be a partial metric space. Then,*

(a) $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in $(X,\, \rho)$ if and only if it is a Cauchy sequence in the metric space $(X,\, \sigma_\rho)$;
(b) $(X,\, \rho)$ is complete if and only if the metric space $(X,\, \sigma_\rho)$ is complete.

Proof:

(a) ($\Longrightarrow$) First, we show that every Cauchy sequence in $(X,\, \rho)$ is a Cauchy sequence in $(X,\, \sigma_\rho)$. To this end, let $\{x_n\}_{n\geq 1}$ be a Cauchy sequence in $(X,\, \rho)$. Then, there exists $a \geq 0$

such that for each $\epsilon > 0$, there is $n_\epsilon \in \mathbb{N}$ such that for all $n,\, m \geq n_\epsilon$, we have

$$|\rho(x_n,\, x_m) - a| < \tfrac{\epsilon}{4}.$$

Without loss of generality, suppose that

$$\max\{\rho(x_n,\, x_n),\, \rho(x_m,\, x_m)\} = \rho(x_m,\, x_m).$$

Then,

$$\begin{aligned}
\sigma_\rho(x_n,\, x_m) &= 2\rho(x_n,\, x_m) - [\rho(x_n,\, x_n) + \rho(x_m,\, x_m)] \\
&\leq 2\rho(x_n,\, x_m) - 2\rho(x_n,\, x_n) \\
&= 2\,[\rho(x_n,\, x_m) - \rho(x_n,\, x_n)] \\
&= 2\,|\rho(x_n,\, x_m) - \rho(x_n,\, x_n)| \\
&\leq 2\,(|\rho(x_n,\, x_m) - a| + |a - \rho(x_n,\, x_n)|) \\
&< 2\left(\tfrac{\epsilon}{4} + \tfrac{\epsilon}{4}\right) = 2\left(\tfrac{\epsilon}{2}\right) = \epsilon
\end{aligned}$$

for all $n,\, m \geq n_\epsilon$. This shows that $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in $(X,\, \sigma_\rho)$.
$(\Longleftarrow)$ Now we prove that every Cauchy sequence $\{x_n\}_{n\geq 1}$ in $(X,\, \sigma_\rho)$ is a Cauchy sequence in $(X,\, \rho)$. Let $\epsilon = 1$. Then, there exists $n_0 \in \mathbb{N}$ such that $\sigma_\rho(x_n,\, x_m) < 1$ for all $n,\, m \geq n_0$. Since

$$\begin{aligned}
\rho(x_n,\, x_n) &\leq \rho(x_n,\, x_{n_0}) \\
&\leq \rho(x_n,\, x_{n_0}) + [\rho(x_n,\, x_{n_0}) - \rho(x_n,\, x_n)] \\
&= 2\rho(x_n,\, x_{n_0}) - \rho(x_n,\, x_n) \\
&= \sigma_\rho(x_n,\, x_{n_0}) + \rho(x_{n_0},\, x_{n_0}),
\end{aligned}$$

then

$$\begin{aligned}
|\rho(x_n,\, x_n)| &\leq |\sigma_\rho(x_n,\, x_{n_0}) + \rho(x_{n_0},\, x_{n_0})| \\
&\leq |\sigma_\rho(x_n,\, x_{n_0})| + |\rho(x_{n_0},\, x_{n_0})| \\
&< 1 + |\rho(x_{n_0},\, x_{n_0})|\,.
\end{aligned}$$

Consequently the sequence $\{\rho(x_n,\, x_n)\}_{n\geq 1}$ is bounded in $\mathbb{R}$, and so there exists $a \in \mathbb{R}$ such that a subsequence $\{\rho(x_{n_k},\, x_{n_k})\}_{k\geq 1}$ is convergent to a, i.e. $\lim\limits_{k\to\infty} \rho(x_{n_k},\, x_{n_k}) = a$. It remains to prove that $\{\rho(x_n,\, x_n)\}_{n\geq 1}$ is a Cauchy sequence in $\mathbb{R}$. Since $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in $(X,\, \sigma_\rho)$, given $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that $\sigma_\rho(x_n,\, x_m) < \frac{\epsilon}{2}$ for all $n,\, m \geq n_\epsilon$.

Thus, for all $n, m \geq n_\epsilon$,

$$\begin{aligned}
|\rho(x_n, x_n) - \rho(x_m, x_m)| &\leq |\rho(x_n, x_n) - \rho(x_n, x_m)| \\
&\quad + |\rho(x_n, x_m) - \rho(x_m, x_m)| \\
&= |\rho(x_m, x_n) - \rho(x_n, x_n)| \\
&\quad + |\rho(x_m, x_n) - \rho(x_m, x_m)| \\
&= \rho(x_m, x_n) - \rho(x_n, x_n) \\
&\quad + \rho(x_m, x_n) - \rho(x_m, x_m) \\
&= 2\rho(x_m, x_n) - [\rho(x_m, x_m) \\
&\quad + \rho(x_n, x_n)] \\
&= \sigma_\rho(x_m, x_n) < \frac{\epsilon}{2} < \epsilon.
\end{aligned}$$

Therefore, $\lim_{n\to\infty} \rho(x_n, x_n) = a$. On the other hand,

$$\begin{aligned}
|\rho(x_n, x_m) - a| &= |\rho(x_n, x_m) - \rho(x_n, x_n) + \rho(x_n, x_n) - a| \\
&\leq |\rho(x_n, x_m) - \rho(x_n, x_n)| + |\rho(x_n, x_n) - a| \\
&\leq \sigma_\rho(x_n, x_m) + |\rho(x_n, x_n) - a| < \epsilon
\end{aligned}$$

for all $n, m \geq n_\epsilon$. Hence $\lim_{n, m\to\infty} \rho(x_n, x_m) = a$ and $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in (X, ρ).

