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THE FUZZY SUBGROUPS FOR THE ABELIAN STRUCTURE $\mathbb{Z}_8 \times \mathbb{Z}_{2^n}, n > 2$

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ABSTRACT. Any finite nilpotent group can be uniquely written as a direct product of p-groups. In this paper, we give explicit formulae for the number of distinct fuzzy subgroups of the cartesian product of two abelian groups of orders 2^n and 8 respectively for every integer $n > 2$.

Keywords and phrases: Finite p-Groups, abelian Group, Fuzzy subgroups, isomorphic, Inclusion-Exclusion Principle, Maximal subgroups.

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1. INTRODUCTION

The natural equivalence relation was introduced in [7], where a method to determine the number and nature of fuzzy subgroups of a finite group G was developed with respect to the natural equivalence. In [2] and [5], a different approach was applied for the classification. In this work, an essential role in solving counting problems is played by adopting the Inclusion-Exclusion Principle. The process leads to some recurrence relations from which the solutions are then finally computed with ease. In the process of our computation, the use of GAP(Group Algorithim and Programming) was actually Invoked. This helped in sorting out various intersections of different degrees. These were used in substitution into our pivotal equation (#) for our calculations.

2. PRELIMINARIES

Suppose that $(G, \cdot, e)$ is a group with identity e. Let $S(G)$ denote the collection of all fuzzy subsets of G. An element $\lambda \in S(G)$ is

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said to be a fuzzy subgroup of G if the following two conditions are satisfied

(i) $\lambda(ab) \geq min\{\lambda(a), \lambda(b)\}, \ \forall\, a, b \in G$;
(ii) $\lambda(a^{-1} \geq \lambda(a)$ for any $a \in G$.

And, since $(a^{-1})^{-1} = a$, we have that $\lambda(a^{-1}) = \lambda(a)$, for any $a \in G$. Also, by this notation and definition, $\lambda(e) = \sup \lambda(G)$. [Marius [4]]. Define by $M_1, M_2, \ldots, M_t$, the maximal subgroups of G, and denote by $h(G)$ the number of chains of subgroups of G which ends in G.

Theorem [Marius [4]: The set $FL(G)$ possessing all fuzzy subgroups of G forms a lattice under the usual ordering of fuzzy set inclusion. This is called the fuzzy subgroup lattice of G.
We define the level subset: $\lambda G_\beta = \{a \in G/\lambda(a) \geq \beta\}$ for each $\beta \in [0, 1]$. The fuzzy subgroups of a finite p-group G are thus, characterized, based on these subsets. In the sequel, λ is a fuzzy subgroup of G if and only if its level subsets are subgroups in G. This theorem gives a link between $FL(G)$ and $L(G)$, the classical subgroup lattice of G.

Moreover, some natural relations on $S(G)$ can also be used in the process of classifying the fuzzy subgroups of a finite q-group G (see [6] and [7]). One of them is defined by: $\lambda \sim \gamma$ iff $(\lambda(a) > \lambda(b) \Longleftrightarrow v(a) > v(b), \ \forall\, a, b \in G)$. Also, two fuzzy subgroups λ, γ of G are said to be distinct if $\lambda \nsim v$.

As a result of this development, let G be a finite p-group and suppose that $\lambda : G \longrightarrow [0, 1]$ is a fuzzy subgroup of G. Put $\lambda(G) = \{\beta_1, \beta_2, \ldots, \beta_k\}$ with the assumption that $\beta_1 < \beta_2 > \cdots > \beta_k$. Then, ends in G is determined by λ.

$$\lambda G_{\beta_1} \subset \lambda G_{\beta_2} \subset \cdots \subset \lambda G_{\beta_k} = G \qquad (a)$$

Also, we have that:

$$\lambda(a) = \beta_t \Longleftrightarrow t = \max\{r/a \in \lambda G_{\beta_r}\} \Longleftrightarrow a \in \lambda G_{\beta_t} \backslash \lambda G_{\beta_{t-1}},$$

for any $a \in G$ and $t = 1, \ldots, k$, where by convention, set $\lambda G_{\beta_0} = \phi$.

3. METHODOLOGY

The method that will be used in counting the chains of fuzzy subgroups of an arbitrary finite p-group G is described. Suppose that $M_1, M_2, \ldots, M_t$ are the maximal subgroups of G, and denote by $h(G)$ the number of chains of subgroups of G which ends in G.

By simply applying the technique of computing $h(G)$, using the application of the Inclusion-Exclusion Principle, we have that:

$$h(G) = 2\left(\sum_{r=1}^{t} h(M_r) - \sum_{1\leq r_1<r_2\leq t} h(M_{r_1} \cap M_{r_2}) + \cdots + (-1)^{t-1}h\right.$$

$$\left. X\left(\bigcap_{r=1}^{t} M_r\right)\right) (\#)$$

In [5], (#) was used to obtain the explicit formulas for some positive integers n.

Theorem (*) [Marius][7]: The number of distinct fuzzy subgroups of a finite p-group of order p^n which have a cyclic maximal subgroup is:

(i) $h(\mathbb{Z}_{p^n}) = 2^n$
(ii) $h(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}) = h(M_{p^n}) = 2^{n-1}(2 + (n-1)p)$

4. THE NUMBER OF FUZZY SUBGROUPS FOR $\mathbb{Z}_8 \times \mathbb{Z}_8$

We begin here with the simplest case for our computation. Here, the given abelian structure was formed and, by the help of GAP, the maximal subroups were computed, and their intersections were made which were substituted into (#).

