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LEGENDRE COLLOCATION METHOD FOR LINEAR SECOND ORDER FREDHOLM VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. This paper discusses the development of a new numerical solution of second order linear Fredholm Volterra integro-differential equations by Legendre collocation method. The Fredholm Volterra integro-differential equation is first converted into integral equation and then transformed into linear algebraic equations which are then solved using matrix inversion method. Numerical solution shows that the method gives better accuracy than the existing methods.

Keywords and phrases: Linear Fredholm Volterra integro - differential equations, Integral equations, Legendre collocation, Second order problem
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1. INTRODUCTION

Integro-differential equations are important equations that have applications in the field of engineering, mechanics, physics, astronomy, potential theory and electrostatics among other areas. These equations are difficult to solve analytically, hence numerical approach are often applied(Irfan, Kumar and Kapoor [9]). Many numerical methods have been developed in recent years for the solution of integro-differential equations, such methods include: operational matrix approach (Yuzbasi and Ismailov [16], Rad, Kazem, Shaban and Parand [13], Akkaya and Yalcinbas [3], Derdik-Yaslan and Akyuz-Dascioglu [5], Akgonullu, Sahin and Sezer [4], Gulus and Sezer [8]), collocation method (Agbolade and Anake [2]), multistep method (Kajani and Gholampoor [11], Medhiyeva, Imanova and Ibrahimov [12]), spectral collocation method (Doha, Abdelkawy and Amin [6], Alipanah and Dehghan [1], El-Kady and Biomy [7]).

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In this paper, we develop a new Legendre collocation method for second order linear Fredholm Volterra integro-differential equation in the form

$$u''(t)+\alpha(t)u'(t)+\beta(t)u(t)=f(t)+\lambda_1\int_0^t k(t,y)u(y)\,dy$$
$$+\lambda_2\int_a^b w(t,y)u(y)\,dy \qquad (1)$$

subject to the initial condition

$$u(0)=u_0,\ u'(0)=u_1 \qquad (2)$$

where $\alpha(t)$, $\beta(t)$, $f(t)$, $k(t,y)$ and $w(t,y)$ are continuous real valued function on the interval $0\le t,y\le b$. λ_1 and λ_2 are given constants.

The remaining part of the paper is organized as follows: sections 2 and 3 considers the development of the method and the convergence analysis of the new method respectively. Section 4 gives numerical examples and discussion of results, finally we conclude in section 5

2. METHOD OF SOLUTION

This section discusses the development of the method, in this approach (1) is first converted to integral equation by integrating twice from 0 to t and using (2) to give

$$\begin{aligned} u(t) &= u_0+tu_1-\int_0^t\left[\int_0^t \alpha(y)u'(y)\,dy\right]dt-\int_0^t\left[\int_0^t \beta(y)u(y)\,dy\right]dt \\ &+\int_0^t\left[\int_0^t f(y)\,dy\right]dt+\int_0^t\int_0^t\left[\int_0^t k(t,y)u(y)\,dy\right]dtdt \\ &+\int_0^t\int_0^t\left[\int_a^b w(t,y)u(y)\,dy\right]dtdt \qquad (3)\end{aligned}$$

we then consider the approximate solution to (3) in the form

$$u_N(t)=\phi(t)T,\ N\in\mathbb{Z}^+ \qquad (4)$$

where $\phi(t)$ is the Legendre interpolating polynomial with the recurrence relation

$$\phi_{n+1}(t)=\frac{(2n+1)t\phi_n(t)-n\phi_{n-1}(t)}{n+1},n>1$$

$\phi_0(t)=1$, $\phi_1(t)=t$ and $T=\left[\begin{array}{cccc} t_0 & t_1 & \cdots & t_N\end{array}\right]^T$ are constants to be determined.