(b) ($\Longrightarrow$) First, we show that the completeness of the partial metric space (X, ρ) implies the completeness of the associated metric space (X, σ_ρ). Let $\{x_n\}_{n\geq}$ be a Cauchy sequence in (X, σ_ρ). Then, $\{x_n\}_{n\geq 1}$ is a Cauchy seuqence in (X, ρ), and so it is convergent to a point $y \in X$ with

$$\lim_{n, m} \rho(x_n, x_m) = \lim_{n\to\infty} \rho(y, x_n) = \rho(y, y).$$

Then, given $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that

$$\rho(x_n, y) - \rho(y, y) < \tfrac{\epsilon}{2} \text{ and } \rho(y, y) - \rho(x_n, x_n) < \tfrac{\epsilon}{2}$$

whenever $n \geq n_\epsilon$. As a consequence, we have that

$$\begin{aligned}
\sigma_\rho(x_n, y) &= 2\rho(x_n, y) - [\rho(x_n, x_n) + \rho(y, y)] \\
&= [\rho(x_n, y) - \rho(x_n, x_n)] + [\rho(x_n, y) - \rho(y, y)] \\
&\leq [\rho(y, y) - \rho(x_n, x_n)] + [\rho(x_n, y) - \rho(y, y)] \\
&\leq |\rho(y, y) - \rho(x_n, x_n)| + |\rho(x_n, y) - \rho(y, y)| \\
&< \epsilon
\end{aligned}$$

whenever $n \geq n_\epsilon$. Therefore, (X, σ_ρ) is complete.

($\Longleftarrow$) Next, we prove that completeness of (X, σ_ρ) implies completeness of (X, ρ). Indeed, if $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in (X, ρ) then it is also a Cauchy sequence in (X, σ_ρ). Since the metric space (X, σ_ρ) is complete we deduce that there exists $y \in X$ such that $\lim_{n\to\infty} \sigma_\rho(y, x_n) = 0$. It then follows that $\{x_n\}_{n\geq 1}$ is a convergent sequence in (X, ρ). Next, we prove that $\lim_{n,m\to\infty} \rho(x_n, x_m) = \rho(y, y)$. Since $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in (X, ρ) it is sufficient to see that $\lim_{n\to\infty} \rho(x_n, x_n) = \rho(y, y)$. Let $\epsilon > 0$ be given. Then, there exists $n_0 \in N$ such that $\sigma_\rho(y, x_n) < \epsilon$ whenever $n \geq n_0$. Thus

$$\begin{aligned}|\rho(y, y) - \rho(x_n, x_n)| &\leq |\rho(y, y) - \rho(y, x_n)| + |\rho(y, x_n) - \rho(x_n, x_n)|\\ &= |\rho(x_n, y) - \rho(y, y)| + |\rho(x_n, y) - \rho(x_n, x_n)|\\ &= \rho(x_n, y) - \rho(y, y) + \rho(x_n, y) - \rho(x_n, x_n)\\ &= 2\rho(x_n, y) - [\rho(x_n, x_n) + \rho(y, y)]\\ &= \sigma_\rho(x_n, y) < \epsilon\end{aligned}$$

whenever $n \geq n_0$. This shows that (X, ρ) is complete.

Lemma 2. *Let (X, σ) be a dislocated metric space. Let $\{x_n\}_{n\geq 1}$ be a sequence in X such that $x_n \to x$. Then, for all $y \in X$, we have*

$$\lim_{n\to\infty} \sigma(x_n, y) = \sigma(x, y).$$

Proof:
Observe that $\forall\ x, y \in X$,

$$\sigma(x_n, y) \leq \sigma(x_n, x) + \sigma(x, y).$$

Thus,

$$\sigma(x_n, y) - \sigma(x, y) \leq \sigma(x_n, x). \tag{2}$$

Again,

$$\sigma(x, y) \leq \sigma(x, x_n) + \sigma(x_n, y).$$

Thus,

$$\begin{aligned}&\sigma(x, y) - \sigma(x_n, y) \leq \sigma(x, x_n)\\ \text{i.e.,}\qquad &-[\sigma(x_n, y) - \sigma(x, y)] \leq \sigma(x_n, x).\end{aligned} \tag{3}$$

Combining (2) and (3), we obtain that

$$|\sigma(x_n, y) - \sigma(x, y)| \leq \sigma(x_n, x).$$

Taking limit as $n \to \infty$ and using Sandwich Theorem, we obtain that

$$\lim_{n\to\infty} \sigma(x_n, y) = \sigma(x, y).$$

Lemma 3. [7] *d-limits in dislocated metric spaces are unique, i.e., if $\{x_n\}_{n\geq 1}$ is a sequence in the dislocated metric space $(X,\, \sigma)$ such that $x_n \to x$ and $x_n \to y$, then $x = y$.*

