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DIRECT INTEGRATION OF GENERAL FOURTH ORDER ORDINARY DIFFERENTIAL EQUATIONS USING FIFTH ORDER RUNGE-KUTTA METHOD

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ABSTRACT. In this work, the fifth-order Runge-Kutta method for the solution of first order Ordinary Differential Equations (ODEs) was modified for direct integration of general fourth-order ODEs via the idea as those invented by Nyström. The theory of Nyström was adopted in the derivation of the method. The method has an explicit structure for efficient implementation, self-starting produces simultaneously approximation of the solution of special and general fourth-order ODEs. The proposed method is direct, does not involve the reduction of higher order ODEs to system of first-order ODEs as in most recent related articles, convergence analysis of the method was presented, was tested with Numerical experiment to illustrate its efficiency, could be extended to solve higher-order differential equations, simple to implement and the approximate results are in good agreement with other methods mentioned in literature.

Keywords and phrases: Fifth-order Runge-Kutta method, The theory of Nyström, First-order, General Fourth-order ODEs
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1. INTRODUCTION

Differentials equation play an important role in modeling virtually every physical, technical or biological process from celestial motion to bridge design, to interaction between neurons[1]. Although it is possible to integrate a third order IVP by reducing it to first order system and apply one of the method available for such system it seems more natural to provide commercial method in order to integrate the problem directly[2]. The advantage of these approaches lies in the fact that they are able to exploit special information about ODEs and this result in an increase in efficiency (that is,

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high accuracy at low cost) For instance, it is well know that Runge-Kutta Nystrm method involve a real improvement as compared to standard Runge-Kutta method for a given number of stages [3]. The RKN method is an extension of Runge- Kutta (RK) method for first order ODEs of the form

$$y' = f(x, y), \qquad y(x_0) = y_0 \tag{1}$$

The most popular RK method or classical RK method of order four is good since it has it has local error bounds 0 ,which is small enough (h< 1 The classical RK methods of order four for the initial value problem (1) is given by

$$y_{n+1} = y_n + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4) \tag{2}$$

where

$$k_1 = f(x, y) \qquad k_2 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1)$$

$$k_3 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2) \qquad k_4 = f(x_n + h, y_n + k_3)$$

h is the step-size vector chosen , usually h< 1 The method (2) in Butcher-array form can be written as

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 & 1 \\ \hline & \frac{1}{6} & \frac{2}{6} & \frac{2}{6} & \frac{1}{6} \end{array}$$

The Runge- Kutta- Nystrm (RKN) method for second order ODEs of the form

$$y'' = f(x, y, y'), \qquad y(x_0) = y_0 \quad y'(x_0) = y'_0 \tag{3}$$

is given by

$$y_{n+1} = y_n + hy'_n + \frac{1}{3}h(k_1 + k_2 + k_3) \tag{4}$$

$$y'_{n+1} = y'_n + \frac{1}{3}h(k_1 + 2k_2 + 2k_3 + k_4) \tag{5}$$

where

$$k_1 = f(x, y, y')$$

$$k_2 = \frac{1}{2}f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}h(y'_n + \frac{1}{2}k_1), y'_n + k_1)$$

$$k_3 = \frac{1}{2}f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}h(y'_n + \frac{1}{2}k_1), y'_n + k_2)$$

$$k_4 = \frac{1}{2}f(x_n + h, y_n + h(y'_n + k_3), y'_n + 2k_3)$$

Both the RK and RKN methods could be expanded into Taylor series.[4]. The work is organized as follows: In section 2 we will show how the Butcher's Runge-kutta methods for the first order differential equations tableau are modified to include second (that is Runge-Kutta Nystrm method) which was extended to fourth derivatives, this idea will be used in section 3 to illustrate the main derivation of an explicit Runge-Kutta method for solution of fourth order ODEs, some numerical experiments are presented in section 4, finally, the conclusion section 5 .

