Journal of the
Nigerian Mathematical Society

Vol. 39, Issue 1, pp. 47-68, 2020

NUMERICAL APPROXIMATIONS OF FOURTH-ORDER PDES USING BLOCK UNIFICATION METHOD

M. I. MODEBEI[1], R. B. ADENIYI AND S. N, JATOR

ABSTRACT In this paper, a continuous linear multistep method is derived and used to formulate a block unification method (BUM), which is applied to solve fourth-order PDEs with appropriate initial and boundary conditions. Specifically, the method is used to solve the fourth order PDEs by first converting the PDEs into system of fourth-order ordinary differential equations (ODEs) via the method of lines, by replacing one of the variables with a finite difference method. The convergence properties of the method is discussed and some test problems are presented to demonstrate the accuracy of the method.

Keywords and phrases: Fourth-order PDEs; Block unification method, Linear Multistep method Boundary and Initial value problem

2010 Mathematical Subject Classification: 34K28, 65L06, 65L12

1. INTRODUCTION

Some physical processes in science and engineering are modeled as fourth-order partial differential equations (PDEs); such as the fourth-order time-dependent problem, the Cahn-Hilliard type equation, Boussinesq equation, the fourth-order KdV equation, the biharmonic problem, and the Kuramoto-Sivashinsky equation. It turns out that must of these equations cannot be solve analytically. Hence, numerical methods to solution of Partial differential equations (PDEs) are of great interest to numerical analyst in recent time, especially higher-order PDEs. This is as a result of the fact that most physical problems in real life sense occur in space and time, and are thus modeled as PDEs. Hence, PDEs with higher-order spatial derivatives, are considered in this paper. Wide range of numerical methods have been used to solve higher-order PDEs with Dirichlet and Neumann boundary conditions that

Received by the editors September 24, 2019; Revised December 30, 2019 ; Accepted: January 07, 2020

www.nigerianmathematicalsociety.org; Journal available online at https://ojs.ictp.it/jnms/

[1]Corresponding author

arise in engineering, physics and mathematics [8]. For example, the Euler-Bernoulli equation is an approximate equation for a long and slender beam, whose solution is a transverse displacement of the beam from an initially horizontal position. This forms a fourth-order parabolic PDE, and have been solved with the use of Sextic spline method, [1, 2]. A typical fourth-order parabolic PDE is described in [3] as follows:

$$y_{tt} + y_{xxxx} = G(x, t), \tag{1}$$

subject to appropriate boundary conditions.

The method of line was used in [3] to solve problems describing nonlinear wave phenomena, like the fourth-order "good" Boussinesq equation, Other method used include Quintic B-spline for the numerical solution of the "good" Boussinesq equation, B-splines methods with redefined basis functions for solving fourth-order parabolic partial differential equations, just to mention, but few References such [17]-[19] have generally studied parabolic PDEs. The standard form of the Boussinesq is described as in [14] as follows:

$$y_{tt} = y_{xx} + q y_{xxxx} + y_{xx}^2 \tag{2}$$

subject to appropriate boundary conditions, where $q = 1$ or -1. In [18] The "good" Boussinesq equation is studied numerically using an iterative implicit finite-difference scheme.

Two new two-level compact implicit variable mesh numerical methods of order two in time and space, and of order two in time, and three in space, was developed for the solution of 1D unsteady quasi-linear biharmonic problem, was discussed in [9], and is of the form

$$A(x, t, u, u_{xx}) \frac{\partial^4 u}{\partial x^4} + \frac{\partial u}{\partial t} = f(x, t, u, u_x, u_{xx}, u_{xxx}) \tag{3}$$

where $(x, t) \in \Omega \equiv \{(x, t) : a < x < b,\ t > 0\}$.

In this work, continuous linear multistep method (CLMM) is used to develop a block unification method (BUM) which is used to solve general fourth order linear and nonlinear PDEs with initial and boundary conditions via the method of line (MOL). The PDE considered is of the form;

$$y_{xxxx} = f(x, t, y_x, y_t, y_{xx}, y_{tt}, y_{xxx}, y_{ttt}) \tag{4}$$

with any of the initial-boundary conditions

$$\left.\begin{array}{l} y(x,0)=\phi(x),\ y_t(x,0)=\varphi(x),\\ y(\eta_0,t)=g_0(t),\ y(\eta_1,t)=g_1(t),\\ y_{xx}(\eta_0,t)=p_0(t),\ y_{xx}(\eta_1,t)=p_1(t),\\ y(x,0)=\phi(x),\ y_t(x,0)=\varphi(x),\\ y(\eta_0,t)=g_0(t),\ y(\eta_1,t)=g_1(t),\\ y_x(\eta_0,t)=p_0(t),\ y_x(\eta_1,t)=p_1(t), \end{array}\right\} \tag{5}$$

where $y(x,t)$ is the dependent variables, x and t are variables such that $\eta_0 \leq x \leq \eta_1$, η_1, η_2 are finite real numbers, $t \geq 0$, $\phi(x)$, $\varphi(x)$, $g_i(t)$, $p_i(t)$, $i = 0, 1$ are continuous functions. The subscript notation denotes partial derivatives, e.g. $y_t = \partial y/\partial t$; $y_x = \partial y/\partial x$, and so on.

In order to apply BUM to PDEs, the problem is first converted into system of ODEs via the method of lines. The method of lines approach is traditionally used for solving partial differential equations (PDEs), whereby the PDE is converted into a system of ODEs, by replacing the appropriate derivatives using finite difference approximations (see Ramos and Vigo-Aguiar [15], and Brugnano and Trigiante [16]). Using the approach in [13, 6], we demonstrate how (1) is converted into a system of ODEs and solved by the BUM. Thus, for real numbers L_1, L_2, L_3, L_4, and solution $y(x,t)$ of (1), where (x,t) is in the rectangle $[L_1, L_2] \times [L_3, L_4]$. The t variable is discretised with mesh spacing

$$\Delta t = \frac{L_4 - L_3}{M},\ t_m = L_3 + m\Delta t,\ m = 0, 1, \ldots, M$$

, and noting that

$$\Delta x = \frac{L_2 - L_1}{N},\ x_n = L_1 + n\Delta x,\ n = 0, 1, \ldots, N$$

with the vector

$$\mathbf{y} = [y_{1,1}, y_{1,2}, y_{2,1}, \ldots, y_{n-1,m-1}]^T$$

and

$$\mathbf{G} = [G_{1,1}, G_{1,2}, G_{2,1}, \ldots, G_{n-1,m-1}]^T$$

, where $y_m \approx y(x, t_m)$ and $G_m \approx G(x, t_m)$; using the central difference method yields

$$y_{tt}(x, t_m) \approx \frac{y(x, t_{m+1}) - 2y(x, t_m) + y(x, t_{m-1})}{(\Delta t)^2}.$$

Then (1) has the following semi-discretized form

$$\frac{dy_m^4}{dx^4} = -\left(\frac{y_{m+1} - 2y_m + y_{m-1}}{(\Delta t)^2}\right) + G_m \tag{6}$$

which can be written in the form

$$y^{(iv)} = f(x, y, y', y'', y''') = Ay + g \tag{7}$$

subject to the appropriate initial and boundary conditions
where $y^{(j)} = (y_{1,1}^{(j)}, y_{1,2}^{(j)}, y_{2,1}^{(j)}, \ldots, y_{n-1,m-1}^{(j)})^T$, $j = 1, 2, 3$, A is an $M - 1 \times M - 1$, matrix arising from the semi-discretized system (6) and solved by the BUM. Following the same procedure, we can rewrite the general form (5) as follows:

A continuous linear multistep method (CLMM) is derived via a block technique, which is used to formulate continuous block finite difference methods (CBFDMs), [5], using Chebyshev polynomials as basis functions. Thus, using multistep interpolation and collocation, a continuous FDM is derived and an additional $k-1$ methods which are assembled and solved simultaneously to obtain approximations $y_{n,m}$, for $n = 1, \ldots, N-1$ to solution the of (4) at points x_n, $n = 1, \ldots, N-1$, which is applied to solve fourth-order PDEs using the method of line.

