

CHEBYSHEV HYBRID MULTISTEP METHOD FOR DIRECTLY SOLVING SECOND-ORDER INITIAL AND BOUNDARY VALUE PROBLEMS

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ABSTRACT. A numerical method had been proposed in this work for directly solving second-order initial and boundary value problems in ordinary differential equations. The approach of collocation of a derivative function at equidistant grid and off-grid points $x = x_{n+\frac{i}{3}}$, $i = 0, 1, \dots, k$, where k is the step number in the interval $[x_n, x_{n+k}]$ was adopted. The derived Chebyshev Hybrid Multistep Method (CHMM) is of order $(2k + 3)$. The continuous scheme was evaluated at different off-step points to obtain multiple hybrid schemes of uniform order which were solved simultaneously for dense approximations that make its computation competitive. Some numerical examples were given to demonstrate the accuracy and efficiency advantages of the proposed method.

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1. INTRODUCTION

The study of numerous physical phenomena in sciences and engineering often lead to second-order ordinary differential equations of the type

$$y'' = f(x, y, y'), \quad (1)$$

subject to the following initial conditions

$$y(a) = \eta_0, y'(a) = \eta_1 \quad (2)$$

or boundary conditions

$$y(a) = y_N, y(b) = y_{N_1}, \quad (3)$$

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where $f \in R$ is continuous and satisfies Lipschitz condition, $\frac{\partial f}{\partial y}$ is continuous and non-negative, $\forall x \in \{a, b\}$ and $\forall y$, then there exists a unique solution to the problem (1) ([2, 21, 39]). There exists an extensive literature on numerical methods for solving ordinary differential equations ODEs (see [1]-[5]). Among these methods, one-step and linear multistep methods are often adopted for the numerical integration of ODEs which are either implemented in predictor-corrector (see e.g. [2]-[6]) or block ([7]-[12]) modes and their stability domain are carefully studied. Backward Difference Formulae (BDF) are the first numerical methods to be proposed for stiff differential equations. [3] reported that LMMs, generally are more efficient in terms of accuracy though with weak stability properties for a given number of evaluations per step, suffered the disadvantage of requiring additional starting values and special procedures for changing step-length. Stomer-Cowell methods which are a subclass of LMMs with step number greater than two exhibit phenomenon which [13] termed as orbital instability. [14] proposed alternative methods which require prior knowledge of the frequencies of the periodic problem. Dahlquist [15] barrier theorem, which has been generalized by [1] has greatly influenced the development of LMMs for second-order in particular.

The main characteristic of all the methods developed in the literature for the numerical solution of (1) is that they belong to the class of multistep and hybrid techniques. Emphasis is now on hybrid methods because of their high order and the p-stability characteristic [14]. Hybrid methods have the advantage of incorporating function evaluation at off-step points which affords the opportunity of circumventing the Dahlquist [15] barrier. The first analysis of instability phenomena and step size restrictions for hyperbolic equations was made in [16], later, many authors undertook a stability analysis very often independently. The size of the stability region of a numerical method is especially important in the choice of methods suitable for solving stiff system. Indeed, for the numerical solution of stiff systems, it requires an interval of stability region to be as large as possible to avoid restricted step size implementation during numerical integration [17]. In order to increase the stability of linear multistep methods, off step points are included and according to [1], the main ingredient for these methods is the adoption of Chebyshev polynomials as the basis function.

This work presented an efficient and accurate numerical method for directly solving initial and boundary value problems in ordinary differential equations. The derived CHMM handled effectively classical problems in physics such as Troesch’s [33]-[31] and Bratu’s [22]-[31] problems and singular problem such as Michaelis-Menten Oxygen diffusion [37].

Theorem 1:(Chebyshev equi-oscillation theorem [18]) Let f be a continuous function on a finite and closed interval $[a, b]$ and let $T_n(x)$ be a polynomial with the oscillation

$$L = \max_{a \leq x \leq b} |f(x) - T_n(x)|, \tag{4}$$

then, $T_n(x)$ is the best approximating polynomial to $f(x)$ on the interval $[a, b]$ if and only if there exist at least $n + 2$ points $a \leq x_0 < x_1 < \dots < x_{n+1} \leq b$ such that

$$|f(x_i) - T_n(x_i)| = L, i = 0, 1, \dots, n + 1, \tag{5}$$

$$f(x_i) - T_n(x_i) = -[f(x_{i+1}) - T_n(x_{i+1})] \text{ for } i=0, 1, \dots, n. \tag{6}$$

The points $x_i = 0, 1, \dots, n + 1$ which satisfy the above conditions (5) and (6) are the knots.