We substitute (4) into (3) to give

$$h(t)T=g(t) \qquad (5)$$

where

$$\begin{aligned} h(t) &= \phi(t) + \int_0^t \left[\int_0^t \alpha(y)\,\phi'(y)\,dy\right] dt + \int_0^t \left[\int_0^t \beta(y)\,\phi(y)\,dy\right] dt \\ &\quad - \int_0^t \int_0^t \left[\int_0^t k(t,y)\,\phi(y)\,dy\right] dtdt \\ &\quad - \int_0^t \int_0^t \left[\int_a^b w(t,y)\,\phi(y)\,dy\right] dtdt \end{aligned}$$

$$g(t) = u_0 + tu_1 + \int_0^t \left[\int_0^t f(y)\,dy\right] dt$$

We then collocate (5) using the standard collocation points

$$t_i = a + \frac{(b-a)\,i}{N} \tag{6}$$

to give

$$[h(t_i)]_{(N+1\times N+1)}\,T = [g(t_i)]_{(N+1\times 1)} \tag{7}$$

We then substitute (7) into (4) to get the numerical solution

3. CONVERGENCE ANALYSIS

In establishing the convergence of this method, we substitute the approximate solution into (1) to give

$$u_N''(t) + \alpha(t)\,u_N'(t) + \beta(t)\,u_N(t) = f(t) + \lambda_1 \int_0^t k(t,y)\,u_N(y)\,dy$$

$$+\lambda_2 \int_a^b w(t,y)\,u_N(y)\,dy \tag{8}$$

subtracting (1) from (8) and using $e_N(t) = (U_N(t) - U(t))$ which is the error gives

$$\frac{|e_N''(t) + \alpha(t)\,e_N'(t) + \beta(t)\,e_N(t)|}{|e_N(t)|} \le \left|\lambda_1 \int_0^t k(t,y)\,dy\right| + \left|\lambda_2 \int_a^b w(t,y)\,dy\right|$$

hence

$$\frac{\|e_N''(t)\|_\infty + |\alpha(t)|\,\|e_N'(t)\|_\infty + |\beta(t)|\,\|e_N(t)\|_\infty}{\|e_N(t)\|_\infty} \le |\lambda_1| \left|\int_0^t k(t,y)\,dy\right|$$

$$+\,|\lambda_2| \left|\int_a^b w(t,y)\,dy\right|$$

Since the error is bounded, then the method converges.

4. NUMERICAL EXAMPLES

In this section, three examples are solved to test the efficiency and simplicity of the new method. Results are given in Tables except where the numerical solution is the same as the exact solution. In the tables,we define $abs - e_N = |u_N(u) - u(t)|$ as the absolute error for N, which is the absolute difference between the numerical solution and the exact solution. All computations are done with the aids of program written in MATLAB(2015a) and run on a PC.

Rahman [14] Consider the second order problem

$$u''(t) = -8+6t-3t^2+t^3+\int_0^t u(y)\,dy+\int_{-1}^1 (1-2ty)\,u(y)\,dy \quad (9)$$

with the condition $u(0) = 2, u'(0) = 6$. Using the procedure in section 2, $\alpha(t) = 0$, $\beta(t) = 0$, $f(t) = -8+6x-3x^2+x^3$, $k(t,y) = 1$, $w(t,y) = 1-2ty$, $a = -1$, $b = 1$, $\lambda_1 = \lambda_1 = 1$. Converting (9) into integral equation gives

$$\begin{aligned} u(t) &= u_0 + tu'(0) + \int_0^t \int_0^t \left(\int_0^t \phi(y)\,dy\right) dtdt \\ &+ \int_0^t \int_0^t \left(\int_{-1}^1 (1-2ty)\,\phi(y)\,dy\right) dtdt \\ &+ \int_0^t \left(\int_0^t \left(-8+6y-3y^2+y^3\right) dy\right) dt \qquad (10) \end{aligned}$$

We solve (10) using the approximate

$$u_2(t) = \phi(t)\,T \quad (11)$$

where $\phi(t) = \left[\begin{array}{ccc} \phi_0(t) & \phi_1(t) & \phi_2(t) \end{array}\right]$, $T = \left[\begin{array}{ccc} t_0 & t_1 & t_2 \end{array}\right]^T$. Hence (5) is defined as

$$h(t) = \phi(t) - \int_0^t \int_0^t \left(\int_0^t \phi(y)\,dy\right) dtdt - \int_0^t \int_0^t \left(\int_{-1}^1 (1-2ty)\,\phi(y)\,dy\right) dtdt$$