2. DERIVATION OF THE METHODS

The exact solution $u_{n,m}(x, t)$ of (6) is approximated by seeking the continuous solution of the form

$$p(x) = \sum_{i=0}^{k=r+s-1} \rho_i T_i^*(x) \cong u(x) \tag{8}$$

with the 4^{th} derivative given as

$$p^{(iv)}(x) = \sum_{i=4}^{r+s-1} \rho_i T_i^{*(iv)}(x) \cong u^{(iv)}(x) \tag{9}$$

where $x \in [a, b]$, ρ_i's are constants to be determined. $T_i^*(x)$ in the interval $[x_n, x_{n+k}]$, $i = 0(1)r + s + 1$, are shifted Chebyshev polynomials in the interval $[0, k]$. The parameters r and s are respectively the number of interpolation points that satisfies $4 \le r \le \mu$ and the number of collocation points satisfying $0 < s \le \mu + 1$, μ is the order of the differential equation.
Interpolating (8) at the points x_{n+i}; $i = 0, 1, 2, \ldots, r-1$ and collocating (9) at the points x_{n+s}; $s = 0, 1, 2, \ldots, s-1$.

2.1 Specification of the Method

The method has the following specifications $k = 5$, $r = 4$, $s = 6$, $T_n^*(x_{n+j})$ for $x_{n+j} \in [x_n, x_{n+5}]$. yield the system noting that $u_{n+i,m} \simeq u(x_{n+i}, t_m)$ and $u^{(iv)}_{n+i,m} \simeq f_{n+i,m} = f(x_{n+i}, t_m, u_{n+i,m}, \ldots, u'''_{n+i,m})$. The interpolation (8) at the points x_{n+j} $j = 0(1)3$ and collocation (9) at the points x_{n+j} $j = 0(1)5$ yields a system which after solving, the values of the coefficients ρ_i, $i = 0, 1, \ldots 9$ are obtained. These values of are then substituted into (8) and after some simplification, the approximate polynomial (8) adopts the continuous form

$$p(x) = \sum_{i=0}^{3} \alpha_i(x) u_{n+i,m} + h^4 \sum_{i=0}^{5} \beta_i(x) f_{n+i,m} \tag{10}$$

where the α_i's, and β_i's are continuous coefficients expressed as functions of ξ and given as:

$$\left.\begin{array}{ll}
\alpha_0 = & \frac{1}{48}\left(3 + 5\xi - 75\xi^2 - 125\xi^3\right), \quad \alpha_1 = \frac{5}{16}\left(-1 - \xi + 25\xi^2 + 25\xi^3\right) \\
\alpha_2 = & -\frac{5}{16}\left(-3 + 7\xi + 35\xi^2 + 25\xi^3\right), \quad \alpha_3 = \frac{5}{48}\left(3 + 23\xi + 45\xi^2 + 25\xi^3\right) \\
\beta_0 = & \frac{1}{185794560}(36729 + 28779\xi - 1049700\xi^2 - 695500\xi^3 + 3543750\xi^4 \\
 & -708750\xi^5 - 6562500\xi^6 + 2812500\xi^7 + 3515625\xi^8 - 1953125\xi^9) \\
\beta_1 = & \frac{1}{37158912}(-430479 - 676299\xi + 10981980\xi^2 + 16840300\xi^3 - 5906250\xi^4 \\
 & +1968750\xi^5 + 10237500\xi^6 - 7312500\xi^7 - 2109375\xi^8 + 1953125\xi^9) \\
\beta_2 = & \frac{1}{18579456}(-533547 - 1478277\xi + 12637020\xi^2 + 37650500\xi^3 + 17718750\xi^4 \\
 & -17718750\xi^5 - 4462500\xi^6 + 9562500\xi^7 + 703125\xi^8 - 1953125\xi^9) \\
\beta_3 = & \frac{1}{18579456}(30933 - 129147\xi - 1474980\xi^2 + 2535100\xi^3 + 17718750\xi^4 \\
 & +17718750\xi^5 - 4462500\xi^6 - 9562500\xi^7 + 703125\xi^8 + 1953125\xi^9) \\
\beta_4 = & \frac{1}{37158912}(-27279 + 15051\xi + 901980\xi^2 - 309100\xi^3 - 5906250\xi^4 \\
 & -1968750\xi^5 + 10237500\xi^6 + 7312500\xi^7 - 2109375\xi^8 - 1953125\xi^9) \\
\beta_5 = & \frac{1}{185794560}(20601 - 1899\xi - 646500\xi^2 + 23500\xi^3 + 3543750\xi^4 \\
 & +708750\xi^5 - 6562500\xi^6 - 2812500\xi^7 + 3515625\xi^8 + 1953125\xi^9)
\end{array}\right\} \tag{11}$$

where $\xi = \frac{1}{h}\left(\frac{2x}{5} - 1\right)$

Evaluating $p(x)$ in (10) at the points $x = x_{n+4,m},\ \ x_{n+5,m}$, which implies $\xi = \frac{3}{5},\ \ 1$, the following 5-step discrete LMMs are obtained

$$\left.\begin{array}{ll}
u_{n+4,m} = & 4u_{n+3,m} - 6u_{n+2,m} + 4u_{n+1,m} - u_{n,m} + h^4\left(\frac{-1}{720} f_{n,m} + \frac{31}{180} f_{n+1,m} + \frac{79}{120} f_{n+2,m}\right. \\
 & \left. + \frac{31}{180} f_{n+3,m} - \frac{1}{720} f_{n+4,m}\right) \\
u_{n+5,m} = & -4u_{n,m} + 15u_{n+1,m} - 20u_{n+2,m} + 10u_{n+3,m} + h^4\left(-\frac{1}{180} f_{n,m} + \frac{11}{16} f_{n+1,m} + \frac{101}{36} f_{n+2,m}\right. \\
 & \left. + \frac{97}{72} f_{n+3,m} + \frac{1}{6} f_{n+4,m} - \frac{1}{720} f_{n+5,m}\right)
\end{array}\right\} \tag{12}$$

From the transformation $x = \frac{(5h\xi+5)}{2}$, $\frac{2}{5}dx = hd\xi$, (11) is differentiated three times respectively and on each differentiation, $p'(x)$, $p''(x)$ and $p'''(x)$ in (10) are evaluated at the points $x = x_{n+i,m}$,

$i = 0(1)5$, which implies $\xi = -1, -\frac{3}{5}, -\frac{1}{5}, \frac{1}{5}, \frac{3}{5}, 1$, the following 5-step discrete LMMs are obtained:

$$\left.\begin{aligned}
hu'_{n,m} =\ & -\tfrac{11u_{n,m}}{6} + 3u_{n+1,m} - \tfrac{3u_{n+2,m}}{2} + \tfrac{u_{n+3,m}}{3} - h^4\left(-\tfrac{937}{100800}f_{n,m} - \tfrac{19}{105}f_{n+1,m} - \tfrac{599}{10080}f_{n+2,m}\right.\\
& \left.-\tfrac{1}{720}f_{n+3} + \tfrac{3}{2240}f_{n+4} - \tfrac{1}{3600}f_{n+5}\right)\\
hu'_{n+1,m} =\ & -u_{n,m} - \tfrac{1}{2}u_{n+1,m} + u_{n+2,m} - \tfrac{1}{6}u_{n+3,m} + h^4\left(-\tfrac{1}{5400}f_{n,m} + \tfrac{2809}{60480}f_{n+1,m} + \tfrac{43}{1008}f_{n+2,m}\right.\\
& \left.-\tfrac{229}{30240}f_{n+3,m} + \tfrac{1}{432}f_{n+4,m} - \tfrac{11}{33600}f_{n+5,m}\right)\\
hu'_{n+2,m} =\ & \tfrac{1}{6}u_{n,m} - u_{n+1,m} + \tfrac{1}{2}u_{n+2,m} + \tfrac{1}{3}u_{n+3,m} + h^4\left(\tfrac{169}{302400}f_{n,m} - \tfrac{311}{10080}f_{n+1,m} - \tfrac{353}{6048}f_{n+2,m}\right.\\
& \left.+\tfrac{1}{135}f_{n+3,m} - \tfrac{7}{2880}f_{n+4,m} + \tfrac{53}{151200}f_{n+5,m}\right)\\
hu'_{n+3,m} =\ & -\tfrac{1}{3}u_{n,m} + \tfrac{3}{2}u_{n+1,m} - 3u_{n+2,m} + \tfrac{11}{6}u_{n+3,m} + h^4\left(-\tfrac{41}{50400}f_{n,m} + \tfrac{173}{2880}f_{n+1,m} + \tfrac{11}{60}f_{n+2,m}\right.\\
& \left.+\tfrac{61}{10080}f_{n+3,m} + \tfrac{17}{10080}f_{n+4,m} - \tfrac{11}{33600}f_{n+5,m}\right)\\
hu'_{n+4,m} =\ & -\tfrac{11}{6}u_{n,m} + 7u_{n+1,m} - \tfrac{19}{2}u_{n+2,m} + \tfrac{13}{3}u_{n+3,m} + h^4\left(-\tfrac{671}{302400}f_{n,m} + \tfrac{169}{540}f_{n+1,m} + \tfrac{12821}{10080}f_{n+2,m}\right.\\
& \left.+\tfrac{7447}{15120}f_{n+3,m} + \tfrac{509}{60480}f_{n+4,m} - \tfrac{1}{3600}f_{n+5,m}\right)\\
hu'_{n+5,m} =\ & -\tfrac{13}{3}u_{n,m} + \tfrac{31}{2}u_{n+1,m} - 19u_{n+2,m} + \tfrac{47}{6}u_{n+3,m} + h^4\left(-\tfrac{31}{5400}f_{n,m} + \tfrac{1663}{2240}f_{n+1,m} + \tfrac{6847}{2160}f_{n+2,m}\right.\\
& \left.+\tfrac{60863}{30240}f_{n+3,m} + \tfrac{2473}{5040}f_{n+4,m} + \tfrac{2041}{302400}f_{n+5,m}\right)
\end{aligned}\right\} \tag{13}$$

