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SOME REMARKS ON A PAPER OF MOGBADEMU ET AL.

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ABSTRACT. In this paper we state and prove the correct forms of the two theorems and the related results proved by Mogbademu , S. Hans, J. A. Adepoju, in the *Journal of Nigerian Mathematical Society* 34(2015) 243-248 and also give a generalization of these results.

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1. INTRODUCTION AND PRELIMINARY

Mogbademu et al [1] proved the following extensions and generalizations of the *Enestrom-Kakeya theorem*:

Theorem 1.1. Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients. If $Re(a_j)=\alpha_j,\quad Im(a_j)=\beta_j$ for $0 \le j \le n$ such that for some real $t>0,\quad \mu \ge 0,\quad \lambda:\quad 0 \le \lambda \le n-1$ and $\rho:\quad 0<\rho\le 1$

$$t^n\alpha_n+\mu t^{n-1} \le t^{n-1}\alpha_{n-1} \le \ldots \le t^{\lambda+1}\alpha_{\lambda+1} \le t^{\lambda}\alpha_{\lambda} \ge t^{\lambda-1}\alpha_{\lambda-1} \ge \ldots \ge \alpha_0$$

then all the zeros of $P(z)$ lie in

$$\left|z-\frac{\mu}{a_n}\right| \le \frac{1}{|a_n|}\left[\frac{2\alpha_\lambda}{t^{n-\lambda-1}}+\mu-\alpha_n t+\frac{(|\alpha_0|-\alpha_0)+|\beta_0|}{t^{n-1}} + \sum_{j=1}^{n}\frac{|\beta_j t-\beta_{j-1}|}{t^{n-j}}\right].$$

Theorem 1.2. Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients. If $Re(a_j)=\alpha_j,\quad Im(a_j)=\beta_j$ for $0 \le$

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$j \leq n$ such that for some real $t > 0$, $\mu \geq 0$, $\lambda : \quad 0 \leq \lambda \leq n-1$ and $\rho : \quad 0 < \rho \leq 1$

$$t^n \alpha_n + \mu t^{n-1} \leq t^{n-1}\alpha_{n-1} \leq ... \leq t^{\lambda+1}\alpha_{\lambda+1} \leq t^{\lambda}\alpha_{\lambda} \geq t^{\lambda-1}\alpha_{\lambda-1} \geq ... \geq \rho\alpha_0$$

then all the zeros of $P(z)$ lie in

$$\left| z - \frac{\mu}{a_n} \right| \leq \frac{1}{|a_n|}\left[\frac{2\alpha_\lambda}{t^{n-\lambda-1}} + \mu - \alpha_n t + \frac{2|\alpha_0| - \rho(|\alpha_0| + \alpha_0) + |\beta_0|}{t^{n-1}} + \sum_{j=1}^{n} \frac{|\beta_j t - \beta_{j-1}|}{t^{n-j}} \right].$$

Firstly, in the statement of Theorem 1.1., the parameter ρ is redundant in the condition of the theorem since it is not involved in its conclusion. Therefore, it has to be removed. Secondly, the statements of the two theorems are not correct. The condition that $\mu \geq 0$ is not true in general since for large μ, $t^n\alpha_n + \mu t^{n-1}$ cannot be less than $t^{n-1}\alpha_{n-1}$.

After Theorem 1.1, the authors make the following ambiguous statement:

"Notice that with $t = 1$ in Theorem 1.2. and suppose the imaginary parts of the coefficients are monotonic and non-negative, then we get the following.

Corollary 2.2. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients. If $Re(a_j) = \alpha_j$, $Im(a_j) = \beta_j$ for $0 \leq j \leq n$ such that for some real $k \geq 1$, $0 \leq \lambda \leq n-1$ and $0 < \rho \leq 1$

$$\alpha_n + \mu \leq \alpha_{n-1} \leq ... \leq \alpha_{\lambda+1} \leq \alpha_\lambda \geq \alpha_{\lambda-1} \geq ... \geq \alpha_0 \qquad (1)$$

and

$$\beta_n \geq \beta_{n-1} \geq ... \geq \beta_1 \geq \beta_0 > 0$$

then all the zeros of $P(z)$ lie in

$$\left| z - \frac{\mu}{a_n} \right| \leq \frac{1}{|a_n|}\left[2\alpha_\lambda + \mu - \alpha_n + (|\alpha_0| - \alpha_0) + |\beta_n|\right]. \qquad (2)$$

Take $\lambda = n$, $\mu = (k-1)\alpha_n$, $\alpha_0 > 0$ and $\beta_j = 0$ for $0 \leq j \leq n$ in Corollary 2.2 the hypothesis becomes

$$k\alpha_n \geq \alpha_{n-1} \geq ... \geq \alpha_1 \geq \alpha_0 > 0$$

therefore we obtain the main result of Aziz and Zargar[9] which in turn is an extension of Theorem 1.3(Enestrom-Kakeya)."