2. PRELIMINARY

Butcher [5], defined an s-stage implicit Runge-Kutta methods for the first order differential equations(ODEs) as

$$y_{n+1} = y_n + h\sum_{j=1}^{s} a_{ij}k_j \tag{6}$$

for $i = 1, 2....s$.

$$k_i = f(x_i + \alpha_j h, y_n + h\sum_{j=1}^{s} a_{ij}k_j$$

in Butcher-array form can be written as

$$\begin{array}{c|c} \alpha & \beta \\ \hline & W^T \end{array}$$

The real parameters α_i, k_j, a_{ij} define the method. Based on (5)and The Taylor's series expansion of and the idea in used to formulate equation (4) using equation(2) presented in [6],[7] and [8].We propose an s-stage Runge-Kutta Type method for direct integration of fourth order ODEs was formulated as

$$y_{n+1} = y_n + \alpha_j h y'_n + \frac{(\alpha_j h)^2}{2}y''_n + \frac{(\alpha_j h)^3}{6}y'''_n + h^4\sum_{j=1}^{s} a_{ij}k_j \tag{7}$$

$$y'_{n+1} = y'_n + \alpha_j h y''_n + \frac{(\alpha_j h)^2}{2}y'''_n + h^3\sum_{j=1}^{s} a_{ij}k_j \tag{8}$$

$$y''_{n+1} = y''_n + \alpha_j h y'''_n + h^2 \sum_{j=1}^{s} \overline{a}_{ij} k_j \tag{9}$$

$$y'''_{n+1} = y'''_n + h \sum_{j=1}^{s} \overline{\overline{a}}_{ij} k_j \tag{10}$$

for $i = 1, 2....s$.

$$k_i = f(x_n + \alpha_j h,\ y_n + \alpha_j h y'_n + \frac{(\alpha_j h)^2}{2} y''_n + \frac{(\alpha_j h)^3}{6} y'''_n + h^4 \sum_{j=1}^{s} a_{ij} k_j$$

$$y'_n + \alpha_j h y''_n + \frac{(\alpha_j h)^2}{2} y'''_n + h^3 \sum_{j=1}^{s} a_{ij} k_j,\ y''_{n+1} = y''_n + \alpha_j h y'''_n + h^2 \sum_{j=1}^{s} \overline{a}_{ij} k_j,$$

$$y'''_{n+1} = y'''_n + h \sum_{j=1}^{s} \overline{\overline{a}}_{ij} k_j)$$

The real parameters α_i, k_j, a_{ij}, $\overline{a}_{ij}$, $\overline{\overline{a}}_{ij}$ $\overline{\overline{\overline{a}}}_{ij}$ define the method. In Butcher - array form method (9) is given by

α	$\overline{\overline{\overline{A}}}$	$\overline{\overline{A}}$	$\overline{A}$	A
	$\overline{\overline{\overline{b}}}^T$	$\overline{\overline{b}}^T$	$\overline{b}^T$	b

$$A = a_{ij} = \beta^4 \qquad \overline{A} = \overline{a}_{ij} = \beta^3 \qquad \overline{\overline{A}} = \overline{\overline{a}}_{ij} = \beta^2 \qquad \overline{\overline{\overline{A}}} = \overline{\overline{\overline{a}}}_{ij} = \beta$$

$$\beta = \beta e \qquad \overline{\overline{\overline{b}}} = w \qquad \overline{\overline{b}} = w^T \qquad \overline{b} = w^T \beta^2 \qquad b = w^T \beta^3$$