$$\left.\begin{aligned}
h^2u''_{n,m} &= 2u_{n,m} - 5u_{n+1,m} + 4u_{n+2,m} - u_{n+3,m} + h^2\left(\tfrac{1411}{20160}f_{n,m} + \tfrac{3091}{4320}f_{n+1,m} + \tfrac{2831}{30240}f_{n+2,m}\right.\\
&\quad \left.+\tfrac{143}{2520}f_{n+3,m} - \tfrac{1391}{60480}f_{n+4,m} + \tfrac{23}{6048}f_{n+5,m}\right)\\
h^2u''_{n+1,m} &= u_{n,m} - 2u_{n+1,m} + u_{n+2,m} + h^4\left(-\tfrac{73}{30240}f_{n,m} - \tfrac{1601}{20160}f_{n+1,m} + \tfrac{1}{7560}f_{n+2,m}\right.\\
&\quad \left.-\tfrac{11}{4320}f_{n+3,m} + \tfrac{11}{10080}f_{n+4,m} - \tfrac{11}{60480}f_{n+5,m}\right)\\
h^2u''_{n+2,m} &= u_{n+1,m} - 2u_{n+2,m} + u_{n+3,m} + h^4\left(\tfrac{11}{60480}f_{n,m} - \tfrac{53}{15120}f_{n+1,m} - \tfrac{773}{10080}f_{n+2,m}\right.\\
&\quad \left.-\tfrac{53}{15120}f_{n+3,m} + \tfrac{11}{60480}f_{n+4,m}\right)\\
h^2u''_{n+3,m} &= -u_{n,m} + 4u_{n+1,m} - 5u_{n+2,m} + 2u_{n+3,m} + h^4\left(-\tfrac{1}{720}f_{n,m} + \tfrac{10427}{60480}f_{n+1,m} + \tfrac{9901}{15120}f_{n+2,m}\right.\\
&\quad \left.+\tfrac{107}{1120}f_{n+3,m} - \tfrac{37}{7560}f_{n+4,m} + \tfrac{11}{60480}f_{n+5,m}\right)\\
h^2u''_{n+4,m} &= -2u_{n,m} + 7u_{n+1,m} - 8u_{n+2,m} + 3u_{n+3,m} + h^4\left(-\tfrac{179}{60480}f_{n,m} + \tfrac{3469}{10080}f_{n+1,m} + \tfrac{6421}{4320}f_{n+2,m}\right.\\
&\quad \left.+\tfrac{3791}{3780}f_{n+3,m} + \tfrac{121}{1344}f_{n+4,m} - \tfrac{23}{6048}f_{n+5,m}\right)\\
h^2u''_{n+5,m} &= -3u_{n,m} + 10u_{n+1,m} - 11u_{n+2,m} + 4u_{n+3,m} + h^4\left(-\tfrac{11}{30240}f_{n,m} + \tfrac{29689}{60480}f_{n+1,m} + \tfrac{499}{210}f_{n+2,m}\right.\\
&\quad \left.+\tfrac{58271}{30240}f_{n+3,m} + \tfrac{4561}{4320}f_{n+4,m} + \tfrac{271}{4032}f_{n+5,m}\right)
\end{aligned}\right\} \tag{14}$$

$$\left.\begin{aligned}
h^3u'''_{n,m} &= -u_{n,m} + 3u_{n+1,m} - 3u_{n+2,m} + u_{n+3,m} - h^3\left(-\frac{19151}{60480}f_{n,m} - \frac{73967}{60480}f_{n+1,m} + \frac{1261}{6048}f_{n+2,m}\right.\\
&\quad \left.-\frac{7439}{30240}f_{n+3,m} + \frac{5549}{60480}f_{n+4,m} - \frac{883}{60480}f_{n+5,m}\right)\\
h^3u'''_{n+1,m} &= -u_{n,m} + 3u_{n+1,m} - 3u_{n+2,m} + u_{n+3,m} + h^4\left(\frac{799}{60480}f_{n,m} - \frac{14033}{60480}f_{n+1,m} - \frac{10453}{30240}f_{n+2,m}\right.\\
&\quad \left.+\frac{2683}{30240}f_{n+3,m} - \frac{1717}{60480}f_{n+4,m} + \frac{251}{60480}f_{n+5,m}\right)\\
h^3u'''_{n+2,m} &= -u_{n,m} + 3u_{n+1,m} - 3u_{n+2,m} + u_{n+3,m} + h^4\left(-\frac{67}{12096}f_{n,m} + \frac{12721}{60480}f_{n+1,m} + \frac{11009}{30240}f_{n+2,m}\right.\\
&\quad \left.-\frac{547}{6048}f_{n+3,m} + \frac{1517}{60480}f_{n+4,m} - \frac{211}{60480}f_{n+5,m}\right)\\
h^3u'''_{n+3,m} &= -u_{n,m} + 3u_{n+1,m} - 3u_{n+2,m} + u_{n+3,m} + h^4\left(\frac{127}{60480}f_{n,m} + \frac{1763}{12096}f_{n+1,m} + \frac{27851}{30240}f_{n+2,m}\right.\\
&\quad \left.+\frac{14107}{30240}f_{n+3,m} - \frac{2389}{60480}f_{n+4,m} + \frac{251}{60480}f_{n+5,m}\right)\\
h^3u'''_{n+4,m} &= -u_{n,m} + 3u_{n+1,m} - 3u_{n+2,m} + u_{n+3,m} + h^4\left(-\frac{67}{12096}f_{n,m} + \frac{12049}{60480}f_{n+1,m} + \frac{22433}{30240}f_{n+2,m}\right.\\
&\quad \left.+\frac{35569}{30240}f_{n+3,m} + \frac{4873}{12096}f_{n+4,m} - \frac{883}{60480}f_{n+5,m}\right)\\
h^3u'''_{n+5,m} &= -u_{n,m} + 3u_{n+1,m} - 3u_{n+2,m} + u_{n+3,m} + h^4\left(\frac{799}{60480}f_{n,m} + \frac{4783}{60480}f_{n+1,m} + \frac{6511}{6048}f_{n+2,m}\right.\\
&\quad \left.+\frac{18811}{30240}f_{n+3,m} + \frac{84299}{60480}f_{n+4,m} + \frac{19067}{60480}f_{n+5,m}\right)
\end{aligned}\right\} \tag{15}$$

The formulae (13),(14) and (15) are the additional methods and are considered for $n = 0, 5, \ldots, N-5$.