Lemma 1: Let G be abelian such that $G = \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^m}, n \leq m$ then, there exists $p+1$ maximal subgroups in G. By lemma 1 , there are three maximal subgroups . Each is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_8$. Hence, by (#,) it follows that,
$\frac{1}{2}h(G) = 3h(\mathbb{Z}_4 \times \mathbb{Z}_8) - 3h(\mathbb{Z}_4 \times \mathbb{Z}_4) + h(\mathbb{Z}_4 \times \mathbb{Z}_4) = 3h(\mathbb{Z}_4 \times \mathbb{Z}_8) - 2h(\mathbb{Z}_4 \times \mathbb{Z}_4)$.

Theorem 2 [1] : Let $H = \mathbb{Z}_4 \times \mathbb{Z}_{2^n}$. Then $h(H) = 2^n(n^2 + 5n - 2)$
By theorem 2, $= h(G) = 864$

Proposition 3: For $G = \mathbb{Z}_{2^4}) \times \mathbb{Z}_8, h(G) = 3200$
Proof : By lemma 1, G has 3 maximal subgroups. One of them is isomorphic to $\mathbb{Z}_8 \times \mathbb{Z}_8$, while two are isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_16$. If (#) is applied, we whould have that : $\frac{1}{2}h(G) = 2h(\mathbb{Z}_4 \times \mathbb{Z}_16) + h(\mathbb{Z}_8 \times \mathbb{Z}_8) - 3h(\mathbb{Z}_4 \times \mathbb{Z}_8) + h(\mathbb{Z}_4 \times \mathbb{Z}_8 = 3200$.

Proposition 4: Suppose that $G = \mathbb{Z}_{2^5}) \times \mathbb{Z}_8, h(G) = 10368$
Proof : By (#), $\frac{1}{2}h(G) = 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^5}) + h(\mathbb{Z}_8 \times \mathbb{Z}_{2^4}) - 3h(\mathbb{Z}_4 \times \mathbb{Z}_{2^4})$ $+ h(\mathbb{Z}_4 \times \mathbb{Z}_{2^4} = 5184$, whence, $h(G) = 10368$.

Lemma 5 : $G = \mathbb{Z}_{2^6} \times \mathbb{Z}_8, h(G) = 30976$

Proof : Using (#), $\frac{1}{2}h(G) = 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^6}) + h(\mathbb{Z}_8 \times \mathbb{Z}_{2^5}) - 3h(\mathbb{Z}_4 \times \mathbb{Z}_{2^5}) + h(\mathbb{Z}_4 \times \mathbb{Z}_{2^5})$
$= 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^6}) - 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^5}) + h(\mathbb{Z}_8 \times \mathbb{Z}_{2^5}) = 30976$

Theorem : Let $G = \mathbb{Z}_{2^n} \times \mathbb{Z}_8$, then $h(G) = \frac{1}{3}(2^{n+1})(n^3 + 12n^2 + 17n - 24)$

Proof : The three maximal subgroups of G have the following properties :
one is isomorphic to $\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}})$, while two are isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_{2^n})$.
We have : $\frac{1}{2}h(G) = 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}}) - 3h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) + h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}})$
$= 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}}) - 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}})$
$= h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}}) + 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) - h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}})$
Hence,
$h(G) = 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) - 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) + 2h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}})$
$= 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) + 8h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-2}}) - 16h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-3}})$
$+ 32h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-3}}) - 32h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-4}}) + 16h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-4}})$
$= 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) + 8h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-2}}) + 16h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-3}})$
$+ 32h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-4}}) - 64h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-5}}) + 32h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-5}}) + \cdots - 2^{j+1}h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-j}})$
$+ 2^j h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-j}})$, for $n - j = 3 = 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + 2^{n-3}h(\mathbb{Z}_8 \times \mathbb{Z}_{2^3}) - 2^{n-1}h(\mathbb{Z}_4 \times \mathbb{Z}_{2^3}) + \sum_{k=1}^{n-3}[2^{k+1}h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-k}}) = 2^{n+2}[n^2 + 5n + 3] + \sum_{k=1}^{n-3} h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-k}}) = 2^{n+2}((n^2 + 5n + 3) + \frac{1}{6}(n-3)(n^2 + 9n + 14))$
$= \frac{1}{3}(2^{n+1})(n^3 + 12n^2 + 17n - 24), n > 2.$ $\square$

REFERENCES

[1] Adebisi S.A. , M. Ogiugo and M. EniOluwafe (2020) Computing the Number of Distinct Fuzzy Subgroups for the Nilpotent p-Group of $D_{2^n} \times C_4$ International J.Math. Combin.Vol.1(2020),86-89.

[2] Mashinchi M. , Mukaidono M.(1992). A classification of fuzzy subgroups. Ninth Fuzzy System Symposium, Sapporo, Japan, 649-652.

[3] Tarnauceanu, Marius. (2009). The number of fuzzy subgroups of finite cyclic groups and Delannoy numbers, European J. Combin. (30), 283-289, doi: 10.1016/j.ejc.2007.12.005.

[4] Tarnauceanu Marius. (2011). Classifying fuzzy subgroups for a class of finite p-groups. "ALL CUZa" Univ. Iasi, Romania.

[5] Tarnauceanu, Marius. (2012). Classifying fuzzy subgroups of finite nonabelian groups. Iran.J.Fussy Systems. (9) 33-43.

[6] Tarnauceanu, Marius. , Bentea, L. (2008). A note on the number of fuzzy subgroups of finite groups, Sci. An. Univ. "ALL.Cuza" Iasi, Matt., (54) 209-220.

[7] Tarnauceanu, Marius. , Bentea, L. (2008). On the number of fuzzy subgroups of finite abelian groups, Fuzzy Sets and Systems (159), 1084-1096, doi:10.1016/j.fss.2017.11.014

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