Though the first statement is correct if k is replaced by μ, $0 \le \mu < 1$, ρ is removed from the condition and (1) is replaced by

$$\alpha_n - \mu \le \alpha_{n-1} \le ... \le \alpha_{\lambda+1} \le \alpha_\lambda \ge \alpha_{\lambda-1} \ge ... \ge \alpha_0$$

the second statement is totally wrong. Since for no values of λ and μ will (1) reduce to (2).

2. MAIN RESULT

In this paper, we give the correct statements and proofs of these theorems.

Theorem 2.1. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients. If $Re(a_j) = \alpha_j$, $Im(a_j) = \beta_j$ for $0 \le j \le n$ such that for some real $t > 0$, $\mu: \quad 0 \le \mu < 1$, $\lambda: 0 \le \lambda \le n-1$,

$$t^n\alpha_n - \mu t^{n-1} \le t^{n-1}\alpha_{n-1} \le ... \le t^{\lambda+1}\alpha_{\lambda+1} \le t^\lambda\alpha_\lambda \ge t^{\lambda-1}\alpha_{\lambda-1} \ge ... \ge \alpha_0$$

then all the zeros of $P(z)$ lie in

$$\left|z - \frac{\mu}{a_n}\right| \le \frac{1}{|a_n|}\left[\frac{2\alpha_\lambda}{t^{n-\lambda-1}} + \mu - \alpha_n t + \frac{(|\alpha_0| - \alpha_0) + |\beta_0|}{t^{n-1}} + \sum_{j=1}^{n} \frac{|\beta_j t - \beta_{j-1}|}{t^{n-j}}\right].$$

Theorem 2.2. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients. If $Re(a_j) = \alpha_j$, $Im(a_j) = \beta_j$ for $0 \le j \le n$ such that for some real $t > 0$, $\mu: \quad 0 \le \mu < 1$, $\lambda: 0 \le \lambda \le n-1$ and $\rho: \quad 0 < \rho \le 1$

$$t^n\alpha_n - \mu t^{n-1} \le t^{n-1}\alpha_{n-1} \le ... \le t^{\lambda+1}\alpha_{\lambda+1} \le t^\lambda\alpha_\lambda \ge t^{\lambda-1}\alpha_{\lambda-1} \ge ... \ge \rho\alpha_0$$

then all the zeros of $P(z)$ lie in

$$\left|z - \frac{\mu}{a_n}\right| \le \frac{1}{|a_n|}\left[\frac{2\alpha_\lambda}{t^{n-\lambda-1}} + \mu - \alpha_n t + \frac{2|\alpha_0| - \rho(|\alpha_0| + \alpha_0) + |\beta_0}{t^{n-1}} + \sum_{j=1}^{n} \frac{|\beta_j t - \beta_{j-1}|}{t^{n-j}}\right].$$

Theorem 2.1. is a special case of Theorem 2.2. by taking $\rho = 1$. So, we proceed to prove Theorem 2.2.