Definition 4 (Karapinar and Salimi [10])**.** *Let A and B be nonempty subsets of a metric space $(X,\, \sigma)$ and $T : A \cup B \mapsto A \cup B$ be a mapping. The mapping T is called a cyclic map if and only if*

$$T(A) \subseteq B \quad \text{and} \quad T(B) \subseteq A.$$

Definition 5 (Kirk, Srinivasan and Veeramani [11])**.** *Let A and B be nonempty subsets of a metric space $(X,\, \sigma)$. A cyclic map $T : A \cup B \mapsto A \cup B$ is said to be a cyclic contraction if there exists $k \in (0,\, 1)$ such that*

$$\sigma(Tx,\, Ty) \leq k\, \sigma(x,\, y) \quad \text{for all } x \in A \text{ and } y \in B.$$

Definition 6 (Karapinar and Erhan [9])**.** *Let A and B be nonempty subsets of a metric space $(X,\, \sigma)$. A cyclic map $T : A \cup B \mapsto A \cup B$ is called a Kannan-type cyclic contraction if there exists $k \in (0,\, \frac{1}{2})$ such that*

$$\sigma(Tx,\, Ty) \leq k\, [\sigma(Tx,\, x) + \sigma(Ty,\, y)] \quad \text{for all } x \in A \text{ and } y \in B.$$

3. THE HEART OF THE MATTER

We begin this section by introducing the new class of 3α-contractive mappings in dislocated metric space studied in this work.

Definition 7. *Let $(X,\, \sigma)$ be a dislocated metric space and $T : X \mapsto X$ be a given mapping. We say that T is a 3α- contractive mapping if there exists a function $\alpha : X \times X \mapsto [0,\, \infty)$ and $k \in [0,\, 1)$ such that*

$$\alpha(x,\, y)\, \sigma(Tx,\, Ty) \leq \tfrac{k}{3} M_1(x,\, y); \qquad \text{for all } x,\, y \in X, \tag{4}$$

where

$$M_1(x,\, y) = \sigma(x,\, y) + 2\sigma(y,\, Ty).$$

The following examples show that Definition 7 is not vacuous.

Example 4. *Let $X = [0,\, \infty)$ be endowed with the dislocated metric given as*

$$\sigma(x,\, y) = \max\{x,\, y\}.$$

Then, $(X,\, \sigma)$ is a complete dislocated metric space. Let the mapping $T_1 : X \mapsto X$ be defined by

$$T_1 x = \begin{cases} \frac{1}{4}x^2 & \text{if } x \in [0,\, 1]; \\ 0 & \text{otherwise.} \end{cases}$$

Consider the map $\alpha : X \times X \mapsto [0, \infty)$ *defined by*

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [0, 1]; \\ 0 & \text{otherwise.} \end{cases}$$

Then, T_1 is a 3α-contractive mapping and α-admissible. Also, T_1 is d-continuous (refer to Example 6).

Example 5. *Consider* $X = \{0, 1, 2\}$*. Take the dislocated metric* $\sigma : X \times X \mapsto [0, +\infty)$ *defined by*

$$\sigma(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0); \\ 1 & \text{if } (x, y) = (0, 1); \\ 1 & \text{if } (x, y) = (0, 2); \\ 1 & \text{if } (x, y) = (1, 0); \\ 0 & \text{if } (x, y) = (1, 1); \\ 2 & \text{if } (x, y) = (1, 2); \\ 1 & \text{if } (x, y) = (2, 0); \\ 2 & \text{if } (x, y) = (2, 1); \\ 2 & \text{if } (x, y) = (2, 2). \end{cases}$$

Note that σ *is not a metric. More so,* σ *is not a partial metric. However,* (X, σ) *is a complete dislocated metric space. Let* $T_2 : X \mapsto X$ *be defined by*

$$T_2x = \begin{cases} 0 & \text{if } x \in \{0, 1\}; \\ 1 & \text{if } x = 2. \end{cases}$$

Define the mapping $\alpha : X \times X \mapsto [0, \infty)$ *by*

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x = 0; \\ 0 & \text{otherwise.} \end{cases}$$

Then, T_2 is a 3α-contractive mapping. Also, T_2 is α-admissible.

Remark 2. *We remark that d-continuity is a d-metric property. That is to say, a mapping that is d-continuous with respect to a given d-metric may fail to be d-continuous with respect to another d-metric. We illustrate this fact in the next example.*

Example 6. *Let* $X = [0, \infty)$*. Endow* X *respectively with the two dislocated metrics* σ_1 *and* σ_2 *defined by* $\sigma_1(x, y) = |x - y|$ *and* $\sigma_2(x, y) = \max\{x, y\}$ *for all* $x, y \in X$*. Define the mapping* $T : X \mapsto X$ *by*

$$Tx = \begin{cases} \frac{1}{4}x^2 & \text{if } x \in [0, 1]; \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, T *is not d-continuous on the dislocated metric space* (X, σ_1)*. However,* T *is d-continuous on* (X, σ_2)*.*

We now present the following fixed point theorems and their accompanying proofs.