Remark 1: The formulae (12),(13),(14) and (15) together form the Block Unification Method (BUM) which is used to solve (5) numerically with the appropriate boundary conditions.

3. ANALYSIS OF THE METHODS

3.1 Order of a Method

The formulae

$$u(x) = \sum_{j=0}^{k_1} \alpha_j(x) u_{n+j,m} + h^4 \sum_{j=0}^{k_2} \beta_j(x) f_{n+j,m} \tag{16}$$

is associated with the linear difference operator $\mathcal{L}[y(x);\ h]$ defined by

$$\mathcal{L}[u(x);\ h] \equiv h^l u(x+jh,t) - \left[\sum_{j=0}^{k_1} \alpha_j u(x+jh,t) + h^4 \sum_{j=0}^{k_2} \beta_j u^{(4)}(x+jh,t) \right] \tag{17}$$

where $k_1 = 3, k_2 = 5\ l = 0(1)3$.
Expanding (17) in Taylor series, the following linear combinations of C_i are obtained

$$\mathcal{L}[u(x);\ h] = C_0 u(x) + C_1 h u'(x) + C_2 h^2 u''(x) + \cdots + C_p h^q u^{(p)}(x) + O(h^{p+1}) \tag{18}$$

where the constants C_i's are constants.

Definition 1: The LMM (16) is of order p if $C_0 = C_1 = C_2 = \cdots = C_{p+3} = 0, and \quad C_{p+4} \neq 0$ in which

$$\mathcal{L}[y(x);\ h] = C_{p+4} h^{p+4} y^{(p+4)}(x) + O(h^{p+5}) \tag{19}$$

Expanding each of the formulae in (12),(13),(14) and (15), the residual is the LTE given as;

$$\mathcal{L}[y(x);\ h] = C_{10} h^{10} y^{(10)}(x) + O(h^{(11)}); with\ \ p = 6 \tag{20}$$

where $C_0 = C_1 = C_2 = \cdots = C_{10} = 0, and \quad C_9 \neq 0$, and $C_{10} = C_{p+4}$. Here p is the order and $\tau_{n+j} = C_{p+4},\ j = 0(1)5$ is the error constant (see [10, 11]). The following table shows the order and error constants of each formulae in (12),(13),(14) and (15).

TABLE 1. Order and error constants

Formulae	C_{p+4}	Formulae	C_{p+4}	Formulae	C_{p+4}
$\tau_{n+4,m}$	$\frac{-1}{3024}$	-	-	-	
$\tau_{n+5,m}$	$\frac{-5}{3024}$	-	-	-	
$\tau'_{n,m}$	$\frac{-1}{2880}$	$\tau''_{n,m}$	$\frac{167}{50400}$	$\tau'''_{n,m}$	$\frac{-283}{24192}$
$\tau'_{n+1,m}$	$\frac{-163}{907200}$	$\tau''_{n+1,m}$	$\frac{-1}{6300}$	$\tau'''_{n+1,m}$	$\frac{311}{120960}$
$\tau'_{n+2,m}$	$\frac{23}{113400}$	$\tau''_{n+2,m}$	$\frac{1}{43200}$	$\tau'''_{n+2,m}$	$\frac{-11}{5760}$
$\tau'_{n+3,m}$	$\frac{-13}{50400}$	$\tau''_{n+3,m}$	$\frac{-31}{100800}$	$\tau'''_{n+3,m}$	$\frac{151}{120960}$
$\tau'_{n+4,m}$	$\frac{-487}{907200}$	$\tau''_{n+4,m}$	$\frac{-29}{25200}$	$\tau'''_{n+4,m}$	$\frac{-391}{120960}$
$\tau'_{n+5,m}$	$\frac{-307}{181440}$	$\tau''_{n+5,m}$	$\frac{251}{151200}$	$\tau'''_{n+5,m}$	$\frac{89}{8064}$

where the formulae $\tau^{(r)}_{n+j,m}$, represent each LMM $u^{(r)}_{n+j,m}$, $r = 0, \ldots, 3;\ \ j = 0, \ldots, 5$. From (20), it follows that for all the formulae, the order $p = 6$

3.2 Convergence Analysis

To establish the convergence, the method can be put in the block form

$$GY - h^4 HF(Y) + C + \tau(h) = 0 \tag{21}$$

where G is a $4N \times 4N$ matrix defined by

$$G = \begin{bmatrix} G_{11} & G_{12} & G_{13} & G_{14} \\ G_{21} & G_{22} & G_{23} & G_{24} \\ G_{31} & G_{32} & G_{33} & G_{34} \\ G_{41} & G_{42} & G_{43} & G_{44} \end{bmatrix}$$

and the entries of G are $N \times N$ matrices given as

$$G_{11} = \begin{bmatrix}
-3 & 3/2 & -1/3 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
5 & -4 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
-3 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
-4 & 6 & -4 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
-15 & 20 & -10 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 11/6 & -3 & 3/2 & -1/3 & 0 & \cdots & 0 \\
0 & 0 & 0 & -2 & 5 & -4 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & -3 & 3 & -1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & -4 & 6 & -4 & 1 & \cdots & 0 \\
0 & 0 & 0 & -15 & 20 & -10 & 0 & 1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 0 & 11/6 & -3 & 3/2 & -1/3 & 0 \\
0 & 0 & 0 & \cdots & 0 & -2 & 5 & -4 & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & -3 & 3 & -1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & -4 & 6 & -4 & 1 \\
0 & 0 & 0 & \cdots & 0 & -15 & 20 & -10 & 0 & 1
\end{bmatrix}$$

G_{11} is the matrix of repeated coefficients of $u_{n+i,m}$, for $i = 1(1)5$, in (12) and (13). G_{21} is the matrix of repeated coefficients of $hu'_{n+i,m}$, for $i = 1(1)5$, in (13). G_{31} is the matrix of repeated coefficients of $h^2u''_{n+i,m}$, for $i = 1(1)5$, in (14) and G_{41} is the matrix of repeated coefficients of $h^3u'''_{n+i,m}$, for $i = 1(1)5$, in (15).

$G_{22} = G_{33} = G_{44} = I$, where I is an $N \times N$ identity matrix, $G_{ij} = \mathbf{0}$, are zero matrices for $i = 1(1)4$, $j = 2(1)4$, $i < j$.

Similarly, let H be a $4N \times 4N$ matrix defined by

$$H = \begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{bmatrix}$$

and the entries of B are $N \times N$ matrices given as

$$H_{11} = \begin{bmatrix}
\frac{19}{105} & \frac{599}{10080} & \frac{1}{720} & \frac{-3}{2240} & \frac{1}{3600} & 0 & 0 & 0 & \cdots & 0 \\
\frac{3091}{4320} & \frac{2831}{30240} & \frac{143}{2520} & \frac{-1391}{60450} & \frac{23}{6048} & 0 & 0 & 0 & \cdots & 0 \\
\frac{73967}{60450} & \frac{-1261}{6048} & \frac{7439}{30240} & \frac{-5549}{60480} & \frac{883}{60480} & 0 & 0 & 0 & \cdots & 0 \\
\frac{31}{180} & \frac{79}{120} & \frac{31}{180} & \frac{-1}{720} & 0 & 0 & 0 & 0 & \cdots & 0 \\
\frac{11}{16} & \frac{101}{36} & \frac{97}{72} & \frac{1}{6} & \frac{-1}{720} & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \frac{937}{100800} & \frac{19}{105} & \cdots & \frac{1}{3600} & 0 & \cdots & 0 \\
0 & 0 & 0 & \frac{1411}{20160} & \frac{3091}{4320} & \cdots & \frac{23}{6048} & 0 & \cdots & 0 \\
0 & 0 & 0 & \frac{19151}{60450} & \frac{73967}{60450} & \cdots & \frac{883}{60480} & 0 & \cdots & 0 \\
0 & 0 & 0 & \frac{-1}{720} & \frac{31}{180} & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \frac{-1}{180} & \frac{11}{16} & \cdots & \frac{-1}{720} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & & \ddots & & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{937}{100800} & \frac{19}{105} & \cdots & \frac{1}{3600} \\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{1411}{20160} & \frac{3091}{4320} & \cdots & \frac{23}{6048} \\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{19151}{60450} & \frac{73967}{60450} & \cdots & \frac{883}{60480} \\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{-1}{720} & \frac{31}{180} & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{-1}{180} & \frac{11}{16} & \cdots & \frac{-1}{720}
\end{bmatrix}$$

H_{11} is the matrix of repeated coefficients of $f_{n+i,m}$, for $i = 0(1)5$, in (12) and (13). H_{21} is the matrix of repeated coefficients of $f_{n+i,m}$, $i = 0(1)5$, for all hu'_{n+i} in (13). H_{31} is the matrix of repeated coefficients of $f_{n+i,m}$, $i = 0(1)5$, for all $h^2u''_{n+i,m}$, in (14) and H_{41} is the matrix of repeated coefficients of $f_{n+i,m}$, $i = 0(1)5$, for all $h^3u'''_{n+i,m}$ (15).