Proof. The proof of the necessary and sufficient conditions for $T_n(x)$ to be the best Chebyshev approximating polynomial to function $f(x)$ is the same as that given in [18]. \square

The paper has been arranged in five sections. Section 2 deals with the mathematical formulation and the development of the scheme, section 3, the computational aspects of the scheme, section 4 with convergence and stability analysis and finally, section 5 numerical experiments and results.

2. MATHEMATICAL FORMULATION

The objective is to construct a two-step continuous formulation with four intermediate points within interval $[0, 2]$. To achieve this, the following conditions are imposed:

$$y(x_{n+\frac{j}{3}}) = y_{n+\frac{j}{3}}, \quad j = 0, 1 \tag{7}$$

$$y''(x_{n+\frac{j}{3}}) = f((x_{n+\frac{j}{3}}, y(x_{n+\frac{j}{3}}), y'(x_{n+\frac{j}{3}})) \quad j = 0, 1, \dots, 3k, \tag{8}$$

where k is the step number. It is assumed that the theoretical solution of (1) denoted by $y(x_{n+j})$ is approximated by a polynomial

of the type

$$\bar{y}(x_{n+j}) \cong y(x_{n+j}) = \sum_{j=0}^{(r+s)-1} a_j T_j(x) \quad (9)$$

where a_j 's are the coefficients to be determined, r and s are the numbers of distinct interpolation and collocation points. The terms $T_j(x)$ are Chebyshev polynomials of the first kind which can be obtained by means of Rodrigues's formula [19]

$$T_j(x) = \frac{(-2)^j j!}{(2j)!} \sqrt{1-x^2} \frac{d^j}{dx^j} (1-x^2)^{j-\frac{1}{2}} \quad j = 0, 1, 2, 3, \dots \quad (10)$$

. We interpolate (9) at $x = x_{n+\frac{j}{3}}$, $j = 0, 1$ and collocate its second-derivative at $x = x_{n+\frac{j}{3}}$, $j = 0, 1, \dots, 3k$ respectively. These and conditions (7) and (8) lead to a system of $(r+s)$ equations and $(r+s)$ unknown coefficients to be determined. Solving for the unknown parameters and substituting into equation (7), after some simplification gives the continuous scheme of the form:

$$\begin{aligned} y(\psi) = & (1-3\psi)y_n + 3\psi y_{n+\frac{1}{3}} + h^2 \left(\left(\frac{81}{4480} \psi^8 \right. \right. \\ & - \frac{441}{320} \psi^5 - \frac{27}{160} \psi^7 - \frac{49}{40} \psi^3 + \frac{203}{120} \psi^4 + \frac{1}{2} \psi^2 - \frac{28549}{362880} \psi + \frac{21}{32} \psi^6 \Big) f_n \\ & + \left(-\frac{10621}{90720} \psi - \frac{81}{224} \psi^8 + \frac{10}{3} \psi^3 + \frac{81}{28} \psi^7 - \frac{127}{12} \psi^4 + \frac{279}{20} \psi^5 - \frac{363}{40} \psi^6 \Big) f_{n+1} \\ & + \left(\frac{81}{4480} \psi^8 - \frac{1}{12} \psi^3 - \frac{27}{224} \psi^7 + \frac{137}{480} \psi^4 - \frac{27}{64} \psi^5 + \frac{199}{72576} \psi + \frac{51}{160} \psi^6 \Big) f_{n+2} \\ & + \left(-\frac{243}{2240} \psi^8 - \frac{261}{40} \psi^4 + \frac{27}{28} \psi^7 - \frac{279}{80} \psi^6 + 3\psi^3 + \frac{261}{40} \psi^5 - \frac{275}{1728} \psi \right) f_{n+\frac{1}{3}} \\ & + \left(\frac{5717}{40320} \psi + \frac{243}{896} \psi^8 - \frac{4149}{320} \psi^5 - \frac{513}{224} \psi^7 + \frac{351}{32} \psi^4 + \frac{1233}{160} \psi^6 - \frac{15}{4} \psi^3 \right) f_{n+\frac{2}{3}} \\ & + \left(-\frac{2763}{320} \psi^5 + \frac{7703}{120960} \psi + \frac{963}{160} \psi^6 + \frac{99}{16} \psi^4 - \frac{459}{224} \psi^7 - \frac{15}{8} \psi^3 + \frac{243}{896} \psi^8 \right) f_{n+\frac{4}{3}} \\ & + \left. \left(-\frac{243}{2240} \psi^8 - \frac{81}{40} \psi^4 + \frac{117}{40} \psi^5 + \frac{3}{5} \psi^3 - \frac{171}{80} \psi^6 - \frac{403}{20160} + \frac{27}{35} \psi^7 \right) f_{n+\frac{5}{3}} \right), \quad (11) \end{aligned}$$