Proof of Theorem 2.2.:
Consider the polynomial

$$
\begin{aligned}
F(z) &= (t-z)P(z) \\
&= a_0 t + \sum_{j=1}^{n}(ta_j - a_{j-1})z^j - a_n z^{n+1} \\
&= -a_n z^{n+1} + \sum_{j=1}^{n}(t\alpha_j - \alpha_{j-1})z^j + t\alpha_0 \\
&+ \iota[t\beta_0 + \sum_{j=1}^{n}(t\beta_j - \beta_{j-1})z^t] \\
&= -a_n z^{n+1} + (t\alpha_n - \alpha_{n-1})z^n + (t\alpha_{n-1} - \alpha_{n-2})z^{n-1} \\
&+ ... + (t\alpha_{\lambda+1} - \alpha_\lambda)z^{\lambda+1} + (t\alpha_\lambda - \alpha_{\lambda-1})z^\lambda + (t\alpha_{\lambda-1} - \alpha_{\lambda-2})z^{\lambda-1} \\
&+ ... + (t\alpha_1 - \alpha_0)z + t\alpha_0 + \iota[t\beta_0 + \sum_{j=1}^{n}(t\beta_j - \beta_{j-1})z^t] \\
&= -a_n z^{n+1} + (t\alpha_n - \mu - \alpha_{n-1})z^n + \mu z^n + (t\alpha_{n-1} - \alpha_{n-2})z^{n-1} \\
&+ ... + (t\alpha_{\lambda+1} - \alpha_\lambda)z^{\lambda+1} + (t\alpha_\lambda - \alpha_{\lambda-1})z^\lambda + (t\alpha_{\lambda-1} - \alpha_{\lambda-2})z^{\lambda-1} \\
&+ ... + (t\alpha_1 - \rho\alpha_0)z + (\rho\alpha_0 - \alpha_0)z + t\alpha_0 \\
&+ \iota[t\beta_0 + \sum_{j=1}^{n}(t\beta_j - \beta_{j-1})z^t].
\end{aligned}
$$

Now, for $|z| \leq t$ so that $\frac{1}{|z|^j} \geq \frac{1}{t^j}, 1 \leq j \leq n$, we have

$$
\begin{aligned}
|F(z)| = |&- a_n z^{n+1} + (t\alpha_n - \mu - \alpha_{n-1})z^n + \mu z^n + \\
&(t\alpha_{n-1} - \alpha_{n-2})z^{n-1} + ... + (t\alpha_{\lambda+1} - \alpha_\lambda)z^{\lambda+1} \\
&+ (t\alpha_\lambda - \alpha_{\lambda-1})z^\lambda + (t\alpha_{\lambda-1} - \alpha_{\lambda-2})z^{\lambda-1} + ... \\
&+ (t\alpha_1 - \rho\alpha_0)z + (\rho\alpha_0 - \alpha_0)z + t\alpha_0
\end{aligned}
$$

$$
\begin{aligned}
&+ \iota[t\beta_0 + \sum_{j=1}^{n}(t\beta_j - \beta_{j-1})z^t]| \\
&\geq |z|^n|a_nz - \mu| - |z|^n\Bigg(|t\alpha_n - \mu - \alpha_{n-1}| + \frac{|t\alpha_{n-1} - \alpha_{n-2}|}{|z|} \\
&+ ... + \frac{|t\alpha_{\lambda+1} - \alpha_\lambda|}{|z|^{n-\lambda-1}} + \frac{|t\alpha_\lambda - \alpha_{\lambda-1}|}{|z|^{n-\lambda}} + ... + \frac{|t\alpha_1 - \rho\alpha_0|}{|z|^{n-1}} \\
&+ \frac{|\rho\alpha_0 - \alpha_0|}{|z|^{n-1}} + \frac{|t\alpha_0|}{|z|^n} + \frac{|t\beta_0|}{|z|^n} + \sum_{j=1}^{n}\frac{|t\beta_j - \beta_{j-1}|}{|z|^{n-j}}\Bigg) \\
&\geq |z|^n|a_nz - \mu| - |z|^n\Bigg(\alpha_{n-1} - t\alpha_n + \mu + \frac{\alpha_{n-2}}{t} - \alpha_{n-1} + \\
&... + \frac{\alpha_\lambda}{t^{n-\lambda-1}} - \frac{\alpha_{\lambda+1}}{t^{n-\lambda-2}} + \frac{\alpha_\lambda}{t^{n-\lambda-1}} - \frac{\alpha_{\lambda-1}}{t^{n-\lambda}} + ... + \frac{\alpha_1}{t^{n-2}} \\
&- \frac{\rho\alpha_0}{t^{n-1}} + \frac{(1-\rho)|\alpha_0|}{t^{n-1}} + \frac{t|\alpha_0|}{t^n} + \frac{t|\beta_0|}{t^n} + \sum_{j=1}^{n}\frac{|t\beta_j - \beta_{j-1}|}{t^{n-j}}\Bigg) \\
&= |z|^n\Bigg[|a_nz - \mu| - \Bigg(\frac{2\alpha_\lambda}{t^{n-\lambda-1}} + \mu - t\alpha_n + \frac{2|\alpha_0|}{t^{n-1}} - \\
&\frac{\rho(|\alpha_0| + \alpha_0)}{t^{n-1}} + \frac{|\beta_0|}{t^{n-1}} + \sum_{j=1}^{n}\frac{|t\beta_j - \beta_{j-1}|}{t^{n-j}}\Bigg)\Bigg] \\
&> 0
\end{aligned}
$$