Definition 1: Order and Error Constant of Runge-Kutta Methods A first, second and third order ODEs methods are said to be of order p if p is the largest integer for which

$$y(x+h) - y(x) - h\phi(x, y(x), h) = O(h^{p+1}) \tag{11}$$

$$y(x+h) - y(x) - h^2\phi(x, y(x), y'(x), h^2) = O(h^{p+2}) \tag{12}$$

$$y(x+h) - y(x) - h^3\phi(x, y(x), y'(x), y''(x), h^3) = O(h^{p+3}) \tag{13}$$

Holds respectively. Where

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2} y'' + ... + \frac{h^s}{s!} y^s(x)$$

$$\phi(x, y(x), h) = y'(x+h) = f(x, y(x))$$

$$\phi(x, y(x), y'(x), h^2) = y''(x+h) = f(x, y(x), y'(x))$$

$$\phi(x, y(x), y'(x), y''(x), h^3) = y'''(x+h) = f(x, y(x), y'(x), y''(x))$$

in the Taylor series expansion about x_0 and compare coefficients of $h^k y^k(x_0), y(x_0)$ is the interval value. The coefficients for which p is the largest integer is known as the error constant (See Lambert [9]).

Proposisition 1: An order P method for a G(N) order ODEs extended for a higher order G(N+1) ODEs has order P-1,where G(1),G(2),.,G(N) denote first order, second order, nth order respectively.

Proof: From definition 1 let $h^k y^k(x_0)$ represent the largest integer for which G(N) equation(11) holds, implies $k = p+1+O(h^{p+1})$ then G(N+1) equation(12)$k = p + 2 + (-1 + O(h^{p+1})$ and for G(N+2) equation(13) $k = p + 3 + (-2 + O(h^{p+1}))$such that $O(h^{p+1}), (-1 + O(h^{p+1}) = O(h^{p+2}) and (-2 + O(h^{p+1}) = O(h^{p+3})$ are the order of G(N), G(N+1) and G(N+2) respectively.

3. DERIVATION OF THE EXPLICIT RUNGE-KUTTA METHOD DIRECT INTEGRATION OF GENERAL FOURTH ORDER

Consider the sixth-order eight-stage Runge-Kutta Method for equation(1)

$$y_{n+1} = y_n + \frac{h}{90}(7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6) \tag{14}$$

where $k_1 = f(x, y) \quad k_2 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1)$
$k_3 = f(x_n + \frac{h}{4}, y_n + \frac{h}{16}(3k_1 + k_2) \quad k_4 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_3)$
$k_5 = f(x_n + \frac{3h}{4}, y_n + \frac{h}{16}(-3k_2 + 6k_3 + 9k_4)$
$k_6 = f(x_n + h, y_n + \frac{h}{7}(k_1 + 43k_2 + 6k_3 - 12k_4 + 8k_5)$

0	0					
$\frac{1}{2}$	$\frac{1}{2}$					
$\frac{1}{4}$	$\frac{3}{16}$	$\frac{1}{16}$				
$\frac{1}{2}$	0	0	$\frac{1}{2}$			
$\frac{3}{4}$	0	$\frac{-3}{16}$	$\frac{6}{16}$	$\frac{9}{16}$		
1	$\frac{1}{7}$	$\frac{4}{7}$	$\frac{6}{7}$	$\frac{-12}{7}$	$\frac{8}{7}$	
	$\frac{7}{90}$	0	$\frac{16}{45}$	$\frac{2}{15}$	$\frac{16}{45}$	$\frac{7}{90}$

Using Butcher - array form of method (9)we obtained

$$
\begin{array}{c|cccccc|ccccc}
0 & & & & & & & & & & & \\
\frac{1}{2} & \frac{1}{2} & & & & & & & & & & \\
\frac{1}{4} & \frac{3}{16} & \frac{1}{16} & & & & & \frac{1}{32} & & & & \\
\frac{1}{2} & 0 & 0 & \frac{1}{2} & & & & \frac{3}{32} & \frac{1}{32} & & & \\
\frac{3}{4} & 0 & \frac{-3}{16} & \frac{6}{16} & \frac{9}{16} & & & \frac{-3}{128} & \frac{3}{128} & \frac{9}{32} & & \\
1 & \frac{1}{7} & \frac{4}{7} & \frac{6}{7} & \frac{-12}{7} & \frac{8}{7} & & \frac{25}{56} & \frac{-9}{56} & \frac{-3}{7} & \frac{9}{14} & \\
\hline
 & \frac{7}{90} & 0 & \frac{16}{45} & \frac{2}{15} & \frac{16}{45} & \frac{7}{90} & \frac{7}{90} & 0 & \frac{4}{15} & \frac{1}{15} & \frac{4}{15}
\end{array}
$$

