Journal of the Nigerian Mathematical Society
Vol. 39, Issue 2, pp. 173-182, 2020

LINEAR SUM OF ANALYTIC FUNCTIONS DEFINED BY A CONVOLUTION OPERATOR

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ABSTRACT. In this paper, a new family $\mathcal{R}_n^\sigma(\beta,\lambda)$ of analytic functions defined by a convolution operator and a linear combination of some geometric expressions is presented. We established some early coefficient bounds, the Fekete-Szegö estimate and the Toeplitz determinant of the family.

Keywords and phrases: Analytic function, Carathéodory function, Babalola convolution operator, Fekete-Szegö functional and Toeplitz determinants
2010 Mathematical Subject Classification: 30C45

1. INTRODUCTION

As usual, let $\mathcal{A}$ denote the family of normalized analytic functions

$$f(z) = z + a_2z^2 + \cdots \tag{1}$$

which are analytic in the unit disk $\mathcal{E} := |z| < 1$.

Let $\mathcal{S}$ denote the subfamily of $\mathcal{A}$ consisting of functions univalent in $\mathcal{E}$. A function $f \in \mathcal{S}$ is called starlike of order β $(0 \le \beta < 1)$, if $Re(zf'/f) > \beta$ and convex of order β $(0 \le \beta < 1)$, if $Re(1 + (zf''/f')) > \beta$. However, if $\beta = 0$, they are respectively and simply called starlike and convex functions.

Also, let $\mathcal{P}$ denote the family of functions

$$p(z) = 1 + c_1z + c_2z^2 + \cdots \tag{2}$$

analytic and having positive real part in $\mathcal{E}$.

The functional $|a_3 - \lambda a_2^2|$ for class $f \in \mathcal{S}$ is well-known for its rich history in the theory of geometric functions. The functional has received great attention particularly in many subclasses of the family of univalent functions.

Received by the editors April 02, 2020; Accepted: May 12, 2020
www.nigerianmathematicalsociety.org; Journal available online at https://ojs.ictp.it/jnms/
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For integers $n, q \geq 1$, the qth Hankel determinant was defined by Pommerenke [15] as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix}$$

which includes the Fekete-Szegö functional as a special case when $\lambda = 1$ has also received the attention of many researchers for wide range of subclasses of functions.

Ramachandra and Kavitha [16] defined the symmetric Toeplitz determinants $T_q(n)$ as

$$T_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_n \end{vmatrix}$$

for $q, n \geq 1$.

Babalola [3] defined the operator $\mathscr{L}_n^\sigma : \mathcal{A} \longrightarrow \mathcal{A}$ as

$$\mathscr{L}_n^\sigma f(z) = (\tau_\sigma * \tau_{\sigma,n}^{-1} * f)(z) \tag{3}$$

where

$$\tau_{\sigma,n}(z) = \frac{z}{(1-z)^{\sigma-(n-1)}}, \quad \sigma - (n-1) > 0, \quad \tau_\sigma = \tau_{\sigma,0} \quad \text{and} \quad \tau_{\sigma,n}^{(-1)}$$

is such that

$$(\tau_{\sigma,n} * \tau_{\sigma,n}^{(-1)})(z) = \frac{z}{1-z}$$

for a fixed real number σ and $n \in \mathbb{N}$. Given $f \in \mathcal{A}$, then (3) is equivalent to

$$\mathscr{L}_n^\sigma f(z) = z + \sum_{k=2}^{\infty} \left\{ \frac{(\sigma+k-1)!}{\sigma!} \frac{(\sigma-n)!}{(\sigma+k-n-1)!} \right\} a_k z^k. \tag{4}$$

The operator was used to introduce families B_n^σ (in [3]) and S_n^σ (in [4]) of functions in $\mathcal{A}$. Hence, a function $f \in \mathcal{A}$ is in $B_n^\sigma(\beta)$ if

$$Re \frac{\mathscr{L}_n^\sigma f(z)}{z} > \beta$$

and is in S_n^σ if

$$Re \frac{\mathscr{L}_{n+1}^\sigma f(z)}{\mathscr{L}_n^\sigma f(z)} > \frac{\sigma-(n+1)}{\sigma-n}.$$

where $0 \leq \beta < 1$ and σ satisfies the inequality $\sigma - (n-1) > 0$ for $n \in \mathbb{N}$. Also by putting some restrictions on σ and n in [4], the following relations hold true.

$$(\sigma - n)\mathscr{L}_{n+1}^{\sigma} f(z) = (\sigma - (n+1))\mathscr{L}_n^{\sigma} f(z) + z(\mathscr{L}_n^{\sigma} f(z))' \quad (5)$$

$$(\sigma - n)(\mathscr{L}_{n+1}^{\sigma} f(z))' = (\sigma - n)(\mathscr{L}_n^{\sigma} f(z))' + z(\mathscr{L}_n^{\sigma} f(z))'' \quad (6)$$

The concept of linear combination of geometric expressions is not new any more. One may see the works in [1, 5, 8, 9, 12, 13, 17].