if

$$
\begin{aligned}
|a_nz - \mu| > \frac{2\alpha_\lambda}{t^{n-\lambda-1}} + \mu - t\alpha_n + \frac{2|\alpha_0|}{t^{n-1}} - \frac{\rho(|\alpha_0| + \alpha_0)}{t^{n-1}} + \frac{|\beta_0|}{t^{n-1}} \\
+ \sum_{j=1}^{n}\frac{|t\beta_j - \beta_{j-1}|}{t^{n-j}}
\end{aligned}
$$

that is, if

$$
\begin{aligned}
\left|z - \frac{\mu}{a_n}\right| > \frac{1}{|a_n|}\Bigg[\frac{2\alpha_\lambda}{t^{n-\lambda-1}} + \mu - t\alpha_n + \frac{2|\alpha_0|}{t^{n-1}} - \frac{\rho(|\alpha_0| + \alpha_0)}{t^{n-1}} + \frac{|\beta_0|}{t^{n-1}} \\
+ \sum_{j=1}^{n}\frac{|t\beta_j - \beta_{j-1}|}{t^{n-j}}\Bigg].
\end{aligned}
$$

This shows that those zeros of $F(z)$ whose modulus is greater than or equal to t lie in

$$\left|z-\frac{\mu}{a_n}\right| \leq \frac{1}{|a_n|}\left[\frac{2\alpha_\lambda}{t^{n-\lambda-1}}+\mu-t\alpha_n+\frac{2|\alpha_0|}{t^{n-1}}-\frac{\rho(|\alpha_0|+\alpha_0)}{t^{n-1}}+\frac{|\beta_0|}{t^{n-1}}\right.$$
$$\left.+\sum_{j=1}^{n}\frac{|t\beta_j-\beta_{j-1}|}{t^{n-j}}\right].$$

Since those zeros of $F(z)$ whose modulus is less than t already satisfy the above inequality and also all the zeros of $P(z)$ are also the zeros of $F(z)$, it follows that all the zeros of $P(z)$ lie in

$$\left|z-\frac{\mu}{a_n}\right| \leq \frac{1}{|a_n|}\left[\frac{2\alpha_\lambda}{t^{n-\lambda-1}}+\mu-t\alpha_n+\frac{2|\alpha_0|}{t^{n-1}}-\frac{\rho(|\alpha_0|+\alpha_0)}{t^{n-1}}+\frac{|\beta_0|}{t^{n-1}}\right.$$
$$\left.+\sum_{j=1}^{n}\frac{|t\beta_j-\beta_{j-1}|}{t^{n-j}}\right]$$

and the proof of Theorem 2.2. is complete. □

Supposing that the imaginary parts of the coefficients are monotonic and positive in Theorem 2.1. and Theorem 2.2. respectively, we get the following results.

Corollary 2.3. Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with complex coefficients. If $Re(a_j)=\alpha_j,\quad Im(a_j)=\beta_j$ for $0\leq j\leq n$ are such that for some real $t>0,\quad \mu:\quad 0\leq\mu<1,\quad \lambda:\quad 0\leq\lambda\leq n-1,$

$$t^n\alpha_n-\mu t^{n-1}\leq t^{n-1}\alpha_{n-1}\leq ...\leq t^{\lambda+1}\alpha_{\lambda+1}\leq t^\lambda\alpha_\lambda\geq t^{\lambda-1}\alpha_{\lambda-1}$$
$$\geq ...\geq\alpha_0$$
$$\beta_n\geq\beta_{n-1}\geq ...\geq\beta_1\geq\beta_0>0,$$

then all the zeros of $P(z)$ lie in

$$\left|z-\frac{\mu}{a_n}\right| \leq \frac{1}{|a_n|}\left[\frac{2\alpha_\lambda}{t^{n-\lambda-1}}+\mu-\alpha_nt+\frac{(|\alpha_0|-\alpha_0)+\beta_n}{t^{n-1}}\right].$$