Theorem 2. *Let $(X,\, \sigma)$ be a complete dislocated metric space and $T : X \mapsto X$ be a 3α contractive mapping. Suppose that*

(i) T is α-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0,\, Tx_0) \geq 1$;
(iii) T is d-continuous,

then there exists $u \in X$ such that $\sigma(u,\, u) = 0$.

(iv) If in addition, there exists $x \in X$ such that whenever $\sigma(x,\, x) = 0$, we have that $\alpha(x,\, x) \geq 1$,

then u is a fixed point of T, that is, $Tu = u$.

Proof:
By assumption (ii), there exists a point $x_0 \in X$ such that $\alpha(x_0,\, Tx_0) \geq 1$. We define a sequence $\{x_n\} \in X$ by $x_{n+1} = Tx_n = T^{n+1}x_0$ for all $n \geq 0$. Suppose that $x_{n_0} = x_{n_0+1}$ for some n_0. Then, the proof is completed since

$$u = x_{n_0} = x_{n_0+1} = Tx_{n_0} = Tu.$$

Consequently, throughout the proof, we assume that

$$x_n \neq x_{n+1} \qquad \text{for all } n.$$

Observe that

$$\alpha(x_0,\, x_1) = \alpha(x_0,\, Tx_0) \geq 1 \implies \alpha(Tx_0,\, Tx_1) = \alpha(x_1,\, x_2) \geq 1,$$

since T is α-admissible. By repeating the process above, we derive that

$$\alpha(x_n,\, x_{n+1}) \geq 1, \qquad \text{for all } n = 0,\, 1,\, 2,\, \ldots \tag{5}$$

Step 1: We shall prove that

$$\lim_{n\to\infty} \sigma(x_n,\, x_{n+1}) = 0.$$

Combining (4) and (5), we find that

$$\begin{aligned} \sigma(x_n,\, x_{n+1}) &= \sigma(Tx_{n-1},\, Tx_n) \\ &\leq \alpha(x_{n-1},\, x_n)\, \sigma(Tx_{n-1},\, Tx_n) \\ &\leq \tfrac{k}{3} M_1(x_{n-1},\, x_n) & (6) \\ &= \tfrac{k}{3}\left[\sigma(x_{n-1},\, x_n) + 2\sigma(x_n,\, x_{n+1})\right] & (7) \end{aligned}$$

for all $n \in \mathbb{N}$. That is, for all $n \in \mathbb{N}$, we have that

$$\begin{aligned} 3\sigma(x_n,\, x_{n+1}) &\leq k\sigma(x_{n-1},\, x_n) + 2k\sigma(x_n,\, x_{n+1}) \\ &\leq k\sigma(x_{n-1},\, x_n) + 2\sigma(x_n,\, x_{n+1}). \end{aligned}$$

Thus,

$$\sigma(x_n,\, x_{n+1}) \leq k\sigma(x_{n-1},\, x_n) \qquad \forall n \in \mathbb{N}. \tag{8}$$

From (8), we find that

$$\sigma(x_n,\, x_{n+1}) \leq k^n \sigma(x_0,\, x_1) \qquad \forall\; n \in \mathbb{N}. \tag{9}$$

Since $k \in [0,\, 1)$, we conclude that

$$\lim_{n\to\infty} \sigma(x_n,\, x_{n+1}) = 0. \tag{10}$$

Step 2: We shall prove that $\{x_n\}$ is a Cauchy sequence. First, by using $(D4)$ and (9), we have that

$$\begin{aligned}
\sigma(x_n,\, x_{n+j}) &\leq \sigma(x_n,\, x_{n+1}) + \sigma(x_{n+1},\, x_{n+2}) + \cdots + \sigma(x_{n+j-1},\, x_{n+j}) \\
&= \sum_{p=n}^{n+j-1} \sigma(x_p,\, x_{p+1}) \\
&\leq \sum_{p=n}^{n+j-1} k^p(\sigma(x_0,\, x_1)) \\
&\leq \sum_{p=n}^{+\infty} k^p(\sigma(x_0,\, x_1)) \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Thus, by the symmetry of σ, we obtain that

$$\lim_{n,\, m\to\infty} \sigma(x_n,\, x_m) = 0. \tag{11}$$

Hence, we conclude that $\{x_n\}$ is a Cauchy sequence in $(X,\, \sigma)$. Since $(X,\, \sigma)$ is complete, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$. By Lemma 2, we obtain that

$$\lim_{n\to\infty} \sigma(x_n,\, x_m) = \sigma(u,\, x_m)$$
$$\text{and} \qquad \lim_{m\to\infty} \sigma(u,\, x_m) = \sigma(u,\, u).$$

Therefore,

$$\begin{aligned}
0 &= \lim_{n,\, m\to\infty} \sigma(x_m,\, x_n) \\
&= \lim_{m\to\infty} \left[\lim_{n\to\infty} \sigma(x_n,\, x_m)\right] \\
&= \lim_{m\to\infty} \sigma(u,\, x_m) \\
&= \sigma(u,\, u).
\end{aligned} \tag{12}$$