In this research we extend these results to linear combination of more general geometric expressions defined by (4).

Definition 1.1. Let $f \in \mathcal{A}$. Then f is in $\mathcal{R}_n^{\sigma}(\beta, \lambda)$ if

$$Re\left((1-\lambda)\frac{\mathscr{L}_n^{\sigma} f(z)}{z} + \lambda \frac{\mathscr{L}_{n+1}^{\sigma} f(z)}{\mathscr{L}_n^{\sigma} f(z)}\right) > \beta, \ z \in \mathcal{E}. \quad (7)$$

where $0 \leq \beta < 1$.

Remark 1.2. By varying some parameters in (7), if

(1) $\lambda = 1$ and $n = 0$, then (7) reduces to a starlike function of order β
(2) $\lambda = n = 0$, then (7) reduces to Yamaguchi [19] functions of order β (see also [18]).

We shall prove that if f satisfies (7), then $Re((\mathscr{L}_n^{\sigma} f(z))/z) > 0$ in $\mathcal{E}$ for all λ such that $\lambda \leq \beta < 1$. For $n \geq 1$, univalence is implied by the results.

2. PRELIMINARY LEMMA

Let $p(z) = 1 + c_1 z + c_2 z^2 + \cdots \in \mathcal{P}$. Then the following lemmas are useful in obtaining our results.

Lemma 2.1.[6] Let $p \in \mathcal{P}$. Then $|c_n| \leq 2, \ n = \mathbb{N}$. The inequality is sharp.

Lemma 2.2.[2] Let $p \in \mathcal{P}$. Then

$$\left|c_2 - \lambda \frac{c_1^2}{2}\right| \leq 2\max\{1, |1-\sigma|\}, \quad \lambda \in \mathbb{C}.$$

Lemma 2.3.[2] Let $u = u_1 + u_2 i, \ v = v_1 + v_2 i$ and $\psi(u, v)$ a complex-valued function satisfying conditions:

a. $\psi(u,v)$ is continuous in a domain Ω of $\mathbb{C}^2$,
b. $(1,0)\in\Omega$ and $Re\ \psi(1,0)>0$,
c. $Re\ \psi(\xi+(1-\xi)u_2 i, v_1)\le\xi$ when $(\xi+(1-\xi)u_2 i, v_1)\in\Omega$ and $2v_1\le-(1-\xi)(1+u_2^2)$ for real number $0\le\xi<1$.

If $p\in P$ such that $(p(z),zp^{'}(z))\in\Omega$ and $Re\ \psi(p(z),zp^{'}(z))>\xi$ for $z\in E$, then $Re\ p(z)>\xi$ in E.

Lemma 2.4.[14] Let $p\in\mathcal{P}$. Then

$$2c_2=c_1^2+x(4-c_1^2)$$

and

$$4c_3=c_1^3+2c_1(4-c_1^2)x-c_1(4-c_1^2)x^2+2(4-c_1^2)(1-|x|^2)z$$

for some x, z such that $|x|\le1$ and $|z|\le1$.

3. MAIN RESULTS

In what follows are the main results on this paper.
Theorem 3.1. If $\lambda\le\beta<1$, then $\mathcal{R}_n^\sigma(\beta,\lambda)\subset B_n^\sigma(\beta)$.

Proof. Let $f\in\mathcal{A}$, then for $p\in\mathcal{P}$ we define the equation

$$\frac{\mathcal{L}_n^\sigma f(z)}{z}=p(z),$$

differentiating $p(z)$ we have

$$z(\mathcal{L}_n^\sigma f(z))'=\mathcal{L}_n^\sigma f(z)+z^2p'(z)$$

and by applying (5),

$$(\sigma-n)\mathcal{L}_{n+1}^\sigma f(z)=(\sigma-n)\mathcal{L}_n^\sigma f(z)+z^2p'(z)$$

so that

$$\frac{\mathcal{L}_{n+1}^\sigma f(z)}{\mathcal{L}_n^\sigma f(z)}=1+\frac{z^2p'(z)}{(\sigma-n)\mathcal{L}_n^\sigma f(z)}.$$

Now let $\mathcal{L}_n^\sigma f(z)=zp(z)$, then condition (7) can be expressed as

$$Re\left((1-\lambda)p(z)+\lambda\left(1+\frac{zp'(z)}{(\sigma-n)p(z)}\right)\right)-\beta>0.$$

Define $\psi(u,v)=(1-\lambda)u+\lambda[1+\frac{v}{(\sigma-n)u}]-\beta$ on a domain $\Omega=[\mathbb{C}-\{0\}]\times\ \mathbb{C}$ of $\mathbb{C}^2$. Clearly $\psi(u,v)$ satisfies the condition (a) of Lemma 2.3.