Corollary 2.4. Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with complex coefficients. If $Re(a_j)=\alpha_j,\quad Im(a_j)=\beta_j$ for $0\leq j\leq n$ are such that for some real $t>0,\quad \mu:\quad 0\leq\mu<1,\quad \lambda:\quad 0\leq\lambda\leq n-1$ and $\rho:\quad 0<\rho\leq 1$

$$t^n\alpha_n-\mu t^{n-1}\leq t^{n-1}\alpha_{n-1}\leq ...\leq t^{\lambda+1}\alpha_{\lambda+1}\leq t^\lambda\alpha_\lambda\geq t^{\lambda-1}\alpha_{\lambda-1}$$
$$\geq ...\geq\rho\alpha_0$$
$$\beta_n\geq\beta_{n-1}\geq ...\geq\beta_1\geq\beta_0>0,$$

then all the zeros of $P(z)$ lie in

$$\left|z-\frac{\mu}{a_n}\right| \le \frac{1}{|a_n|}\left[\frac{2\alpha_\lambda}{t^{n-\lambda-1}}+\mu-\alpha_n t+\frac{2|\alpha_0|-\rho(|\alpha_0|+\alpha_0)+\beta_n}{t^{n-1}}\right].$$

Taking $t=1$ in Corollary 2.3. and Corollary 2.4., we get the following results.

Corollary 2.5. Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients. If $Re(a_j)=\alpha_j, \quad Im(a_j)=\beta_j$ for $0\le j\le n$ are such that for some real $\mu: \quad 0\le\mu<1, \quad \lambda: \quad 0\le \lambda\le n-1$,

$$\alpha_n-\mu\le\alpha_{n-1}\le ...\le\alpha_{\lambda+1}\le\alpha_\lambda\ge\alpha_{\lambda-1}\ge ...\ge\alpha_0$$
$$\beta_n\ge\beta_{n-1}\ge ...\ge\beta_1\ge\beta_0>0,$$

then all the zeros of $P(z)$ lie in

$$\left|z-\frac{\mu}{a_n}\right| \le \frac{1}{|a_n|}\left[2\alpha_\lambda+\mu-\alpha_n+(|\alpha_0|-\alpha_0)+\beta_n\right].$$

Corollary 2.6. Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients. If $Re(a_j)=\alpha_j, \quad Im(a_j)=\beta_j$ for $0\le j\le n$ are such that for some real $\mu: \quad 0\le\mu<1, \quad \lambda: \quad 0\le \lambda\le n-1$ and $\rho: \quad 0<\rho\le 1$

$$\alpha_n-\mu\le\alpha_{n-1}\le ...\le\alpha_{\lambda+1}\le\alpha_\lambda\ge\alpha_{\lambda-1}\ge ...\ge\rho\alpha_0$$
$$\beta_n\ge\beta_{n-1}\ge ...\ge\beta_1\ge\beta_0>0,$$

then all the zeros of $P(z)$ lie in

$$\left|z-\frac{\mu}{a_n}\right| \le \frac{1}{|a_n|}\left[2\alpha_\lambda+\mu-\alpha_n+2|\alpha_0|-\rho(|\alpha_0|+\alpha_0)+\beta_n\right].$$

Taking $\mu=0$ in Theorem 2.1. and Theorem 2.2., we get the following results.

Corollary 2.7. Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients. If $Re(a_j)=\alpha_j, \quad Im(a_j)=\beta_j$ for $0\le j\le n$ such that for some real $t>0, \quad \lambda: \quad 0\le\lambda\le n-1$,

$$t^n\alpha_n\le t^{n-1}\alpha_{n-1}\le ...\le t^{\lambda+1}\alpha_{\lambda+1}\le t^\lambda\alpha_\lambda\ge t^{\lambda-1}\alpha_{\lambda-1}\ge ...\ge\alpha_0$$

then all the zeros of $P(z)$ lie in

$$|z|\le\frac{1}{|a_n|}\left[\frac{2\alpha_\lambda}{t^{n-\lambda-1}}-\alpha_n t+\frac{(|\alpha_0|-\alpha_0)+|\beta_0|}{t^{n-1}}+\sum_{j=1}^{n}\frac{|\beta_j t-\beta_{j-1}|}{t^{n-j}}\right].$$