Thus, d-continuity of T and Lemma 2 give

$$\lim_{n\to\infty} \sigma(x_{n+1},\, Tu) = \lim_{n\to\infty} \sigma(Tx_n,\, Tu) = \sigma(Tu,\, Tu). \tag{13}$$

On the other hand, since $\lim_{n\to\infty} x_n = u$, it follows that $\lim_{n\to\infty} x_{n+1} = u$. Thus, applying Lemma 2, we obtain that

$$\lim_{n\to\infty} \sigma(x_{n+1}, Tu) = \sigma(u, Tu). \tag{14}$$

Comparing (13) and (14) and using Lemma 3, we get

$$\sigma(u, Tu) = \sigma(Tu, Tu).$$

From hypothesis *(iv)* and the fact that $\sigma(u, u) = 0$, we have $\alpha(u, u) \geq 1$. Therefore, by Definition 7,

$$\begin{aligned} \sigma(u, Tu) &= \sigma(Tu, Tu) \\ &\leq \alpha(u, u)\, \sigma(Tu, Tu) \\ &\leq \tfrac{k}{3} M_1(u, u) \\ &= \tfrac{k}{3}\, [\sigma(u, u) + 2\sigma(u, Tu)] \\ &= \tfrac{2k}{3} \sigma(u, Tu). \end{aligned}$$

Hence,

$$(1 - \tfrac{2k}{3})\sigma(u, Tu) \leq 0. \tag{15}$$

But $k \in [0, 1) \implies 1 - \frac{2k}{3} \in (\frac{1}{3}, 1]$. So, (15) gives

$$\sigma(u, Tu) \leq 0.$$

Hence, $\sigma(u, Tu) = 0$, that is $Tu = u$. So, we conclude that u is a fixed point of T.

Remark 3. *Theorem 2 remains true if we replace the d-continuity hypothesis by the following property:*

"If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\alpha(x_{n_j}, x) \geq 1$ for all j."

Thus, we have the following theorem:

Theorem 3. *Let (X, σ) be a complete dislocated metric space and $T : X \mapsto X$ be a 3α-contractive mapping. Suppose that*

(i) T is α-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\alpha(x_{n_j}, x) \geq 1$ for all j.

Then, there exists $u \in X$ such that $Tu = u$.

Proof:
Following the proof of Theorem 2, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \geq 0$ is Cauchy in (X, σ) and converges to some $u \in X$. So,

$$\lim_{j\to\infty} \sigma(x_{n_j+1}, Tu) = \sigma(u, Tu). \tag{16}$$

We shall show that $Tu = u$. Now, suppose, on the contrary, that $Tu \neq u$ (which implies that $\sigma(Tu, u) \neq 0$). From (5) and Condition (iii), there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\alpha(x_{n_j}, u) \geq 1$ for all j. By Definition 7, we obtain that

$$\begin{aligned}\sigma(x_{n_j+1}, Tu) &\leq \alpha(x_{n_j}, u)\, \sigma(x_{n_j+1}, Tu)\\ &= \alpha(x_{n_j}, u)\, \sigma(Tx_{n_j}, Tu)\\ &\leq \tfrac{k}{3} M_1(x_{n_j}, u) \end{aligned} \tag{17}$$

where

$$M_1(x_{n_j}, u) = \sigma(x_{n_j}, u) + 2\sigma(u, Tu).$$

But $\lim_{j\to\infty} x_{n_j} = u \implies \lim_{j\to\infty} \sigma(x_{n_j}, u) = 0$ and using (12), we have that

$$\lim_{j\to\infty} M_1(x_{n_j}, u) = 2\sigma(u, Tu).$$

Letting $j \to \infty$ in (17) gives $\sigma(u, Tu) \leq \frac{2k}{3}\sigma(u, Tu)$, which implies that $(1 - \frac{2k}{3})\sigma(u, Tu) \leq 0$. Thus, $\sigma(u, Tu) = 0$, a contradiction. Hence, $Tu = u$, that is, u is a fixed point of T.

Corollary 4. *Let (X, σ) be a complete dislocated metric space and $T : X \mapsto X$ be such that*

$$\sigma(Tx, Ty) \leq \tfrac{k}{3} M_1(x, y)$$

where $k \in [0, 1)$. Then, T has a fixed point.

Proof:
To prove the above corollary, it suffices to take $\alpha(x, y) = 1$ in Theorem 3.