Moreso, $(1,0)\in\Omega$ so $\psi(1,0)=(1-\lambda)+\lambda-\beta$ and $Re\psi(1,0)=1-\beta>0$,
$0\le\beta<1$.

Thus with $\xi = 0$ in Lemma 2.3 we have $\psi(u_{2i}, v_1) = (1-\lambda)u_{2i} + \lambda\left[1 + \frac{v_1}{(\sigma-n)u_{2i}}\right] - \beta$. So that $Re\psi(u_{2i}, v_1) = \lambda - \beta \le 0$ if $\lambda \le \beta < 1$. Therefore ψ satisfies all the conditions of the Lemma 2.3 so that $Re(\mathscr{L}_n^\sigma f(z)/z) > 0$.

If $n = 0$ we have

Corollary 3.2. Let $n = 0$ and $\lambda \le \beta < 1$. If

$$Re\ \left((1-\lambda)\frac{f(z)}{z} + \lambda\frac{zf'(z)}{f(z)}\right) > \beta, \quad z \in \mathcal{E}$$

then

$$Re\frac{f(z)}{z} > 0.$$

If $n = 1$ we have

Corollary 3.3. Let $n = 1$ and $\lambda \le \beta < 1$.

$$Re\ \left((1-\lambda)f'(z) + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) > \beta, \quad z \in \mathcal{E},$$

then f is the Yamaguchi function in [19].

Theorem 3.4. Let $f \in \mathcal{R}_n^\sigma(\beta, \lambda)$. Then,

$$|a_2| \le \frac{2(\sigma-n)(1-\beta)}{[(\sigma-n)(1-\lambda)+\lambda]M_2}$$

$$|a_3| \le \frac{4\lambda(\sigma-n)^2(1-\beta)^2 + 2[(\sigma-n)(1-\lambda)+\lambda]^2(\sigma-n)(1-\beta)}{[(\sigma-n)(1-\lambda)+\lambda]^2[(\sigma-n)(1-\lambda)+2\lambda]M_3}$$

where

$$M_k = \frac{(\sigma+k-1)!}{\sigma!}\frac{(\sigma-n)!}{(\sigma+k-n-1)!} \tag{8}$$

Proof. Let $f \in \mathcal{R}_n^\sigma(\beta, \lambda)$, then in view of $p \in \mathcal{P}$, then define the equation

$$(1-\lambda)\frac{\mathscr{L}_n^\sigma f(z)}{z} + \lambda\frac{\mathscr{L}_{n+1}^\sigma f(z)}{\mathscr{L}_n^\sigma f(z)} = \beta + (1-\beta)p(z). \tag{9}$$

Considering (5), (6), (8), then comparing the coefficients of (9) we have

$$a_2 = \frac{(\sigma-n)(1-\beta)c_1}{[(\sigma-n)(1-\lambda)+\lambda]M_2} \tag{10}$$

and

$$a_3 = \frac{\lambda(\sigma-n)^2(1-\beta)^2c_1^2 + [(\sigma-n)(1-\lambda)+\lambda]^2(\sigma-n)(1-\beta)c_2}{[(\sigma-n)(1-\lambda)+\lambda]^2[(\sigma-n)(1-\lambda)+2\lambda]M_3} \tag{11}$$

and by applying Lemma 2.1, completes the proof.

Corollary 3.5. If $f \in \mathcal{S}^*(\beta)$, then $|a_2| \leq 2(1-\beta)$ and $|a_3| \leq (1-\beta)(3-2\beta)$ which are well-known.

Theorem 3.6. Let $f \in \mathcal{R}_n^\sigma(\beta,\lambda)$ and $\mu \in \mathbb{C}$. Then,

$$\left|a_3 - \mu a_2^2\right| \leq \frac{2(\sigma-n)(1-\beta)}{[(\sigma-n)(1-\lambda)+2\lambda]M_3} \max\left\{1, \left|1 - \frac{2(\sigma-n)(1-\beta)A}{[(\sigma-n)(1-\lambda)+\lambda]^2M_2^2}\right|\right\}$$

where $A = [\mu[(\sigma-n)(1-\lambda)+2\lambda]M_3 - \lambda M_2^2]$ and M_k is as defined in (8).