$$
\begin{array}{c|cccccc|ccccc}
 & & & & & & & & & & & \\
 & & & & & & & & & & & \\
 & & & & & & & & & & & \\
 & \frac{1}{64} & & & & & & & & & & \\
 & \frac{33}{512} & \frac{9}{512} & & & & & \frac{9}{1024} & & & & \\
 & \frac{1}{7} & \frac{4}{7} & \frac{6}{7} & \frac{-12}{7} & \frac{8}{7} & & \frac{3}{64} & \frac{9}{448} & & & \\
\hline
 & \frac{7}{90} & 0 & \frac{16}{45} & \frac{2}{15} & \frac{16}{45} & \frac{7}{90} & \frac{7}{90} & 0 & \frac{4}{15} & \frac{1}{15} & \frac{4}{15}
\end{array}
$$

Putting the array coefficients in equation form of method (9)we have the third-order explicit Runge-Kutta method for direct integration of forth-order ODEs given by

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2}y''_n + \frac{(h)^3}{6}y'''_n + h^4(\frac{1}{80}k_1 + \frac{1}{240}k_2 + \frac{1}{40}k_3) \quad (15)$$

Where

$$y'_{n+1} = y'_n + hy''_n + \frac{h^2}{2}y'''_n + h^3(\frac{1}{20}k_1 + \frac{1}{15}k_3 + \frac{1}{20}k_4)$$

$$y''_{n+1} = y''_n + hy'''_n + h^2(\frac{7}{90}k_1 + \frac{4}{15}k_3 + \frac{1}{15}k_4 + \frac{4}{15}k_5 + \frac{4}{45}k_6)$$

$$y'''_{n+1} = y'''_n + h(\frac{7}{90}k_1 + \frac{16}{45}k_3 + \frac{2}{15}k_4 + \frac{16}{45}k_5 + \frac{7}{90}k_6)$$