4. APPLICATION

Here, we consider the following two-point boundary value problem for the second-order differential equation.:

$$\frac{d^2v}{dt^2} = f(t, v(t)), \quad t \in [0, 1], \qquad v(0) = v(1) = 0, \tag{18}$$

where $f : [0, 1] \times \mathbb{R} \mapsto \mathbb{R}$ is a continuous function. Let $X = \mathcal{C}(I)(I = [0, 1])$ be the space of all continuous functions defined on I. We endow on X, the dislocated metric ρ given by

$$\rho(v_1, v_2) = ||v_1 - v_2||_\infty + ||v_1||_\infty + ||v_2||_\infty \qquad \text{for all } v_1, v_2 \in X,$$

where

$$||u||_\infty = \max_{t\in[0, 1]} |u(t)| \qquad \text{for each } u \in X.$$

Note that ρ is also a partial metric on X since

$$\sigma_\rho(v_1, v_2) := 2\rho(v_1, v_2) - \rho(v_1, v_1) - \rho(v_2, v_2) = 2||v_1 - v_2||_\infty.$$

So by Lemma 1, (X, σ_ρ) is complete since the metric space $(X, ||\cdot||_\infty)$ is complete. It is well known (see Pascali and Shurlan[17]) that $v \in \mathcal{C}^2(I)$ being a solution of (18) is equivalent to that $v \in X = \mathcal{C}(I)$ is a solution of the Hammerstein equation

$$v(t) + \int_0^1 \kappa(t, x)\, f(x, x\, v(x))\, dx = 0, \qquad \text{for all } t \in I,$$

i.e,

$$v(t) = -\int_0^1 \kappa(t, x)\, f(x, x\, v(x))\, dx, \qquad \text{for all } t \in I. \tag{19}$$

Theorem 5. *Suppose that the following conditions are satisfied:*

(i) there exists a continuous function $\eta : I \mapsto [0, \infty)$ such that

$$|f(x, a) - f(x, b)| \le 8\eta(x)|a - b|,$$

for each $x \in I$ and $a, b \in \mathbb{R}$;

(ii) there exists a continuous function $\zeta : I \mapsto [0, \infty)$ such that

$$|f(x, a)| \le 8\zeta(x)|a|,$$

for each $x \in I$ and $a \in \mathbb{R}$;

(iii) $\sup_{x\in I} \eta(x) = k_1 < \frac{1}{9}$;

(iv) $\sup_{x\in I} \zeta(x) = k_2 < \frac{1}{9}$.

(v) the Green's function associated to (18) is

$$\kappa(t, x) = \begin{cases} t(1 - x); & \text{if } 0 \le t \le x \le 1 \\ x(1 - t); & \text{if } 0 \le x \le t \le 1. \end{cases} \tag{20}$$

Then, Problem (18) has a solution $v \in X = \mathcal{C}(I, \mathbb{R})$.

Proof:
Consider the mapping $T : X \mapsto X$ defined by

$$Tv(t) = -\int_0^1 \kappa(t,\, x)\, f(x,\, v(x))\, dx,$$

for all $t \in I$ and $v \in X$. Then, Problem 18 is equivalent to finding $u \in X$ that is a fixed point of T.

Now, let $v_1,\, v_2 \in X$. We have

$$\begin{aligned}
|Tv_1(t) - Tv_2(t)| &= \left| \int_0^1 \kappa(t,\, x)\, f(x,\, v_1(x))\, dx - \int_0^1 \kappa(t,\, x)\, f(x,\, v_2(x))\, dx \right| \\
&= \left| \int_0^1 \kappa(t,\, x)\, \left[f(x,\, v_1(x)) - f(x,\, v_2(x)) \right] dx \right| \\
&\leq \int_0^1 \kappa(t,\, x)\, \left| f(x,\, v_1(x)) - f(x,\, v_2(x)) \right| dx \\
&\leq 8 \int_0^1 \kappa(t,\, x)\, \eta(x)\, |v_1(x) - v_2(x)|\, dx \\
&\leq 8k_1 ||v_1 - v_2||_\infty \sup_{t \in I} \int_0^1 \kappa(t,\, x)\, dx \\
&= k_1 ||v_1 - v_2||_\infty .
\end{aligned}$$

In the above equality, we used the fact that for each $t \in I$, we have

$$\int_0^1 \kappa(t,\, x)\, dx = -\tfrac{1}{2}(t^2 - t),$$

and so

$$\sup_{t \in I} \int_0^1 \kappa(t,\, x)\, dx = \sup_{t \in I} \left\{ -\tfrac{1}{2}(t^2 - t) \right\} = \tfrac{1}{8}.$$

Therefore,

$$||Tv_1 - Tv_2||_\infty \leq k_1 ||v_1 - v_2||_\infty . \tag{21}$$

Again, we have

$$\begin{aligned}
|Tv_1(t)| &= \left| \int_0^1 \kappa(t,\, x)\, f(x,\, v_1(x))\, dx \right| \\
&\leq \int_0^1 \kappa(t,\, x)\, |f(x,\, v_1(x))|\, dx \\
&\leq 8 \int_0^1 \kappa(t,\, x)\, \zeta(x)\, |v_1(x)|\, dx \\
&\leq 8k_2 ||v_1||_\infty \sup_{t \in I} \int_0^1 \kappa(t,\, x)\, dx \\
&= k_2 ||v_1||_\infty .
\end{aligned}$$

Thus,

$$||Tv_1||_\infty \le k_2||v_1||_\infty. \tag{22}$$

By symmetry,

$$||Tv_2||_\infty \le k_2||v_2||_\infty. \tag{23}$$

Take $k = 3(k_1 + 2k_2)$. Under assumptions in Theorem 5, we have $k \in [0, 1)$. Summing (21), (22) and (23), we get

$$\begin{aligned}
\sigma(Tv_1, Tv_2) &= ||Tv_1 - Tv_2||_\sigma + ||Tv_1||_\sigma + ||Tv_2||_\sigma \\
&\le k_1||v_1 - v_2||_\infty + k_2||v_1||_\infty + k_2||v_2||_\infty \\
&\le (k_1 + 2k_2)\left(||v_1 - v_2||_\infty + ||v_1||_\infty + ||v_2||_\infty\right) \\
&= \tfrac{k}{3}\sigma(v_1, v_2) \\
&\le \tfrac{k}{3}M(v_1, v_2).
\end{aligned}$$

Now, we see clearly that the hypotheses of Corollary 4 are satisfied, and so T has a fixed point $u \in X$, that is, the Problem 18 has a solution $v \in \mathcal{C}^2(I)$.