Proof. From (10) and (11) we have

$$\left|a_3 - \mu a_2^2\right| = \left|\frac{(\sigma-n)\left((1-\beta)c_2 + B\right)}{[(\sigma-n)(1-\lambda)+2\lambda]M_3}\right|$$

where

$$B = \frac{\lambda(\sigma-n)(1-\beta)^2M_2^2c_1^2 - \mu(\sigma-n)(1-\beta)^2[(\sigma-n)(1-\lambda)+2\lambda]M_3c_1^2}{[(\sigma-n)(1-\lambda)+\lambda]^2M_2^2}$$

and

$$\left|a_3 - \mu a_2^2\right| \leq \frac{(\sigma-n)(1-\beta)}{[(\sigma-n)(1-\lambda)+2\lambda]M_3}\left|c_2 - \eta\frac{c_1^2}{2}\right|$$

where $\eta = \dfrac{2(\sigma-n)(1-\beta)[\mu[(\sigma-n)(1-\lambda)+2\lambda]M_3 - \lambda M_2^2]}{[(\sigma-n)(1-\lambda)+\lambda]^2M_2^2}$.
Thus using Lemma 2.3, then the results follow.

For $\lambda = \sigma = 1$ and $n = 0$,we have

Corollary 3.7. If $f \in \mathcal{S}^*(\beta)$, then

$$\left|a_3 - \mu a_2^2\right| \leq (1-\beta)\max\{1, |1-(4\mu-2)(1-\beta)|\}.$$

Theorem 3.8. Let $f \in \mathcal{R}_n^\sigma(\beta,\lambda)$. Then,

$$T_2(2) = \left|a_3^2 - a_2^2\right| \leq 16\left(\frac{(\sigma-n)^2(1-\beta)^2}{4L^2M_3^2} + \frac{\lambda(\sigma-n)^3(1-\beta)^3((\sigma-n)(1-\beta)\lambda+K^2)}{L^2K^4M_3^2}\right) - \frac{4(\sigma-n)(1-\beta)^2}{K^2M_2^2}$$

where

$$K = [(\sigma-n)(1-\lambda)+\lambda] \quad \text{and} \quad L = [(\sigma-n)(1-\lambda)+2\lambda]. \tag{12}$$

Proof. Using (10), (11) and (8) we have

$$\left|a_3^2 - a_2^2\right| = \left|\frac{K^4(\sigma-n)^2(1-\beta)^2c_2^2}{L^2K^4M_3^2} + \frac{2\lambda K^2(\sigma-n)^3(1-\beta)^3c_1^2c_2}{L^2K^4M_3^2} + \frac{\lambda^2(\sigma-n)^4(1-\beta)^4c_1^4}{L^2K^4M_3^2} - \frac{(\sigma-n)^2(1-\beta)^2c_1^2}{K^2M_2^2}\right| \tag{13}$$

Now applying Lemma 2.4, (12) and let $X = 4 - c_1^2$, then

$$\left|a_3^2 - a_2^2\right| = \left|\frac{(\sigma-n)^2(1-\beta)^2c_1^4}{4L^2M_3^2} + \frac{(\sigma-n)^2(1-\beta)^2xXc_1^2}{2L^2M_3^2} + \frac{(\sigma-n)^2(1-\beta)^2x^2X^2}{4L^2M_3^2} + \frac{\lambda(\sigma-n)^3(1-\beta)^3c_1^4}{L^2K^2M_3^2} + \frac{\lambda(\sigma-n)^3(1-\beta)^3xXc_1^2}{L^2K^2M_3^2} + \frac{\lambda^2(\sigma-n)^4(1-\beta)^4c_1^4}{L^2K^4M_3^2} - \frac{(\sigma-n)^2(1-\beta)^2c_1^2}{K^2M_3^2}\right|$$