$$\begin{aligned}
k_1 &= f(x, y_n, y'_n, y''_n, y'''_n) \\
k_2 &= f(x_n + \frac{1}{2}h,\ y_n + \frac{1}{2}hy'_n + \frac{(\frac{1}{2}h)^2}{2}y''_n + \frac{(\frac{1}{2}h)^3}{6}y'''_n + h^4(0), \\
&\quad y'_n + \frac{1}{2}hy''_n + \frac{(\frac{1}{2}h)^2}{2}y'''_n + h^3(0), \\
&\quad y''_n + \frac{1}{2}hy'''_n + h^2(0), y'''_n + h(\frac{1}{2}k_1)) \\
k_3 &= f(x_n + \frac{1}{4}h,\ y_n + \frac{1}{4}hy'_n + \frac{(\frac{1}{4}h)^2}{2}y''_n + \frac{(\frac{1}{4}h)^3}{6}y'''_n + h^4(0), \\
&\quad y'_n + \frac{1}{4}hy''_n + \frac{(\frac{1}{4}h)^2}{2}y'''_n + h^3(0), \\
&\quad y''_n + \frac{1}{4}hy'''_n + h^2(\frac{1}{32}k_1), y'''_n + h(\frac{3}{16}k_1 + \frac{1}{16}k_2)) \\
k_4 &= f(x_n + \frac{1}{2}h,\ y_n + \frac{1}{2}hy'_n + \frac{(\frac{1}{2}h)^2}{2}y''_n + \frac{(\frac{1}{2}h)^3}{6}y'''_n + h^4(0), \\
&\quad y'_n + \frac{1}{2}hy''_n + \frac{(\frac{1}{2}h)^2}{2}y'''_n + h^3(\frac{1}{64}k_1), \\
&\quad y''_n + \frac{1}{2}hy'''_n + h^2(\frac{3}{32}k_1 + \frac{1}{32}k_2), y'''_n + h(\frac{11}{2}k_3)) \\
k_5 &= f(x_n + \frac{3}{4}h,\ y_n + \frac{3}{4}hy'_n + \frac{(\frac{3}{4}h)^2}{2}y''_n + \frac{(\frac{3}{4}h)^3}{6}y'''_n + h^4(\frac{9}{1024}), \\
&\quad y'_n + \frac{3}{4}hy''_n + \frac{(\frac{3}{4}h)^2}{2}y'''_n + h^3(\frac{33}{512}k_1 + \frac{9}{512}k_2), \\
&\quad y''_n + \frac{3}{4}hy'''_n + h^2(\frac{-3}{128}k_1 + \frac{3}{128}k_2 + \frac{9}{32}k_3), \\
&\quad y'''_n + h(\frac{-3}{16}k_2 + \frac{3}{8}k_3 + \frac{9}{16}k_4)) \\
k_6 &= f(x_n + h,\ y_n + hy'_n + \frac{(\frac{1}{2}h)^2}{2}y''_n + \frac{h^3}{6}y'''_n + h^4(\frac{9}{448}k_1 + \frac{3}{64}k_2), \\
&\quad y'_n + hy''_n + \frac{h^2}{2}y'''_n + h^3(\frac{-9}{56}k_1 + \frac{-3}{112}k_2 + \frac{9}{28}k_3), \\
&\quad y''_n + hy'''_n + h^2(\frac{25}{56}k_1 + \frac{-9}{56}k_2 + \frac{-3}{7}k_3 + \frac{9}{14}k_4),
\end{aligned}$$

$$y_n''' + h(\frac{1jk}{7}k_1 + \frac{4}{7}k_2 + \frac{6}{7}k_3 + \frac{-12}{7}k_4 + \frac{8}{7}k_5))$$

4. NUMERICAL EXPERIMENT

To study the efficiency of method we present some numerical examples using the new method for direct integration of fourth order ODEs.

Problem 1: (Habtamu and Masho (2017))

$y^{iv} = x \qquad y(0) = y''(0) = y'''(0) = 0$
$y'(0) = 1 \quad h = 0.1 \quad 0 \le x \le 1$ Theoretical Solution:$y(x) = \frac{x^5}{120} + x$

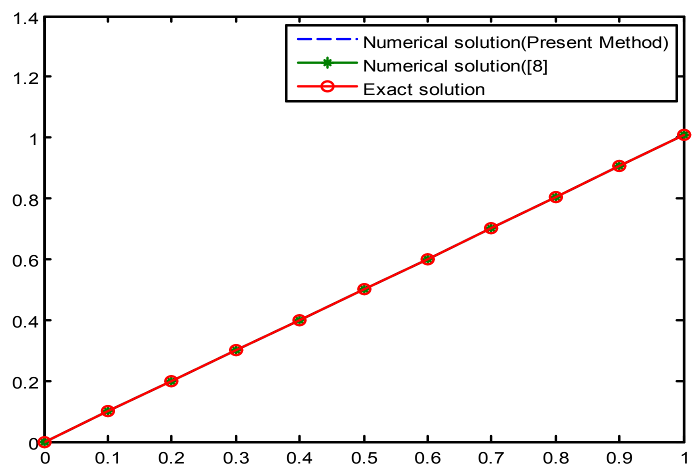

FIGURE 1. The Relationship between the Exact,Numerical solutions by present method and Numerical solutions by [10] for example 1

4.1 Discussion of Results of Problem 1

Figure 1 shows the results using both classical fifth Order Runge-Kutta (Habtamu and Masho) method by reducing the problem to system of first order ODEs and the present method (15); the present method (15) solved the problem 1 faster, it provides direct solution to the problem without any need for reduction and exhibited a higher degree of accuracy.

