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APPLICATION OF THE BRIOT-BOUQUET DIFFERENTIAL EQUATION TO A CLASS OF ANALYTIC FUNCTIONS

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ABSTRACT. In this paper, using the technique of the Briot-Bouquet differential subordination, we find the real number ρ such that $Re\,[D^n f(z)^\alpha/\alpha^n z^\alpha] > \rho$ implies univalence of certain analytic functions.

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1. INTRODUCTION

We denote as usual by A the class of functions f such that

$$f(z) = z + a_2 z^2 + ...$$

which are analytic in the unit disk E and by P the class of Carathéodory functions such that if $p \in P$ then p has the form

$$p(z) = 1 + c_1 z + c_2 z^2 + ...$$

that is analytic in E and satisfies $Re\ p(z) > 0,\ z \in E$. Furthermore, we denote by $P(\beta)$ the subclass of P satisfying the condition $Re\ p(z) > \beta$ for $\beta \in [0, 1)$.
If p and q are two analytic functions in the unit disk E, we say that the function $p(z)$ is subordinate to $q(z)$ i. e.

$$p(z) \prec q(z) \tag{1}$$

if and only if there exists an analytic function w $(|w(z)| < 1,\ z \in E)$ satisfying $w(0) = 0$ such that $p(z) = q(w(z))$. In particular if $q(z)$ is univalent in E then the subordination condition (1) holds if and only if $p(0) = q(0)$ and $p(E) \subseteq q(E)$.

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The class $J_n^\alpha(\beta)$ is a subclass of A defined as follows:
Definition 1: An analytic function $f \in J_n^\alpha(\beta)$ if and only if

$$Re\,\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}\frac{D^{n+1} f(z)}{D^n f(z)} > \beta, \quad z \in E \tag{2}$$

for non-negative real number α, $0 \leq \beta < 1$, $n \geq 0$ and D^n is the Salagean operator.
It was established in [5] that analytic functions satisfying (2) also satisfy

$$Re\,\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} > \beta.$$

This last condition implies univalence of functions in $J_n^\alpha(\beta)$ for $n \geq 1$.

In the present paper, by using the technique of the Briot-Bouquet differential subordination, we find the largest real number ρ such that

$$Re\,\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} > \rho$$

given that the function $f \in A$ satisfies (2).
A function $p(z) \in P$ is said to satisfy a Briot-Bouquet differential subordination if

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \tag{3}$$

for complex constants β and γ, complex functions $h(z)$ with $h(0) = 1$ and $Re\,(\beta h(z) + \gamma) > 0$ in E.
A dominant of the the differential subordination (3) is a univalent function $q(z)$ such that $p(0) = q(0)$ and $p \prec q$. If a dominant $\tilde{q}$ is such that $\tilde{q} \prec q$ for all dominants q of the differential subordination, then $\tilde{q}$ is said to be the best dominant. More results on the subject of differential subordinations can be found in [4, 6, 7].

2. PRELIMINARY LEMMAS

The following lemmas are fundamental in the proof of our main results.
Lemma 1 [3]**:** Let $f(z) \in A$, and $\alpha > 0$ be real. If $\frac{D^{n+1} f(z)^\alpha}{D^n f(z)^\alpha}$ takes a value which is independent of n, then

$$\frac{D^{n+1} f(z)^\alpha}{D^n f(z)^\alpha} = \alpha\frac{D^{n+1} f(z)}{D^n f(z)}.$$

Lemma 2 [4]**:** Let $p(z)$ be analytic in E and let $h(z)$ satisfy the differential subordination

$$p(z)+\frac{zp'(z)}{\eta p(z)+\gamma}\prec h(z),$$

and $Re\ (\eta h(z)+\gamma)>0$ in E. Then the differential equation

$$q(z)+\frac{zq'(z)}{\eta q(z)+\gamma}=h(z),\ q(0)=0$$

has a univalent solution $q(z)$. In addition $p(z)\prec q(z)\prec h(z)$ and $q(z)$ is the best dominant.
The next lemma gives some well-known properties of the Gaussian hypergeometric function

$$_2F_1(a,b;c;z)=1+\frac{ab}{c}\frac{z}{1!}+\frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!}+...,\tag{4}$$

where a, b, and c are complex constants with $c\neq 0,-1,-2,...$
Lemma 3 [1]**:** For real or complex numbers a, b and c ($c\neq 0,-1,-2,...$), we have