Corollary 2.8. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients. If $Re(a_j) = \alpha_j, \quad Im(a_j) = \beta_j$ for $0 \le j \le n$ such that for some real $t > 0, \quad \lambda : \quad 0 \le \lambda \le n-1$ and $\rho : \quad 0 < \rho \le 1$

$$t^n\alpha_n \le t^{n-1}\alpha_{n-1} \le ... \le t^{\lambda+1}\alpha_{\lambda+1} \le t^{\lambda}\alpha_{\lambda} \ge t^{\lambda-1}\alpha_{\lambda-1} \ge ... \ge \rho\alpha_0$$

then all the zeros of $P(z)$ lie in

$$|z| \le \frac{1}{|a_n|}[\frac{2\alpha_\lambda}{t^{n-\lambda-1}} - \alpha_n t + \frac{2|\alpha_0| - \rho(|\alpha_0| + \alpha_0) + |\beta_0}{t^{n-1}} + \sum_{j=1}^{n} \frac{|\beta_j t - \beta_{j-1}|}{t^{n-j}}].$$

Next we prove the following result from which we can easily get Theorems 2.1 and 2.2.

Theorem 2.9. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients. If $Re(a_j) = \alpha_j, \quad Im(a_j) = \beta_j$ for $0 \le j \le n$ such that for some real $t > 0, \quad \mu : \quad 0 \le \mu < 1, \quad \lambda : 0 \le \lambda \le n-1$,

$$t^n\alpha_n - \mu t^{n-1} \le t^{n-1}\alpha_{n-1} \le ... \le t^{\lambda+1}\alpha_{\lambda+1} \le t^{\lambda}\alpha_{\lambda},$$

then all the zeros of $P(z)$ lie in

$$\left|z - \frac{\mu}{a_n}\right| \le \frac{1}{|a_n|}\left[\frac{\alpha_\lambda}{t^{n-\lambda-1}} + \mu - \alpha_n t + \frac{|\alpha_0| + |\beta_0|}{t^{n-1}} + L + M\right]$$

where $L = \sum_{j=1}^{\lambda} \frac{|\alpha_j t - \alpha_{j-1}|}{t^{n-j}}$ and $M = \sum_{j=1}^{n} \frac{|\beta_j t - \beta_{j-1}|}{t^{n-j}}$.

Proof of Theorem 2.9. Consider the polynomial

$$\begin{aligned} F(z) &= (t-z)P(z) \\ &= a_0 t + \sum_{j=1}^{n}(ta_j - a_{j-1})z^j - a_n z^{n+1} \\ &= -a_n z^{n+1} + \sum_{j=1}^{n}(t\alpha_j - \alpha_{j-1})z^j + t\alpha_0 + \iota[t\beta_0 + \sum_{j=1}^{n}(t\beta_j - \beta_{j-1})z^j] \end{aligned}$$

$$= -a_n z^{n+1} + (t\alpha_n - \alpha_{n-1})z^n + (t\alpha_{n-1} - \alpha_{n-2})z^{n-1} + ...$$
$$+ (t\alpha_{\lambda+1} - \alpha_\lambda)z^{\lambda+1} + (t\alpha_\lambda - \alpha_{\lambda-1})z^\lambda + (t\alpha_{\lambda-1} - \alpha_{\lambda-2})z^{\lambda-1} + ...$$
$$+ (t\alpha_1 - \alpha_0)z + t\alpha_0 + \iota[t\beta_0 + \sum_{j=1}^{n}(t\beta_j - \beta_{j-1})z^j]$$
$$= -a_n z^{n+1} + (t\alpha_n - \mu - \alpha_{n-1})z^n + \mu z^n +$$
$$(t\alpha_{n-1} - \alpha_{n-2})z^{n-1} + ... + (t\alpha_{\lambda+1} - \alpha_\lambda)z^{\lambda+1}$$
$$+ \sum_{j=1}^{\lambda}(t\alpha_j - \alpha_{j-1})z^j + t\alpha_0 + \iota[t\beta_0 + \sum_{j=1}^{n}(t\beta_j - \beta_{j-1})z^j].$$