Example 7. *Consider the problem of the forced oscillations of finite amplitude of a pendulum. The amplitude of oscillation $v(t)$ is a solution of the problem*

$$\begin{cases} \frac{d^2v}{dt^2} + a^2 \sin v(t) = z(t), & \text{if } t \in [0, 1] \\ v(0) = v(1) = 0, \end{cases} \tag{24}$$

where the driving function $z(t)$ is odd and periodic (of period, say, P). The constant $a \neq 0$ depends on the length of the pendulum and on gravity. Observe that (24) can be put in the form of (18) where $f(t, v(t)) = z(t) - a^2 \sin v(t)$. If in addition, the conditions of Theorem 5 are satisfied, then (24) has a solution $v \in X = \mathcal{C}^2(I)$.

5. FURTHER FINDINGS

Further Finding 1: Karapinar and Erhan [9] claimed that a map can be a cyclic contraction but not a Kannan-type cyclic contraction using the following example.

Example 8 (Karapinar and Erhan [9])**.** *Take $X = \mathbb{R}$, with the usual metric. Suppose that $A = [-1, 0]$ and $B = [0, 1]$. Let $T : A \cup B \mapsto A \cup B$ be defined by*

$$Tz = -\frac{z}{3} \quad \textit{for all } z \in A \cup B.$$

They claimed that T is cyclic contraction but not Kannan-type cyclic contraction. This example, however, is misleading. To see this, we consider the following proposition:

Proposition 6. *Let $X = \mathbb{R}$ with σ the usual metric on $\mathbb{R}$. Suppose that $A = [-1, 0]$ and $B = [0, 1]$. Let $T_\lambda : A \cup B \mapsto A \cup B$ be defined by*

$$T_\lambda z = -\lambda z \quad \text{for all } z \in A \cup B,$$

where $\lambda \in (0, 1)$. Then, T_λ is a cyclic contraction and also a Kannan-type cyclic contraction.

Proof: We begin by showing that T_λ is a cyclic contraction, that is, T_λ is a cyclic map and that

$$\sigma(T_\lambda x, T_\lambda y) \leq k_\lambda \sigma(x, y) \text{ where } x \in A,\, y \in B \text{ and } k_\lambda \in (0, 1). \tag{25}$$

We first show that T_λ is a cyclic map. Now, T_λ is a cyclic map because

$$T_\lambda[A] = [0, \lambda] \subseteq [0, 1] = B \quad \text{and} \quad T_\lambda[B] = [-\lambda, 0] \subseteq [-1, 0] = A.$$

We then show that T_λ is a cyclic contraction.

$$\sigma(x, y) = |x - y| = y - x$$

since $x \in A = [-1, 0]$ and $y \in B = [0, 1]$. Moreover,

$$\sigma(T_\lambda x, T_\lambda y) = |(-\lambda x) - (-\lambda y)| = |-\lambda|\,|x - y| = \lambda|x - y| = \lambda(y - x).$$

For fixed $\lambda \in (0, 1)$, choose $k_\lambda = \frac{1}{2}(1 + \lambda)$. Then, obviously, $k_\lambda \in (0, 1)$ and $\lambda \leq k_\lambda$. We then see that (25) is satisfied. Hence, T_λ is a cyclic contraction.

Next, we show that T_λ is a Kannan-type cyclic contraction, that is,

$$\sigma(T_\lambda x, T_\lambda y) \leq k'_\lambda[\sigma(T_\lambda x, x) + \sigma(T_\lambda y, y)] \tag{26}$$

where $x \in A$, $y \in B$ and $k'_\lambda \in (0, \frac{1}{2})$. But

$$\begin{aligned}
\sigma(T_\lambda x, x) &= |T_\lambda x - x| \\
&= |-\lambda x - x| \\
&= (\lambda + 1)|-x| \\
&= -(\lambda + 1)x \qquad\qquad \text{since } x \in A = [-1, 0]
\end{aligned}$$

and

$$\begin{aligned}
\sigma(T_\lambda y, y) &= |T_\lambda y - y| \\
&= |-\lambda y - y| \\
&= (\lambda + 1)|-y| \\
&= (\lambda + 1)y \qquad\qquad \text{since } y \in B = [0, 1]
\end{aligned}$$

so that

$$\sigma(T_\lambda x, x) + \sigma(T_\lambda y, y) = -(\lambda + 1)x + (\lambda + 1)y = (\lambda + 1)(y - x).$$

Now, $\lambda \in (0,\, 1) \iff \frac{\lambda}{\lambda+1} \in (0,\, \frac{1}{2})$. Choose $k'_\lambda = \frac{1+3\lambda}{4(1+\lambda)} = \frac{1}{2}(\frac{1}{2} + \frac{\lambda}{\lambda+1})$. Then, obviously, $k'_\lambda \in (\frac{1}{4},\, \frac{1}{2}) \subset (0,\, \frac{1}{2})$ and $\frac{\lambda}{\lambda+1} \le k'_\lambda$. We then see that (26) is satisfied. Hence, T_λ is a Kannan-type cyclic contraction.