$$g(t) = 2+6t++\int_0^t \left(\int_0^t \left(-8+6y-3y^2+y^3\right) dy\right) dt$$

using the collocation points $t_i = \left[\begin{array}{ccc} 0 & \frac{1}{2} & 1 \end{array}\right]$ in (5), then the parameters in (7) give

$$h(t_i) = \left[\begin{array}{ccc} 1 & 0 & \frac{-2}{3} \\ \frac{35}{48} & \frac{605}{1152} & \frac{-241}{1152} \\ \frac{-1}{6} & \frac{85}{72} & \frac{43}{36} \end{array}\right] \quad g(t_1) = \left[\begin{array}{ccc} 2 & \frac{2631}{640} & \frac{24}{5} \end{array}\right]^T$$

hence $T = \begin{bmatrix} \frac{4}{5} & 6 & \frac{-9}{5} \end{bmatrix}^T$. Substituting T into the (11) gives $u_2(t) = 2 + 6t^2 - 3t^2$ which is the exact solution

Yuzbasi and Ismailov [16], Rohaninasab, Maleknejad and Ezzati [15] consider a linear second order Fredholm Volterra integro differential equation

$$\begin{aligned} u''(t) + tu'(t) - tu(t) &= e^t - \sin(t) + \frac{1}{2}t\cos(t) \\ &+ \int_0^1 \sin(t)\, e^{-y} u(y)\, dy \qquad (12) \\ &- \frac{1}{2}\int_0^t \cos(t)\, e^{-y} u(y)\, dy \end{aligned}$$

with the initial condition $u(0) = 1$, $u'(0) = 1$. So, $\alpha(t) = t$, $\beta(t) = -t$, $f(t) = e^t - \sin(t) + \frac{1}{2}t\cos(t)$, $k(t,y) = \cos te^{-y}$, $\lambda_1 = \frac{-1}{2}$ $w(t,y) = \sin te^{-y}$, $a = 0$, $b = 1$, $\lambda_1 = 1$. Converting (12) to integral equation gives

$$\begin{aligned} u(t) &= 1 + t + \int_0^t \left(\int_0^t yu'(y)\, dy\right) dt \\ &+ \int_0^t \left(\int_0^t e^y - \sin(y) + \frac{1}{2}y\cos(y)\, dy\right) dt \qquad (13) \\ &- \int_0^t \left(\int_0^t yu(y)\, dy\right) dt \\ &+ \int_0^t \int_0^t \left(\int_0^1 \sin(t)\, e^{-y} u(y)\, dy\right) dtdt \\ &+ \int_0^t \int_0^t \left(\int_0^t \cos(t)\, e^{-y} u(y)\, dy\right) dtdt \end{aligned}$$

we solve the integral equation (13) for N =5 and 10, the numerical solutions are

$$u_5(t) = \left(1.3832 \times 10^{-2}\right) t^5 + \left(3.4918 \times 10^{-2}\right) t^4 + 0.17037t^3 + 0.49908t^2 + 1.0001t + 1.0$$

$$\begin{aligned} u_{10}(t) &= \left(4.5605 \times 10^{-7}\right) t^{10} + \left(2.2859 \times 10^{-6}\right) t^9 \\ &+ \left(2.5473 \times 10^{-5}\right) t^8 + \left(1.9782 \times 10^{-4}\right) t^7 \\ &+ \left(1.3892 \times 10^{-3}\right) t^6 + \left(8.3332 \times 10^{-3}\right) t^5 \\ &+ \left(4.1667 \times 10^{-2}\right) t^4 + 0.16667t^3 + 0.5t^2 + t + 1 \end{aligned}$$

The numerical solutions are given in Table 1. We compare our solution with the solution of operational matrix method developed by Yuzbasi and Ismailov [16], the new method gives better accuracy as shown in Table 1.

Table 1: Comparison of the exact solutions, numerical solution and absolute errors for (13).