Without loss of generality, let $c = c_1$, implies $|c| = |c_1| \leq 2$, hence $c \in [0,2]$ Now by triangle inequality,

$$\left|a_3^2 - a_2^2\right| \leq \left|\left(\frac{(\sigma-n)^2(1-\beta)^2}{4L^2M_3^2} + \frac{\lambda(\sigma-n)^3(1-\beta)^3((\sigma-n)(1-\beta)\lambda+K^2)}{L^2K^4M_3^2}\right)c^4 - \frac{(\sigma-n)^2(1-\beta)^2c^2}{K^2M_2^2}\right| \tag{14}$$

$$+\frac{(\sigma-n)^2(1-\beta)^2(K^2+2\lambda(\sigma-n)(1-\beta))|x|Xc^2}{2L^2K^2M_3^2}$$

$$+\frac{(\sigma-n)^2(1-\beta)^2|x|^2X^2}{4L^2M_3^2} =: \phi(|x|).$$

Now,

$$\phi'(|x|) = \frac{(\sigma-n)^2(1-\beta)^2(K^2+2\lambda(\sigma-n)(1-\beta))Xc^2}{2L^2K^2M_3^2} + \frac{(\sigma-n)^2(1-\beta)^2|x|X^2}{2L^2M_3^2}$$

so that

$$\begin{aligned}\phi(1) =& \Bigg|\Bigg(\frac{(\sigma-n)^2(1-\beta)^2}{4L^2M_3^2} \\ &+ \frac{\lambda(\sigma-n)^3(1-\beta)^3((\sigma-n)(1-\beta)\lambda+K^2)}{L^2K^4M_3^2}\Bigg)c^4 \\ &- \frac{(\sigma-n)^2(1-\beta)^2c^2}{K^2M_2^2}\Bigg| \\ &+ \frac{(\sigma-n)^2(1-\beta)^2(K^2+2\lambda(\sigma-n)(1-\beta))Xc^2}{2L^2K^2M_3^2} \\ &+ \frac{(\sigma-n)^2(1-\beta)^2X^2}{4L^2M_3^2} \end{aligned} \tag{15}$$

Clearly $\phi'(|x|) > 0$ on $[0,1]$ and $\phi(|x|) \leq \phi(1)$, hence

$$\begin{aligned}\left|a_3^2 - a_2^2\right| \leq& \Bigg|\Bigg(\frac{(\sigma-n)^2(1-\beta)^2}{4L^2M_3^2} \\ &+ \frac{\lambda(\sigma-n)^3(1-\beta)^3((\sigma-n)(1-\beta)\lambda+K^2)}{L^2K^4M_3^2}\Bigg)c^4 \\ &- \frac{(\sigma-n)^2(1-\beta)^2c^2}{K^2M_2^2}\Bigg| \\ &+ \frac{(\sigma-n)^2(1-\beta)^2(K^2+2\lambda(\sigma-n)(1-\beta))Xc^2}{2L^2K^2M_3^2}\end{aligned}$$

$$+\frac{(\sigma-n)^2(1-\beta)X^2}{4L^2M_3^2}\frac{(\sigma-n)^2(1-\beta)^2(K^2+2\lambda(\sigma-n)(1-\beta))Xc^2}{2L^2K^2M_3^2}$$

and

$$\left|a_3^2 - a_2^2\right| \leq \Bigg| \left(\frac{(\sigma - n)^2(1-\beta)}{4L^2M_3^2} + \frac{\lambda(\sigma-n)^3(1-\beta)^3((\sigma-n)(1-\beta)\lambda + K^2)}{L^2K^4M_3^2} \right) c^4 - \frac{(\sigma-n)^2(1-\beta)^2c^2}{K^2M_2^2} \Bigg|$$

$$+ \frac{(\sigma-n)^2(1-\beta)^2\left(\frac{2(K^2+2\lambda(\sigma-n)(1-\beta))c^2}{K^2} - \frac{(K^2+2\lambda(\sigma-n))c^4}{2K^2} + 4 - 2c^2 + \frac{c^4}{4}\right)}{L^2M_3^2}$$

Thus and trivially $\phi(|x|)$ has a maximum value at

$$\left|a_3^2 - a_2^2\right| \leq 16 \left(\frac{(\sigma - n)^2(1-\beta)^2}{4L^2M_3^2} + \frac{\lambda(\sigma-n)^3(1-\beta)^3((\sigma-n)(1-\beta)\lambda + K^2)}{L^2K^4M_3^2} \right) - \frac{4(\sigma-n)(1-\beta)^2}{K^2M_2^2}$$

Corollary 3.9. If $f \in \mathcal{S}^*(\beta)$, then

$$T_2(2) = \left|a_3^2 - a_2^2\right| \leq (1-\beta)\Big(4(1-\beta)(2-\beta) - 3\Big).$$

4. CONCLUDING REMARKS

In this paper we have defined a class of analytic function using Babalola convolution operator introduced in [3]. We then established some initial coefficient bounds, the Fekete-Szegö estimate and the Toeplitz determinant of the family.

ACKNOWLEDGEMENTS

The authors would like to thank the referee for making useful corrections in the original version of this manuscript.

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