Further Finding 2: Recently, Aydi and Karapinar[3] introduced the classes of type A and type B generalized $\alpha - \psi$ contractions in the following definitions 8 and 9.

Definition 8 (Aydi and Karapinar [3])**.** *Let $(X,\, \sigma)$ be a dislocated metric space and $T : X \mapsto X$ be a given mapping. T is called a generalized $\alpha - \psi$ contractive mapping of type A if there exist two functions $\alpha : X \times X \mapsto [0,\, \infty)$ and $\psi \in \Psi$ such that*

$$\alpha(x,\, y)\, \sigma(Tx,\, Ty) \le \psi(M(x,\, y)); \qquad \textit{for all } x,\, y \in X$$

where

$$M(x,\, y) = \max\left\{\sigma(x,\, y),\, \sigma(x,\, Tx),\, \sigma(y,\, Ty),\, \tfrac{\sigma(x,\, Ty)+\sigma(y,\, Tx)}{4}\right\}. \tag{27}$$

Definition 9 (Aydi and Karapinar [3])**.** *Let $(X,\, \sigma)$ be a dislocated metric space and $T : X \mapsto X$ be a given mapping. T is called a generalized $\alpha - \psi$ contractive mapping of type B if there exist two functions $\alpha : X \times X \mapsto [0,\, \infty)$ and $\psi \in \Psi$ such that*

$$\alpha(x,\, y)\, \sigma(Tx,\, Ty) \le \psi(M_0(x,\, y)); \qquad \textit{for all } x,\, y \in X$$

where

$$M_0(x,\, y) = \max\left\{\sigma(x,\, y),\, \sigma(x,\, Tx),\, \sigma(y,\, Ty)\right\}. \tag{28}$$

However, to show that these definitions are misleading, we consider the following proposition:

Proposition 7. *Let σ be a dislocated metric on a nonempty set X. Let $M(x,\, y)$ and $M_0(x,\, y)$ be defined as in (27) and (28) respectively. Then,*

$$M(x,\, y) = M_0(x,\, y).$$

Proof: If we can show that $\frac{\sigma(x,\, Ty)+\sigma(y,\, Tx)}{4} \le M_0$, then we are done. Now, observe that

$$\begin{aligned} \tfrac{\sigma(x,\, Ty)+\sigma(y,\, Tx)}{4} &\le \tfrac{\sigma(x,\, y)+\sigma(y,\, Ty)+\sigma(y,\, x)+\sigma(x,\, Tx)}{4} \\ &= \tfrac{2\sigma(x,\, y)+\sigma(y,\, Ty)+\sigma(x,\, Tx)}{4}. \end{aligned} \tag{29}$$

Now, from the definition of M_0, three cases arise:

Case 1: For $M_0 = \sigma(x, y)$, (29) gives

$$\frac{\sigma(x,Ty)+\sigma(y,Tx)}{4} \le \sigma(x, y) = M_0.$$

Case 2: For $M_0 = \sigma(y, Ty)$, (29) reduces to

$$\frac{\sigma(x,Ty)+\sigma(y,Tx)}{4} \le \sigma(y, Ty) = M_0.$$

Case 3: For $M_0 = \sigma(x, Tx)$, (29) gives

$$\frac{\sigma(x,Ty)+\sigma(y,Tx)}{4} \le \sigma(x, Tx) = M_0.$$

Thus, for each case, $\frac{\sigma(x,Ty)+\sigma(y,Tx)}{4} \le M_0$. We then conclude that

$$M(x, y) = M_0(x, y).$$

6. CONCLUDING REMARKS

It is of interest to note here that the mappings in Examples 4 and 5 are two concrete examples of mappings satisfying the conditions in Theorems 2 and 3 respectively.

As can be seen from Proposition 6, if we take $\lambda = \frac{1}{3} \in (0, 1)$, then T is indeed cyclic contraction *but also Kannan-type cyclic contraction.* Many authors (See for example, Zoto, Hoxha and Kumari[22]) have been misled by this particular example and they keep on citing it as an example of a map that is cyclic contraction but not Kannan-type cyclic contraction.

From Proposition 7, we conclude that Type A and Type B are equivalent. Thus, we observe the following implications in the work of Aydi and Karapinar[3]:

(1) Definition 2.3 is a mere repetition of Definition 1.10.
(2) Theorems 2.4 and 2.5 are mere repetitions of Theorems 2.1 and 2.2 respectively.
(3) Corollaries 3.3 and 3.4 are mere repetitions of Corollaries 3.1 and 3.2 respectively.

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