Now, for $|z| \leq t$ so that $\frac{1}{|z|^j} \geq \frac{1}{t^j}, 1 \leq j \leq n$, we have

$$|F(z)| = | - a_n z^{n+1} + (t\alpha_n - \mu - \alpha_{n-1})z^n + \mu z^n +$$
$$(t\alpha_{n-1} - \alpha_{n-2})z^{n-1} + ... + (t\alpha_{\lambda+1} - \alpha_\lambda)z^{\lambda+1}$$
$$+ \sum_{j=1}^{\lambda}(t\alpha_j - \alpha_{j-1})z^j + t\alpha_0 +$$
$$\iota[t\beta_0 + \sum_{j=1}^{n}(t\beta_j - \beta_{j-1})z^j]|$$
$$\geq |z|^n|a_n z - \mu| - |z|^n\Bigg(|t\alpha_n - \mu - \alpha_{n-1}| + \frac{|t\alpha_{n-1} - \alpha_{n-2}|}{|z|}$$
$$+ ... + \frac{|t\alpha_{\lambda+1} - \alpha_\lambda|}{|z|^{n-\lambda-1}} + \frac{|t\alpha_0|}{|z|^n} + \frac{|t\beta_0|}{|z|^n} + \sum_{j=1}^{\lambda}\frac{|t\alpha_j - \alpha_{j-1}|}{|z|^{n-j}} +$$
$$\sum_{j=1}^{n}\frac{|t\beta_j - \beta_{j-1}|}{|z|^{n-j}}\Bigg)$$
$$\geq |z|^n|a_n z - \mu| - |z|^n\Bigg(\alpha_{n-1} - t\alpha_n + \mu + \frac{\alpha_{n-2}}{t} - \alpha_{n-1}$$
$$+ ... + \frac{\alpha_\lambda}{t^{n-\lambda-1}} - \frac{\alpha_{\lambda+1}}{t^{n-\lambda-2}} + \frac{t|\alpha_0|}{t^n} + \frac{t|\beta_0|}{t^n} + \sum_{j=1}^{\lambda}\frac{|t\alpha_j - \alpha_{j-1}|}{|z|^{n-j}}$$
$$+ \sum_{j=1}^{n}\frac{|t\beta_j - \beta_{j-1}|}{|z|^{n-j}}\Bigg)$$

$$= |z|^n \Bigg[|a_n z - \mu| - \Bigg(\frac{\alpha_\lambda}{t^{n-\lambda-1}} + \mu - t\alpha_n + \frac{|\alpha_0|}{t^{n-1}} + \frac{|\beta_0|}{t^{n-1}}$$
$$+ \sum_{j=1}^{\lambda} \frac{|t\alpha_j - \alpha_{j-1}|}{t^{n-j}} + \sum_{j=1}^{n} \frac{|t\beta_j - \beta_{j-1}|}{t^{n-j}} \Bigg) \Bigg]$$
$$= |z|^n \Bigg[|a_n z - \mu| - \Bigg(\frac{\alpha_\lambda}{t^{n-\lambda-1}} + \mu - t\alpha_n + \frac{|\alpha_0|}{t^{n-1}} + \frac{|\beta_0|}{t^{n-1}}$$
$$+ L + M \Bigg) \Bigg]$$
$$> 0$$

if

$$|a_n z - \mu| > \frac{\alpha_\lambda}{t^{n-\lambda-1}} + \mu - t\alpha_n + \frac{|\alpha_0|}{t^{n-1}} + \frac{|\beta_0|}{t^{n-1}} + L + M$$

that is, if

$$\left| z - \frac{\mu}{a_n} \right| > \frac{1}{|a_n|} \left[\frac{\alpha_\lambda}{t^{n-\lambda-1}} + \mu - t\alpha_n + \frac{|\alpha_0|}{t^{n-1}} + \frac{|\beta_0|}{t^{n-1}} + L + M \right].$$

This shows that those zeros of $F(z)$ whose modulus is greater than or equal to t lie in

$$\left| z - \frac{\mu}{a_n} \right| \le \frac{1}{|a_n|} \left[\frac{\alpha_\lambda}{t^{n-\lambda-1}} + \mu - t\alpha_n + \frac{|\alpha_0|}{t^{n-1}} + \frac{|\beta_0|}{t^{n-1}} + L + M \right].$$

Since those zeros of $F(z)$ whose modulus is less than t already satisfy the above inequality and also all the zeros of $P(z)$ are also the zeros of $F(z)$, it follows that all the zeros of $P(z)$ lie in

$$\left| z - \frac{\mu}{a_n} \right| \le \frac{1}{|a_n|} \left[\frac{\alpha_\lambda}{t^{n-\lambda-1}} + \mu - t\alpha_n + \frac{|\alpha_0|}{t^{n-1}} + \frac{|\beta_0|}{t^{n-1}} + L + M \right].$$

and the proof of Theorem 2.9. is complete. □

Remark 2.10. For

$$t^\lambda \alpha_\lambda \ge t^{\lambda-1}\alpha_{\lambda-1} \ge ... \ge \alpha_0$$

Theorem 2.9. reduces to Theorem 2.1. and for

$$t^\lambda \alpha_\lambda \ge t^{\lambda-1}\alpha_{\lambda-1} \ge ... \ge \rho\alpha_0$$

Theorem 2.9. reduces to Theorem 2.2.