$H_{ij} = \mathbf{0}_{N\times N}$, $i = 1(1)4$, $j = 2(1)4$, $i < j$.

$$\begin{aligned} U = \ & (u_m(x_1), \ldots, u_m(x_N), hu'_m(x_1), \ldots, hu'_m(x_N), h^2u''_m(x_1), \\ & \ldots, h^2u''_m(x_N), h^3u'''_m(x_1), h^3u'''_m(x_N))^T \end{aligned}$$

$$F(U) = (f_{1,m}, \ldots, f_{N,M})^T$$

$$\begin{aligned} C = & \ (-hu'_{0,m}, -\tfrac{11u_{0,m}}{6}, 0\ldots, 0, -h^2u''_{0,m}, 2u_{0,m}, 0, \ldots, 0, -h^3u'''_{0,m}, \\ & u_{0,m}, u_{0,m}, -u_{0,m}, \tfrac{1}{6}u_{0,m}, -\tfrac{1}{3}u_{0,m}, \tfrac{11}{6}u_{0,m}, u_{0,m}, \\ & -u_{0,m}, u_{0,m}, -u_{0,m}, -u_{0,m}, -u_{0,m}, u_{0,m}, 0\ldots, 0)^T \end{aligned}$$

$$\begin{aligned} \tau(h) = & \ (\tau_{1,m}\ldots, \tau_{N,M}, h\tau'_{1,m}\ldots, h\tau'_{N,M}, h^2\tau''_{1,m}\ldots, h^2\tau''_{N,M}, \\ & h^3\tau'''_{1,m}\ldots, h^3\tau'''_{N,M})^T \end{aligned}$$

where $\tau(h)$ is the local truncation error. We also define E as

$$\begin{aligned} E = & \ \overline{U} - U = (e_{1,m}, \ldots, e_{N,M}, he'_{1,m}, \ldots, he'_{N,M}, h^2e''_{1,m}, \\ & \ldots, h^2e''_{N,M}, h^3e'''_{1,m}, \ldots, h^3e'''_{N,M})^T \end{aligned}$$

where

$$\begin{aligned} \overline{U} = & \ (u_{1,m}, \ldots, u_{N,M}, hu'_{1,m}, \ldots, hu'_{N,M}, h^2u''_{1,m}, \ldots, h^2u''_{N,M}, \\ & h^3u'''_{1,m}, \ldots h^3u'''_{N,M})^T \end{aligned}$$

Theorem 1: [4]. Let $\overline{U}$ be an approximation of the solution vector U for the system obtained on a partition $\pi_N = \{a = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = b\}$, and $E = U - \overline{U}$. Define $h^r e_i^{(r)} = |h^r u^{(r)}(x_i) - h^r u_i^{(r)}|$ for $r = 0(1)3$, $i = 1, \ldots, N$. Then the block method (21) is a sixth-order convergent method. That is $||E||_\infty = O(h^6)$.

The proof follows from [4]. Let the exact form of the system be given as in (21) formed from (12),(13),(14) and (15), while the approximate form is defined by the block

$$G\hat{U} - h^4H\ f(\hat{U}) + C = 0 \tag{22}$$

subtracting (21) from (22), gives

$$G(\hat{U} - U) - h^4H(f(\hat{U} - f(U)) = \tau(h) \tag{23}$$

Using $e = \hat{U} - U$, we can write (23) as

$$G\ e - h^4H(f(\hat{U}) - f(U)) = \tau(h) \tag{24}$$

Using Jacobian matrix, we can approximate

$$f(\hat{U}) = f(U) + J_f(U)(\hat{U} - U) + o||\hat{U} - U|| \tag{25}$$

where J_f as the approximate value $\frac{f(\overline{U})-f(U)}{\overline{U}-U} = J_f(U)$. Without lost of generality, the first derivative term can be used so that

$$f(\overline{U}) - f(U) = J_f(U)(\overline{U} - U) = J_f(U)\ e = J_f\ e \tag{26}$$

where

$$J_f = \begin{pmatrix} J_{11} & \cdots & J_{14} \\ \vdots & & \vdots \\ J_{41} & \cdots & J_{44} \end{pmatrix}$$

whose entries J_{ij} are $N \times N$ matrices written as

$$J_{11} = \begin{pmatrix} \frac{\partial f_{1,m}}{\partial u_{1,m}} & \cdots & \frac{\partial f_{1,m}}{\partial u_{N,M}} \\ \vdots & & \vdots \\ \frac{\partial f_{N,M}}{\partial u_{1,m}} & \cdots & \frac{\partial f_{N,M}}{\partial u_{N,M}} \end{pmatrix}$$

;

$$J_{1,j+1} = \begin{pmatrix} \frac{\partial f_{1,m}}{\partial u^{(j)}_{1,m}} & \cdots & \frac{\partial f_{1,m}}{\partial u^{(j)}_{N,M}} \\ \vdots & & \vdots \\ \frac{\partial f_{N,M}}{\partial u^{(j)}_{1,m}} & \cdots & \frac{\partial f_{N,M}}{\partial u^{(j)}_{N,M}} \end{pmatrix} \quad \text{for} \ \ j = 1, 2, 3$$

$$J_{i,j+1} = h^i \begin{pmatrix} \frac{\partial f^{(i)}_{1,m}}{\partial u^{(j)}_{1,m}} & \cdots & \frac{\partial f^{(i)}_{1,m}}{\partial u^{(j)}_{N,M}} \\ \vdots & & \vdots \\ \frac{\partial f^{(i)}_{N,M}}{\partial y^{(j)}_{1,m}} & \cdots & \frac{\partial f^{(i)}_{N,M}}{\partial u^{(j)}_{N,M}} \end{pmatrix} ; for j = 0, \ldots, 3, \quad i = 2, \ldots, 4$$

Substituting (26) in (24), we get

$$G\ e - h^4 H J_f\ e = \tau(h) \tag{27}$$

$$(G - h^4 H(J_f))e = \tau(h)$$

Now, consider the matrix

$$\nu = G - h^4 H\ J_f \tag{28}$$

we claim that ν is invertible for sufficiently small h. First, we claim that G is invertible. To see this, since $G_{ii} = I$, for $i = 3, 4$ and $G_{ij} = 0$ for $i = 1, 2, 3, 4$; $j = 3, 4$; $i \neq j$, we thus write P as

$$A = \begin{bmatrix} G_{11} & 0 & 0 & 0 \\ G_{21} & G_{22} & 0 & 0 \\ G_{31} & 0 & G_{33} & 0 \\ G_{41} & 0 & 0 & G_{44} \end{bmatrix} \tag{29}$$

Observe that G_{11} contains nonzero diagonal submatrices which are identical, so its determinant exists, hence its non singular. This is true for G_{21}, G_{31} and G_{41}. Note also that G_{ii}, $i = 2, 3, 4$ are identity matrices. Therefore G has nonzero for all diagonal elements. It is known that a matrix with nonzero main diagonal is invertible.