Problem 2: (Habtamu and Masho (2017))

$y^{iv} = (y')^2 - yy'' - 4x - e^x(1 - 4x + x^2)$ $\quad y(0) = y'(0) = y'''(0) = 1$
$y''(0) = 3 \quad h = 0.1 \quad 0 \le x \le 1$
Theoretical Solution: $y(x) = \frac{x^5}{120} + x$

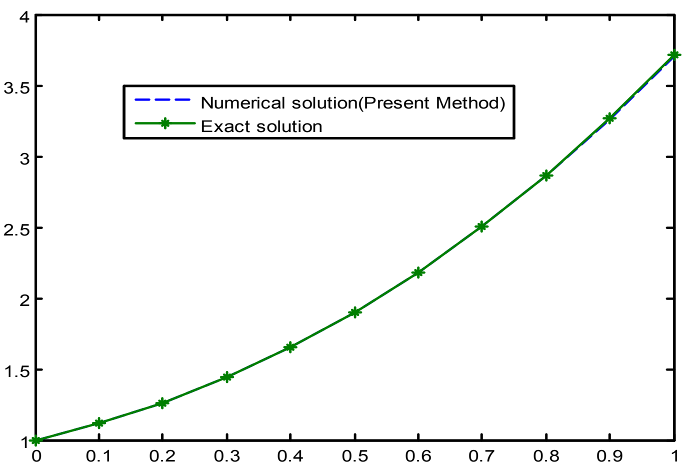

Figure 2. The Relationship between the Exact and Numerical solutions for example 2

4.2 Discussion of Results of Problem 2

Figure 2 shows the relationship between the exact and numerical solutions using the present method. The present approximate results are found to be in good agreement with the exact solutions than the other methods mentioned in the literatures.

4. CONCLUDING REMARKS

Through the approach presented in this paper, the RK method can be extended to solve higher order differential equations. The method requires less work with very little cost (when compared with classical RK).

REFERENCES

[1] Tanner, L. H., *The spreading of silicone oil drops on horizontal surfaces*, Phys. D: Appl. Phys. **12** 1473-1484, 1979.

[2] Vigo - Aguinar J. Ramos H, *Dissipative chebyshev exponential fitted method for numerical solution of second order differential equation*, J. comp. Appl. Math **158** 187-211, 2003.
[3] Butcher J.C and Wright W.M., *The Construction of Practical General Linear Methods*, BIT **43** 695-721, 2003.
[4] Agam S.A and Irhebbhude, *Modification of the Fourth Order Runge-Kutta Method for Third Order Ordinary Differential Equations*, Abacus **38** (2) 87-95, 2011.
[5] Butcher J.C. *General Linear Methods*, Acta Numerical 157-256, 2005.
[6] Hairer E. and Warner G., *A Theory for Nystrom Method*, Numerical Mathematical **25** 383-400, 1976.
[7] Butcher J.C and Hojjat G., *Second Derivative Methods with Runge-Kutta Stability*, Num. alg. **40** 415-429, 2005.
[8] Jain R.K., Rajive Kumar and Rakesh Goel, *Super Stable Implicit Runge-Kutta method for second Order initial value problems*, BIT **29** 518-526, 1989.
[9] Lambert J D, *Computational Methods in Ordinary Differential Equations*, John Willey, New York, 1973.
[10] Habtamu G. D. and Masho J. K., *Numerical Solution of Fourth Order Ordinary Differential Equations Using Fifth Order Runge-Kutta Method* , Asian Journal of Science and Technology **08**(02) 4332-4339, 2017.

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