noting that $\psi = \frac{x-x_{n+1}}{h}$, that is $x = \zeta h + h$. The main discrete scheme is obtained by evaluating (11) at $\psi = 2$ which gives

$$\begin{aligned} y_{n+2} - 6y_{n+\frac{1}{3}} + 5y_n = & \frac{h^2}{36288} \left(1375 f_n + 15004 f_{n+1} + 199 f_{n+2} \right. \\ & \left. + 19554 f_{n+\frac{1}{3}} + 13401 f_{n+\frac{2}{3}} + 6177 f_{n+\frac{4}{3}} + 4770 f_{n+\frac{5}{3}} \right) \quad (12) \end{aligned}$$

The scheme (12) is consistent, zero-stable and of order $p = 7$ with error constant $c_{p+2} = \frac{349}{595213920}$. The local truncation error is

obtained by assuming that $E(x)$ is a sufficiently differentiable function, the linear difference operators associated with the formulas in (12) can be written in the form

$$\mathfrak{L}_{\frac{j}{3}} [E(x); h] \equiv E(x + \frac{j}{3}h) - \left[\alpha_0 + \alpha_{\frac{j}{3}} E(x + \frac{1}{3}h) + h^2 \sum_{m=0}^6 \beta_m (\frac{j}{3}) E^{(2)}(x + \frac{j}{3}h) \right], \quad (13)$$

where $j = 1, 2, \dots, 6$. Taking $E(x)$ as the true solution of the problem in (1), after expanding (13) in Taylor series about x yields the truncation errors of the form

$$\mathfrak{L}_{\frac{j}{3}} [y(x); h] = c_0^j y(x) + c_1^j h y'(x) + \dots + c_r^j h^r y^{(r)}(x) + O(h^{(r+1)}) \quad (14)$$

where c_r^j are constants. It is worthy to note that the first $p + 1$ constants will be equal to zero, which means that

$$c_0^j = c_1^j = c_2^j = \dots = c_{p+1}^j = 0, \text{ and } c_{p+2}^j \neq 0,$$

this implies that

$$\mathfrak{L}_{\frac{j}{3}} [y(x); h] = c_{p+2}^j h^{p+2} y^{p+2}(x) + O(h^{p+3})$$

where p and c_{p+2}^j are respectively known as the order and local principal error constant of the corresponding formula. The principal local truncation error of the main formulas in (12) is obtained to be:

$$\mathfrak{L}_{\frac{j}{3}} [y(x); h] = \frac{349}{595213920} h^9 y^9(x) + O(h^{10}).$$

Method (12) is said to be consistent if it has order at least one. If the first and second characteristic polynomial is defined such that

$$\rho(d) = \sum_{j=0}^k \alpha_j d^j, \quad \sigma(d) = \sum_{j=0}^k \beta_j d^j,$$

it is easily verified that method (12) is consistent if and only if $\rho(1) = \rho'(1)$, $\rho''(1) = 2\sigma(1)$. For the zero-stability, the following definition is pertinent.

Definition 1: (See [4, 21]) The CHMM (12) is said to be zero-stable if no root of the first characteristic polynomial $\rho(d)$ has modulus greater than one, and if every root of modulus one has multiplicity not greater than two.