	Exact	[16]	Present method			
t_i	e^{t_i}	abs-e_{10},	N=5	abs-e_5	N=10	abs-e_{10}
0.2	1.22140275816	7.70E-13	1.221402768261	1.01E-08	1.22140275816	1.15E-15
0.4	1.49182469764	4.11E-13	1.491824778711	8.10E-08	1.49182469764	1.86E-15
0.6	1.82211880039	8.66E-13	1.822118891491	9.11E-08	1.82211880039	2.53E-15
0.8	2.22554092849	5.57E-13	2.225541126502	1.98E-07	2.22554092849	3.00E-15
1.0	2.71828182845	2.52E-13	2.718281812838	1.56E-08	2.71828182845	4.47E-15

Yuzbasi, Sahin and Sezer [17] Consider the second order Volterra integro differential equation

$$u''(t)+tu'(u)-tu(t)=e^{t}+\frac{1}{2}t\cos(t)-\frac{1}{2}\int_{0}^{t}\cos(t)\,e^{-y}u(y)\,dy, \quad (14)$$

$0\leq x\leq 1$ with the condition $u(0)=1$, $u'(0)=1$. So, $\alpha(t)=t$, $\beta(t)=-t$, $g(t)=e^{t}+\frac{1}{2}t\cos t$, $\lambda_1=\frac{-1}{2}$, $k(t,y)=\cos te^{-y}$. We convert (14) into integral equation to give

$$\begin{aligned} u(t) &= 1+t-\int_{0}^{t}\int_{0}^{t}\left(yu'(y)\,dy\right)dt \\ &\quad +\int_{0}^{t}\int_{0}^{t}\left(yu(y)\,dy\right)dt \\ &\quad +\int_{0}^{t}\int_{0}^{t}\left(e^{y}+\frac{1}{2}y\cos y dy\right)dt \\ &\quad -\frac{1}{2}\int_{0}^{t}\int_{0}^{t}\int_{0}^{t}\left(\cos te^{-y}dy\right)dtdt \end{aligned} \quad (15)$$

We solve (15) for N=5 and 10, the numerical solutions are

$$\begin{aligned} u_5(t) &= \left(1.3832\times 10^{-2}\right)t^{5}+\left(3.4918\times 10^{-2}\right)t^{4}+0.17037t^{3} \\ &\quad +0.49908t^{2}+1.0001t+1.0 \end{aligned}$$

$$\begin{aligned} u_{10}(t) &= \left(4.5605\times 10^{-7}\right)t^{10}+\left(2.2859\times 10^{-6}\right)t^{9} \\ &\quad +\left(2.5473\times 10^{-5}\right)t^{8}+\left(1.9782\times 10^{-4}\right)t^{7} \\ &\quad +\left(1.3892\times 10^{-3}\right)t^{6}+\left(8.3332\times 10^{-3}\right)t^{5} \\ &\quad +\left(4.1667\times 10^{-2}\right)t^{4}+0.16667t^{3}+0.5t^{2}+t+1 \end{aligned}$$

We compared the solution of the new method with the operational method developed by Yuzbasi, Sahin and Sezer [17]. Tables 2 shows clearly that the new method gives better accuracy.

Table 2: Comparison of the exact solutions, numerical solutions and absolute errors for (15).

	Exact	[17]	Present method			
t_i	e^{t_i}	abs-e_{10}	N=5	abs-e_5	N=10	abs-e_{10}
0.2	1.22140275816	1.69e-013	1.221402769	1.18e-08	1.2214027581	1.63e-15
0.4	1.49182469764	3.67e-013	1.491824780	8.28e-08	1.4918246976	3.46e-15
0.6	1.82211880039	5.80e-013	1.822118887	8.74e-08	1.8221188003	5.42e-15
0.8	2.22554092849	9.47e-013	2.225541110	1.82e-07	2.2255409284	6.75e-15
1.0	2.71828182845	3.11e-011	2.718281775	5.26e-08	2.7182818284	1.86e-16

5. CONCLUDING REMARKS

We have presented a new collocation method for the solution of second order linear Fredholm Volterra integro-differential equations. Three numerical examples are presented to test the efficiency of this method. The approach adopted in this research has lesser computational burden when compared with the existing methods. Computer programs are easier to write with faster time of execution. Numerical solutions show that this method is accurate with good stability which is measured by the difference between the maximum and minimum absolute errors.

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