If we take $t = 1$, $\lambda = 0$, Theorem 2.9 gives the following result:

Corollary 2.11. Let $P(z) = \sum\limits_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients. If $Re(a_j) = \alpha_j$, $Im(a_j) = \beta_j$ for

$0 \le j \le n$ such that for some real $\mu \ge 0$,

$$\alpha_n - \mu \le \alpha_{n-1} \le ... \le \alpha_1 \le \alpha_0,$$

then all the zeros of $P(z)$ lie in

$$\left| z - \frac{\mu}{a_n} \right| \le \frac{1}{|a_n|} \left[\mu - \alpha_n + |\alpha_0| + \alpha_0 + |\beta_0| + \sum_{j=1}^{n} |\beta_j - \beta_{j-1}| \right].$$

If in addition

$$\beta_n \ge \beta_{n-1} \ge ... \ge \beta_1 \ge \beta_0 > 0$$

Corollary 2.11. gives the following result.

Corollary 2.12. Let $P(z) = \sum\limits_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients. If $Re(a_j) = \alpha_j, \quad Im(a_j) = \beta_j$ for $0 \le j \le n$ such that for some real $\mu \ge 0$,

$$\alpha_n - \mu \le \alpha_{n-1} \le ... \le \alpha_1 \le \alpha_0$$
$$\beta_n \ge \beta_{n-1} \ge ... \ge \beta_1 \ge \beta_0 > 0,$$

then all the zeros of $P(z)$ lie in

$$\left| z - \frac{\mu}{a_n} \right| \le \frac{1}{|a_n|} \left[\mu - \alpha_n + |\alpha_0| + \alpha_0 + \beta_n \right].$$

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REFERENCES

[1] A. A Mogbademu, S. Hans, J.A. Adepoju, *Some extensions and generalizations of Enestrom-Kakeya Theorem*, journal of the Nigerian Mathematical Society **34** 243-248 2015.

[2] K. F. Gauss, *Beitrage zur theorie der algebraischen Gleichungen*, Abh Ges Wiss Gottingen **4** 73-102 1850.

[3] A. L. Cauchy, *Exercise de mathematique*, Oeuvres **2**(9) 122 1829.

[4] N. K. Govil, Q. I. Rahman, *On the Enestrom-Kakeya Theorem*, Tohoku Mat J, **20** 126-138 1968.

[5] A. Joyal, G. Labelle, Q. I. Rahman, *On the location of zeros of polynomials*, Canadian Math Bull **10** 55-63 1967.

[6] M. Marden, *Geometry of polynomials*, Math. Surveys ,vol.3,Providence,er. RI; Amer. Math. Soc. , 1949.

[7] G. V. Milovanovic, D. S. Mitrinovic, T. M. Rassias, *Topics in polynomials*, Extremal Properties, Inequalities, zeros, Singapore World Scientific Publishing Co. 1994.

[8] A. Aziz, B. A. Zargar, *Some extensions of Enestrom-Kakeya theorem*, Glasnik Mathematiki **31** 23-244 1996.
[9] N. K. Govil, G. N. Mctume, *Some extensions of Enestrom-Kakeya theorem*, Int. J. Appl. Math. **11** (3) 245-253 2002.
[10] Aziz A., Zargar B. A., *Bounds for the zeros of a polynomial with restricted coefficients*, Appl Math **3** 30-33 2012.
[11] N. Anderson, E. B. Saff, Verga R. S., *An extension of Enestrom-Kakeya theorem and its sharpness*, SIAM Math Anal **12** 10-22 1981.
[12] K. K. Dewan, N. K. Govil, *On the Enestrom-Kakeya theorem* , J.Approx Theory **42** 239-244 1984.
[13] K. K. Dewan, N. K. Govil, *On the Enestrom-Kakeya theorem* . J.Math Anal Appl **180** 29-36 1993.
[14] G. B. Gardener, N. K. Govil, *Some generalizations of the Enestrom-Kakeya theorem* , Acta Math Hungar **4** 125-134 1997.

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