Thus, G is a matrix with nonzero diagonal, and so G^{-1} exists. Now, (28) can be written as

$$|\nu| = |G - h^4 H\ J_f| = |G||I - C| \tag{30}$$

where $C = h^4 H\ J_f G^{-1}$, then $|\lambda I - C| = 0$ is a characteristic polynomial of C, so that

$$|\lambda I - C| = (\lambda - \lambda_1) \cdots (\lambda - \lambda_{4N})$$

where λ_i are eigenvalues of the matrix C. When $\lambda = 1$, we have

$$|I - C| = (1 - \lambda_1) \cdots (1 - \lambda_{4N})$$

for $|I - C| \neq 0$, then each $\lambda_i \neq 0$. If $\hat{\lambda}_i$ is an eigenvalue of C, so is $h^4\lambda_i$, thus we need $h^4\lambda_i \neq 1$. So we choose h such that $h^4 \notin \left\{ \frac{1}{\hat{\lambda}_i} | \hat{\lambda}_i \text{ are nonzero eigenvector of } HJ_FG^{-1} \right\}$ For such h, $|I - C| \neq 0$, so that

$$|\nu| = |G||I - h^4 H\ J_f| \neq 0 \tag{31}$$

Hence ν is invertible. Then

$$\begin{aligned} \nu\, e &= \tau(h) \\ e &= \nu^{-1}\tau(h) \\ ||e|| &= ||\nu^{-1}\tau(h)|| \\ &\leq ||\nu^{-1}||||\tau(h)|| \\ &\leq O(h^{-4})O(h^{10}) \\ &\leq O(h^6) \end{aligned}$$

This shows that the Method (21) is 6^{th} order convergent.

3.3 Computational procedure

Assume the following boundary conditions are known (2)

$$u_0 = \alpha_{00}, \quad u_N = \alpha_{0N}, \quad u'_0 = \alpha_{10}, \quad u'_N = \beta_{1N}$$

the vector of unknowns u is given by

$$\begin{aligned} u &= \big(u_1, \ldots, u_{N-1}, u'_1, \ldots, u'_{N-1}, \\ &\quad u''_0, \ldots, u''_N, u'''_0, \ldots, u'''_N\big)^T . \end{aligned}$$

This makes a total of $(N-1)+(N-1)+(N+1)+(N+1) = 4N$ unknowns.

On the other hand, we have two formulae in (12) which for $n = 0(5)N-5$ make a total of $2N/5$ formulas. Also there are six formulae in (13) and taking $n = 0(5)N-5$ gives $6N/5$ formulae. In (14) and (15) similarly there are $6N/5$ each. Therefore, the total number of equations becomes $2N/5 + 6N/5 + 6N/5 + 6N/5 = 4N$.

Hence we have a system with $4N$ equations and $4N$ unknowns, whose solution provides a set of approximate values of the BVP in (7). Now, the formulae (12), (13), (14) and (15) in the form (7) to form a block and solved simultaneously using codes written in *Mathematica*, enhanced by the feature ***NSolve[]*** for linear problems while nonlinear problems were solved by Newton's method enhanced by the feature ***FindRoot[]***, as shown in the algorithm below.

Algorithm 1 Block Algorithm for BUM

1 **begin procedure** ENTER Partitions a, b (integration interval), N (number of steps), u_a, u'_a, u_b, u'_b (assumed boundary values), f
2 *sol*, discrete approximate solution of the BVP (12)-(15) in the form (7)
3 Let $n = 0, 5, N-5$, $\quad x_n = a$, $\quad h = \frac{b-a}{N}$
4 Let sol=$\{(x_n, u_n)\}$.
5 Generate block system
6 Set timing and solve [System, variables] to get $u_{n+j}, u'_{n+j}, u''_{n+j}, u'''_{n+j}$, $j = 1(1)N-5$ and CPU time in seconds (s) for obtaining roots,
7 Let $sol = \{(x_{n+i}, u_{n+i})\}_{i=1(1)N-5}$.
8 **end procedure**

4. NUMERICAL EXAMPLES

In this section, some numerical examples are presented and error comparisons are made between the method derived above and those in existing literatures. The code used was based on Newton method which uses the feature FindRoot or NSolve for linear problems in Mathematica. In all examples, a uniform step-size was used and the maximum absolute errors were computed as $max|(y_{ij} - y(x_i, t_j))|$, $0 \leq i \leq M$, $0 \leq j \leq N$. where y_{ij} is the numerical approximation of the exact solution $y(x_i, t_j)$ at the mesh point (x_i, t_j).

Problem 1. Consider the linear fourth-order parabolic equation with constant coefficient discussed in [2]

$$y_{tt} + y_{xxxx} = (\pi^4 - 1)\sin \pi x \cos t,\ \ 0 \leq x \leq 1,\ \ t \geq 0$$

subject to the initial conditions

$$y(x, 0) = \sin \pi x,\ \ y_t(x, 0) = 0$$

and with appropriate boundary conditions

$$y(0, t) = y(1, t) = y_{xx}(0, t) = y_{xx}(1, t) = 0$$

with the exact solution $y(x, t) = \sin \pi x \cos t$

Here, the problem on semi-discretizing the time variable, becomes

$$\frac{y_{m+1} - 2y_m + y_{m-1}}{(\Delta t)^2} + \frac{d^4 y_m}{dx^4} = g_m,\ \ 0 \leq x \leq 1,\ \ \ m = 1, \ldots M - 1 \tag{32}$$

where $\Delta t = (L_4 - L_3)/M,\ t_m = L_3 + m\Delta t,\ m = 0, 1, \ldots, M,$ $y = [y_1(x), \ldots, y_M(x)]^T,\ y_m(x) \approx y(x, t_m),\ g = [g_1(x), \ldots, g_m(x)]^T$ and $g_m(x) \approx g(x, t_m) = (\pi^4 - 1)\sin \pi x \cos t_m$, which is expressed in the form

$$y^{(4)} = f(x, y, y', y'', y''') = Ay + g \tag{33}$$

A is an $(M - 1) \times (M - 1)$ matrix arising from the discretized system, and g is a vector of constants.

Table 2. Problem 1. Maximum error with $h = 0.05$

Method	k	$x = 0.1$	$x = 0.2$	$x = 0.3$	$x = 0.4$	$x = 0.5$
BUM	10	4.88E-10	8.56E-10	1.28E-9	1.39E-9	1.58E-9
	16	8.28E-10	1.50E-9	2.17E-9	2.43E-9	2.48E-9
Spline [2]	10	2.91E-6	1.73E-6	1.60E-6	2.23E-6	2.60E-7
	16	4.47E-7	2.66E-7	1.37E-7	1.55E-7	1.57E-7

Table 2 shows the maximum errors for the BUM compared to Spline method in [2]. For the BUM, the CPU time in seconds for $k = 10$ is 0.71, and for $k = 16$ is 0.98. Clearly the proposed method performed better and accurately when compared to the exact solution. The graphical representation of the solutions are shown in Figure .

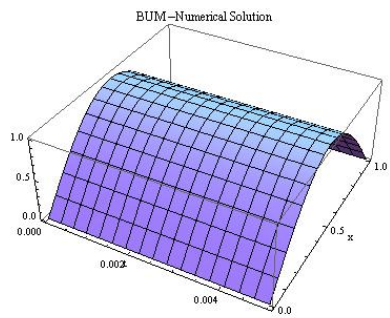

Figure 1. Graphical representation for (32) showing surface plots for the error (e), where $e = |y_{n,m} - y(x_n, t_m)|$

Problem 2. Consider the following homogenous fourth-order parabolic equation:

$$120xy_{tt} + (120 + x^5)y_{xxxx} = 0,\ \frac{1}{2} \le x \le 1,\ t \ge 0$$

subject to the initial conditions

$$y(x,0) = 0,\ y_t(x,0) = 1 + \frac{x^5}{120}$$

and with appropriate boundary conditions

$$\left.\begin{array}{l} y(\frac{1}{2},t) = \frac{3841}{3840}\sin t,\ y(1,t) = \frac{121}{120}\sin t, \\ y_{xx}(\frac{1}{2},t) = \frac{1}{48}\sin t,\ y_{xx}(\frac{1}{2},t) = \frac{1}{6}\sin t \end{array}\right\}$$

The exact solution for this problem is $y(x,t) = \left(1 + \frac{x^5}{120}\right)\sin t$.

Upon semi-discretization of the time variable, we have

$$\frac{y_{m+1}-2y_m+y_{m-1}}{(\Delta t)^2}+\frac{(120+x^5)}{120x}\frac{d^4y_m}{dx^4}=g_m,\quad \frac{1}{2}\leq x\leq 1,\quad m=1,\dots M-1 \tag{34}$$

where Δt, t_m, $m=0,1,\dots,M$, y, $y_m(x)\approx y(x,t_m)$, g and g_m are as expressed in Problem 1, which is put in the form

$$y^{(4)}=f(x,y,y',y'',y''')=Ay+g \tag{35}$$

A is as expressed in problem 1 and $g_m=0$.