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{(-a)}dt=\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}{}_2F_1(a,b;c;z)$$

$$(Re(c)>Re(b)>0)$$

$$_2F_1(a,b;c;z)={}_2F_1(b,a;c;z)$$

$$_2F_1(a,b;c;z)=(1-z)^{-a}{}_2F_1\left(a,c-b;c;\frac{z}{z-1}\right).$$

3. MAIN RESULTS

Theorem 1: Let $f(z)\in J_n^\alpha(\beta)$. Then we have the subordination

$$\frac{D^nf(z)^\alpha}{\alpha^nz^\alpha}\prec q(z)\prec h(z)=\frac{1+(1-2\beta)z}{1-z}\tag{5}$$

and $q(z)$ is the best dominant where $q(z)$ is given by

$$q(z)=(1-z)^{-1}\left[{}_2F_1\left(1,1;1+\alpha;\frac{z}{z-1}\right)\right.$$
$$\left.+\frac{(1-2\beta)\alpha z}{\alpha+1}{}_2F_1\left(1,1;\alpha+2;\frac{z}{z-1}\right)\right].$$

Furthermore,

$$Re\ \frac{D^nf(z)^\alpha}{\alpha^nz^\alpha}>\rho,$$

where

$$\rho = \frac{1}{2}\left[{}_2F_1\left(1,1;1+\alpha;\frac{1}{2}\right) - \frac{(1-2\beta)\alpha}{\alpha+1}{}_2F_1\left(1,1;\alpha+2;\frac{1}{2}\right)\right]$$

is the best possible.

proof: Since $f(z) \in J_n^\alpha(\beta)$. Then

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}\frac{D^{n+1}f(z)}{D^n f(z)} \prec h(z) = \frac{1+(1-2\beta)z}{1-z},\ n \in N. \tag{6}$$

Let

$$p(z) = \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}, \tag{7}$$

where $p(z)$ is analytic with $p(0) = 1$. Taking logarithmic differentiation of both sides of (7), we obtain

$$\frac{zp'(z)}{p(z)} + \alpha = \frac{D^{n+1} f(z)^\alpha}{D^n f(z)^\alpha}. \tag{8}$$

Since the left hand side of (8) is independent of n, we use Lemma 1 and divide through by α, to obtain

$$\frac{zp'(z)}{\alpha p(z)} + 1 = \frac{D^{n+1} f(z)}{D^n f(z)}. \tag{9}$$

Using (7), and (9) in (6) gives

$$p(z) + \frac{zp'(z)}{\alpha} \prec h(z) = \frac{1+(1-2\beta)z}{1-z}.$$

Then by Lemma 2 with $\eta = 0$ and $\gamma = \alpha$, the differential equation

$$q(z) + \frac{zq(z)'}{\alpha} = h(z) = \frac{1+(1-2\beta)z}{1-z}$$

has a univalent solution given by

$$q(z) = \frac{\alpha}{z^\alpha}\int_0^z t^{\alpha-1}\frac{1+(1-2\beta)t}{1-t}dt.$$

In addition, $p(z) = \frac{D^n f^\alpha(z)}{\alpha^n z^\alpha} \prec q(z) \prec h(z)$.
By a change of variable $t = sz$, $q(z)$ can be written as

$$\begin{aligned} q(z) &= \frac{\alpha}{z^\alpha}\int_0^1 (sz)^{\alpha-1}\frac{1+(1-2\beta)sz}{1-sz}zds \\ &= \alpha\int_0^1 s^{\alpha-1}[1+(1-2\beta)sz][1-sz]^{-1}ds \\ &= \alpha\int_0^1 s^{\alpha-1}[1-sz]^{-1}ds + \alpha z[1-2\beta]\int_0^1 s^\alpha[1-sz]^{-1}ds. \end{aligned}$$

Using the first property of hypergeometric functions given in Lemma 3, we can rewrite $q(z)$ as

$$q(z) = \alpha\frac{\Gamma(\alpha)\Gamma(1)}{\Gamma(\alpha+1)}{}_2F_1(1,\alpha,\alpha+1;z)$$
$$+\alpha z(1-2\beta)\frac{\Gamma(\alpha+1)\Gamma(1)}{\Gamma(\alpha+2)}{}_2F_1(1,\alpha+1,\alpha+2;z).$$

By using $\Gamma(n)=(n-1)!$ and the third property of Lemma 3, we have

$$q(z) = (1-z)^{-1}\left[{}_2F_1\left(1,1;\alpha+1;\frac{z}{z-1}\right)\right.$$
$$\left. + \alpha z\frac{1-2\beta}{\alpha+1}{}_2F_1\left(1,1;\alpha+2;\frac{z}{z-1}\right)\right]$$

as desired.