The graphical representation of the solutions are shown in Figure .

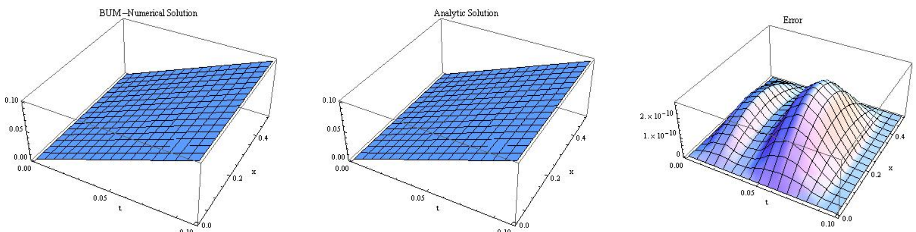

FIGURE 2. Graphical representation for (34) showing surface plots for the numerical solution using BUM, the analytic solution and the error (e), where $e=|y_{n,m}-y(x_n,t_m)|$

Problem 3. Consider the following homogenous fourth-order parabolic equation:

$$y_{tt}=y_{xxxx}=0,\ 0\leq x\leq 1,\ t\geq 0$$

subject to the initial conditions

$$y(x,0)=\sin\pi x,\ y_t(x,0)=-\pi^2\sin\pi x$$

and with appropriate boundary conditions

$$\left.\begin{array}{r} y(0,t)=0,\ y(1,t)=0,\\ y_{xx}(0,t)=0,\ y_{xx}(1,t)=0\end{array}\right\}$$

The exact solution for this problem is $y(x,t)=e^{-\pi^2t}\sin\pi x$.

Upon semi-discretization of the time variable, we obtain,

$$\frac{y_{m+1}-2y_m+y_{m-1}}{(\Delta t)^2}-\frac{d^4y_m}{dx^4}=g_m,\quad 0\leq x\leq 1,\quad m=1,\dots M-1 \tag{36}$$

where Δt, t_m, $m = 0, 1, \ldots, M$, y, $y_m(x) \approx y(x, t_m)$, g and g_m are as expressed in example 1, which is expressed in the form

$$y^{(4)} = f(x, y, y', y'', y''') = Ay + g \tag{37}$$

A is as expressed in problem 1 and $g_m = 0$.

The graphical representation of the solutions are shown in Figure .

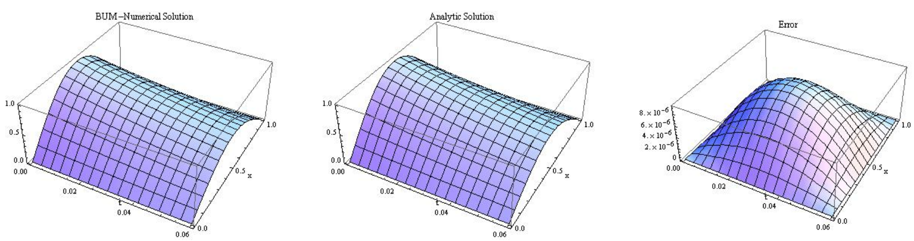

FIGURE 3. Graphical representation for (40) showing surface plots for the numerical solution using BUM, the analytic solution and the error (e), where $e = |y_{n,m} - y(x_n, t_m)|$

Problem 4. Consider the following nonhomogenous fourth-order parabolic equation:

$$y_{tt} + (1+x)y_{xxxx} = \left(x^3 + x^4 - \frac{6}{7!}x^7\right)\cos t, \ \frac{1}{2} \le x \le 1, \ t \ge 0$$

subject to the initial conditions

$$y(x, 0) = \frac{6}{7!}x^7, \ y_t(x, 0) = 0$$

and with appropriate boundary conditions

$$\left.\begin{array}{r} y(0,t) = 0, \ y(1,t) = \frac{6}{7!}\cos t, \\ y_{xx}(0,t) = 0, \ y_{xx}(1,t) = \frac{1}{20}\cos t \end{array}\right\}$$

The exact solution for this problem is $y(x, t) = \frac{6}{7!}x^7 \cos t$. Here, the problem has the semi-discretized form

$$\frac{y_{m+1} - 2y_m + y_{m-1}}{(\Delta t)^2} + (1+x)\frac{d^4 y_m}{dx^4} = g_m, \ \ 0 \le x \le 1, \ \ m = 1, \ldots M-1 \tag{38}$$

where Δt, t_m, $m = 0, 1, \ldots, M$, y, $y_m(x) \approx y(x, t_m)$, g and g_m are as expressed in example 1, $g_m = \left(x^3 + x^4 - \frac{6}{7!}x^7\right)\cos t_m$, which is expressed in the form

$$y^{(iv)} = f(x, y, y', y'', y''') = Ay + g \tag{39}$$

A is as expressed in Problem 1.

TABLE 3. Problem 4. Maximum error with $h = 0.05$

Method	k	$x = 0.1$	$x = 0.2$	$x = 0.3$	$x = 0.4$	$x = 0.5$
BUM	10	0	7.21E-14	1.23E-13	1.22E-13	4.61 E-14
	16	0	4.40E-13	2.35E-13	4.34E-14	9.57E-14
Spline [2]	10	7.19E-10	2.26E-10	7.32E-10	6.91E-10	7.31E-10
	16	6.25E-10	2.22E-10	4.53E-10	4.41E-10	5.03E-10

Table 4 shows the maximum errors for $k = 10, 16$ and at $t = 0.1(0.1)0.5$ for the BUM compared to Spline method in [2]. For the BUM, the CPU time in seconds for $k = 10$ is 0.75, and for $k = 16$ is 1.02. Evidently, the proposed method performed better and accurately when compared to the exact solution and the method in [2]. Figure shows the graphical representation of the solutions and error.

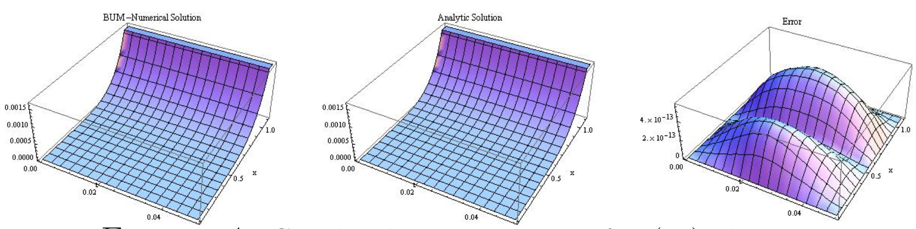

FIGURE 4. Graphical representation for (38) showing surface plots for the numerical solution using BUM, the analytic solution and the error (e), where $e = |y_{n,m} - y(x_n, t_m)|$

Problem 5. Consider the "good" Boussinesq equation:

$$y_{tt} = y_{xx} + y_{xx}^2 - y_{xxxx},\ \ 0 \le x \le 1,\ \ t \ge 0$$

with appropriate boundary conditions

$$\left.\begin{array}{r} y(0,t) = 0,\ \ y(1,t) = 0, \\ y_{xx}(0,t) = 0,\ \ y_{xx}(1,t) = 0, t > 0 \end{array}\right\}$$

The exact solution for this problem is

$$y(x,t) = -A \sec h^2 \left(\sqrt{\frac{A}{6}} (x - ct + v_0) \right) - \left(b + \frac{1}{2} \right)$$

.

Here c is the velocity, A is amplitude of the pulse, b is an arbitrary parameter and v_0 is the initial position. Using the same theoretical parameters as in [8]: $A = 0.369$, $b = -1/2$ and $c = 0.868$.

Here, semi-discretizing the time variable becomes

$$\frac{y_{m+1} - 2y_m + y_{m-1}}{(\Delta t)^2} - \frac{d^2 y_m}{dx^2} - \frac{d^2 y_m^2}{dx^2} + \frac{d^4 y_m}{dx^4} = g_m, \quad 0 \leq x \leq 1, \quad m = 1, \ldots M - 1 \tag{40}$$

where Δt, t_m, $m = 0, 1, \ldots, M$, y, $y_m(x) \approx y(x, t_m)$, g and g_m are as expressed in example 1, which is expressed in the form

$$y^{(4)} = f(x, y, y', y'', y''') = Ay + g \tag{41}$$

A is as expressed in problem 1 and $g_m = 0$.