To prove the second part of the theorem, we only need to show that

$$\inf_{|z|<1} Re\ q(z) = q(-1),\ z\in E.$$

The function $\frac{1+(1-2\beta)z}{1-z}$ is convex univalent in E, therefore for $|z|\le r<1$

$$Re\ \frac{1+(1-2\beta)z}{1-z} \ge \frac{1-(1-2\beta)r}{1+r}.$$

Setting

$$q(s,z) = \frac{1+(1-2\beta)sz}{1-sz},\ 0\le s\le 1,$$

$z\in E$ and $d\mu(s)=\alpha s^{\alpha-1}ds$ which is a positive measure on $[0,1]$, we obtain

$$q(z) = \int_0^1 q(s,z)d\mu(s),$$

so that

$$Re\ q(z) = \int_0^1 Re\ \left[\frac{1+(1-2\beta)sz}{1-sz}\right]d\mu(s)$$
$$\ge \int_0^1 \frac{1-(1-2\beta)sr}{1+sr}d\mu(s).$$

If $r\to 1^-$, then we have

$$Re\ q(z) \ge q(-1).$$

Hence,

$$Re\ \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} > \rho$$

where

$$\rho = q(-1) = \frac{1}{2}\left[{}_2F_1\left(1,1;1+\alpha;\frac{1}{2}\right) - \frac{(1-2\beta)\alpha}{\alpha+1}{}_2F_1\left(1,1;\alpha+2;\frac{1}{2}\right)\right].$$

This ρ is the best possible since $q(z)$ is the the best dominant.

Corollary 1: Let $f(z) \in A$. If

$$f' \prec \frac{1+(1-2\beta)z}{1-z}, \text{ i.e } \text{ Re } f'(z) > \beta.$$

Then

$$\frac{f(z)}{z} \prec \frac{2(\beta-1)\ln(1-z)}{z} + 2\beta - 1$$

and $q(z)$ is given by

$$q(z) = \frac{1}{z}\int_0^z \frac{1+(1-2\beta)t}{1-t}dt$$

is the best dominant.

Moreso,

$$Re\ \frac{f(z)}{z} \geq 2(1-\beta)\ln 2 + 2\beta - 1.$$

Corollary 2: Let $f \in A$ and $Re\ f'(z) > 0$. Then

$$Re\ \frac{f(z)}{z} \geq 2\ln 2 - 1.$$

Corollary 3: Let $f \in A$ and

$$f'(z) + zf''(z) \prec \frac{1+(1-2\beta)z}{1-z}, \quad \text{i.e } \text{ Re } (f'(z) + zf''(z)) > \beta.$$

Then

$$f'(z) \prec q(z) = \frac{(2\beta-2)\ln(1-z)}{z} + 2\beta - 1,$$

and

$$Re\ f'(z) \geq 2(1-\beta)\ln 2 + 2\beta - 1.$$

4. REMARKS

Remark 1: If $\beta = \frac{1}{2}$ in Theorem 1, then

$$\rho = \frac{1}{2}\left[F_1\left(1,1;1+\alpha;\frac{1}{2}\right)\right].$$

By virtue of equation (4),

$$
{}_2F_1\left(1,1;1+\alpha;\frac{1}{2}\right) = 1 + \frac{1}{2(\alpha+1)} + \frac{1}{2(\alpha+1)(\alpha+2)} + \frac{3}{4(\alpha+1)(\alpha+2)(\alpha+3)} + ...
$$

So that

$$
\rho > \frac{1}{2}.
$$

This implies that for each $f \in J_n^\alpha(\frac{1}{2})$,

$$
Re\, \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} > \frac{1}{2}.
$$

Remark 2: Corollaries 1 and 2 agree with existing results (see[2]).

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REFERENCES

[1] M. Abramowitz and I. Stegun, *Handbook of Mathematical formulas, Graph and Mathematical Tables.* Dover Publications, New York. 1971

[2] Babalola, K. O., Opoola T. O. *A sharp inclusion theorem for certain class of Analytic function defined by Salagean derivaties.* International Journal of Mathematics and Computer Science, **2**(4), 371–375, 2007.

[3] Babalola, K. O. and Opoola, T. O. *Radius problems for certain classes of analytic functions involving Salagean derivative.* Advances in Inequalities for Series, 15-21, 2008.

[4] Eenigenburg, P., Miller, S. S., Mocanu, P. T., Reade, M. O. *On Briot- Bouquet Differential subordinations.* Rev. Roumanie Math. Pures Appl., **29**, 567-573, 1984.

[5] Jimoh, F. M. and Babalola, K. O. *Functions defined by products of geometric expressions.* Nigerian Journal of Mathematics and Applications, **24**, 176-183, 2015.

[6] Miller, S. S. and Mocanu, P. T. *Differential subordinations and univalent functions.* Michigan Math. J., **28**, 157-171, 1981.

[7] Miller, S. S. and Mocanu, P. T. *On some classes of first-order differential subordinations.* Michigan Math. J., **32**, 185-195, 1985.

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