TABLE 4. Problem 5. Maximum absolute error

Time	Parameter	BUM	CPU (s.)	Method in [8]	Method in [12]
$t = 0.5$	$h = \frac{1}{40}$	1.7274(-10)	0.61	7.8998(-07)	8.2943(-07)
$t = 1.0$	$h = \frac{1}{60}$	1.7531(-12)	1.92	7.5071(-09)	7.3326(-09)
$t = 1.5$	$h = \frac{1}{80}$	2.2691(-14)	2.21	5.7588(-11)	5.7588(-11)
$t = 2.0$	$h = \frac{1}{100}$	1.6471(-16)	3.51	2.9068(-13)	2.9068(-13)

Table 4 shows the maximum errors for h and t for the BUM and the methods in [8] and [12]. The parameter $h = \frac{1}{2j}$, $j = 20, 30, 40, 50$ is computed for $v_0 = 30, 40, 50, 60$ respectively. The proposed method shows a good performance for such problem as compared to the other methods. The graphical representation of the solutions are shown in Figure .

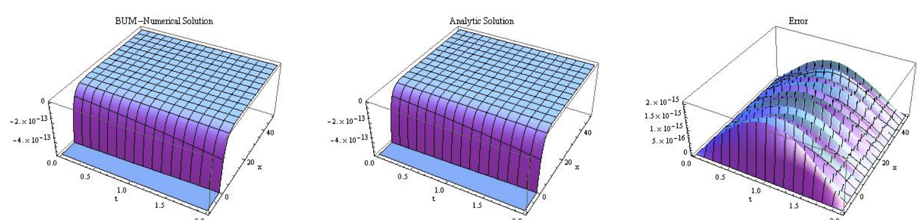

FIGURE 5. Graphical representation for (40) showing surface plots for the numerical solution using BUM, the analytic solution and the error (e), where $e = |y_{n,m} - y(x_n, t_m)|$

Verifying the numerical convergence rate, we determine the order p of the derived block method. To see this, the step-size h is halved and thus the ratios of the maximum errors $Error = u_i^h - u(x_i)$ and $|u_i^{h/2} - u(x_i)|$ are estimated as

$$\frac{u_i^h - u(x_i)}{u_i^{h/2} - u(x_i)} = \frac{Ch^p + O(h^{p+4})}{C(h/2)^p + O((h/2)^{p+4})} = 2^p + O(h) \tag{42}$$

so that

$$\log_2 \left| \frac{u_i^h - u(x_i)}{u_i^{h/2} - u(x_i)} \right| = p + O(h) \tag{43}$$

where C is a number depending on the exact solution $u(x_i)$. Table 5 shows the rate of convergence using problem 1.

Table 5. Rate of convergence (ROC)

N	Error	ROC
4	5.464×10^{-8}	
8	7.432×10^{-10}	6.2
16	1.083×10^{-11}	6.1
32	1.814×10^{-13}	5.9
64	$2.3.3 \times 10^{-15}$	6.3
128	3.549×10^{-17}	6.0

The column with the ROC in Table 5 agrees with the theoretical order p of the derived scheme.

5. CONCLUSION

A block unification method based on the continuous linear multistep methods is proposed and applied via the method of lines technique to solve fourth order PDEs. It is shown that the method is very flexible, easy to derive using computer programs written in Mathematica 11.0 and less ambiguouswithout any subroutine files. The derived scheme can be applied to solve diverse kinds of parabolic PDEs with either Neumann or Dirichlet boundary conditions as seen in the examples presented. The method shows a very high accuracy when compared to the exact solution and with existing methods.

References

[1] A. Khan, I. Khan and T. Aziz, *Sextic spline solution for solving a fourth-order parabolic partial differential equation*, Int. J. of Comp. Math. **82** (7) 871-879, 2005.

[2] J. Rashidinia and R. Mohammadi, *Sextic spline solution of variable coefficient fourth-order parabolic equations*, Int. J. of Comp. Math. **87** (15), 3443-3454, 2010.

[3] P. Saucez, W. A. Vande, W. E. Schiesser and P. Zegeling, *Method of lines study of nonlinear dispersive waves*, J. of Comp. and App Math. **168**, 413-423, 2004.

[4] S. N. Jator and V. Manathunga, *Block Nyström type integrator for Bratu's equation*, J. Comp. App. Math. **327** 341-349, 2018.

[5] T. A. Biala, *A Computational Study of the Boundary Value Methods and the Block Unification Methods for $y'' = f(x, y, y')$*, Abstract and Applied Analysis. Article ID 8465103, 14 pages, 2016. http://dx.doi.org/10.1155/2016/8465103.

[6] T. A. Biala, S. N. Jator and R. B. Adeniyi, *Numerical approximations of second order PDEs by boundary value methods and the method of lines*, SpringerPlus 2015, DOI:10.1186/s40064-015-1348-1.

[7] R. C. Mittal and R. K. Jain, *B-splines methods with redefined basis functions for solving fourth order parabolic partial differential equations*, Appl. Math. and Comp. **217** 9741-9755, 2011.

[8] R. K. Mohanty and D. Kaur, *High accuracy implicit variable mesh methods for numerical study of special types of four thorder nonlinear parabolic equations*, Appl. Math. and Comp. **273** 678-696, 2016.

[9] R. K. Mohanty, D. J. Evans and D. Kumar, *High Accuracy Difference Formulae For A Fourth Order Quasi-Linear Parabolic Initial Boundary Value Problem Of First Kind*, Inter. J. Computer Math. **80** (3) 381-398, 2003.

[10] J. D. Lambert, *Computational Methods in Ordinary Differential Equations*, John Wiley, New York, 1973.

[11] M. I. Modebei, R. B. Adeniyi, S. N. Jator and H. Ramos, *A block hybrid integrator for numerically solving fourth-order Initial Value Problems*, App. Math. and Comp. **346** 680-694, 2019.

[12] S. S. Siddiqi and S. Arshed, *Quintic B-spline for the numerical solution of the good Boussines qequation*, J. Egypt. Math. Soc. **22** 209-213, 2014.

[13] S. Jator, *Block Unification Scheme for Elliptic, Telegraph, and Sine-Gordon Partial Differential Equations*. American Journal of Comp. Math. **5** 175-185, 2015. http://dx.doi.org/10.4236/ajcm.2015.52014.

[14] A. Shokri and M. Dehghan, *A Not-a-Knot meshless method using radial basis functions and predictorcorrector scheme to the numerical solution of improved Boussinesq equation*, Comp. Phy. Comm. **181** 1990-2000, 2010.

[15] J. Vigo-Aguiar and H. Ramos, *A family of A-stable collocation methods of higher order for initial-value problems*, IMA J. Numer. Anal. **27** 798-817, 2007.

[16] L. Brugnano and D. Trigiante, *Solving Differential Problems by Multistep Initial and Boundary Value Methods*, Gordon and Breach Science Publishers, Amsterdam, 1998.

[17] H. Sirajul, B. Nagina, S. I. A. Tirmizi and M. Usman, *Meshless method of lines for the numerical solution of generalized Kuramoto-Sivashinsky equation*, Appl. Math. and Comp. **217** 2404-2413, 2010.

[18] H. El-Zoheiry, *Numerical investigation for the solitary waves interaction of the good Boussinesq equation*, Appl. Num. Math. **45** 161173, 2003.
[19] U. Marja, H. Sirajul, and I. Sirajul, *A mesh-free numerical method for solution of the family of KuramotoSivashinsky equations*, Appl. Math. and Comp. **212** 458-469, 2009.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILORIN, ILORIN, NIGERIA.
E-mail address: gmarc.ify@gmail.com

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILORIN, ILORIN, NIGERIA.
E-mail address: raphadelives@yahoo.com

DEPARTMENT OF MATHEMATICS AND STATISTICS, AUSTIN PEAY STATE UNIVERSITY CLARKSVILLE, TN 37044, USA.
E-mail address: Jators@apsu.edu