The roots of the first characteristic polynomial of (12) has modulus one (1), whose multiplicity is one. This implies that the method is

zero-stable. Noting that, $\frac{d\psi}{dx} = \frac{1}{h}$, we obtain the following additional methods from (11)

$$\left. \begin{aligned} y_{n+\frac{5}{3}} - 5y_{n+\frac{1}{3}} + 4y_n &= h^2 \left(\frac{1669}{54432}f_n + \frac{3751}{13608}f_{n+1} - \frac{95}{54432}f_{n+2} \right. \\ &\quad \left. + \frac{3875}{9072}f_{n+\frac{1}{3}} + \frac{5069}{18144}f_{n+\frac{2}{3}} + \frac{1457}{18144}f_{n+\frac{4}{3}} + \frac{179}{9072}f_{n+\frac{5}{3}} \right), \\ y_{n+\frac{4}{3}} - 4y_{n+\frac{1}{3}} + 3y_n &= h^2 \left(\frac{2089}{90720}f_n + \frac{3457}{22680}f_{n+1} - \frac{19}{18144}f_{n+2} \right. \\ &\quad \left. + \frac{4813}{15120}f_{n+\frac{1}{3}} + \frac{5461}{30240}f_{n+\frac{2}{3}} - \frac{419}{30240}f_{n+\frac{4}{3}} + \frac{109}{15120}f_{n+\frac{5}{3}} \right), \\ y_{n+1} - 3y_{n+\frac{1}{3}} + 2y_n &= h^2 \left(\frac{2803}{181440}f_n + \frac{1777}{45360}f_{n+1} - \frac{137}{181440}f_{n+2} \right. \\ &\quad \left. + \frac{1265}{6048}f_{n+\frac{1}{3}} + \frac{1657}{20160}f_{n+\frac{2}{3}} - \frac{1049}{60480}f_{n+\frac{4}{3}} + \frac{11}{2016}f_{n+\frac{5}{3}} \right), \\ y_{n+\frac{2}{3}} - 2y_{n+\frac{1}{3}} + y_n &= h^2 \left(\frac{863}{108864}f_n + \frac{1987}{136080}f_{n+1} - \frac{221}{544320}f_{n+2} \right. \\ &\quad \left. + \frac{8999}{90720}f_{n+\frac{1}{3}} - \frac{769}{181440}f_{n+\frac{2}{3}} - \frac{1609}{181440}f_{n+\frac{4}{3}} + \frac{263}{90720}f_{n+\frac{5}{3}} \right). \end{aligned} \right\} (15)$$

The first derivative of (11) is obtained and evaluated at $x = x_{n+i}$, $i = 0, 2, \dots, k$ to produce more additional schemes.

3. COMPUTATIONAL ASPECTS OF THE PROPOSED METHOD

The main scheme (12), the additional schemes (15) and a derivative scheme obtained at x_n are combined to give a single matrix of finite difference equations which are solved simultaneously in block form:

$$A^1 Y_{m+1} = A^0 Y_m + hA^2 Y_{m+2} + h^\mu [DF(Y_m) + BF(Y_{m+1})] \quad (16)$$

where

$$Y_{m+1} = (y_{n+\frac{1}{3}}, y_{n+\frac{2}{3}}, y_{n+1}, y_{n+\frac{4}{3}}, y_{n+\frac{5}{3}}, y_{n+2})^T,$$

$$Y_m = (y_{n-\frac{1}{3}}, y_{n-\frac{2}{3}}, y_{n-1}, y_{n-\frac{4}{3}}, y_{n-\frac{5}{3}}, y_n)^T,$$

$$Y_{m+2} = (y'_{n-\frac{1}{3}}, y'_{n-\frac{2}{3}}, y'_{n-1}, y'_{n-\frac{4}{3}}, y'_{n-\frac{5}{3}}, y'_n)^T,$$

$$F(Y_m) = (f_{n-\frac{5}{3}}, f_{n-\frac{4}{3}}, f_{n-1}, f_{n-\frac{2}{3}}, f_{n-\frac{1}{3}}, f_n)^T,$$

$$F(Y_{m+1}) = (f_{n+\frac{1}{3}}, f_{n+\frac{2}{3}}, f_{n+1}, f_{n+\frac{4}{3}}, f_{n+\frac{5}{3}}, f_{n+2})^T$$

A^1 identity matrix, μ is the order of the differential equation A^0 , A^2 , D and B are defined as follows:

$$A^0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{4}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{3} \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix},$$

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{28549}{1088640} \\ 0 & 0 & 0 & 0 & 0 & \frac{1027}{17010} \\ 0 & 0 & 0 & 0 & 0 & \frac{253}{2688} \\ 0 & 0 & 0 & 0 & 0 & \frac{1088}{8505} \\ 0 & 0 & 0 & 0 & 0 & \frac{35225}{217728} \\ 0 & 0 & 0 & 0 & 0 & \frac{41}{210} \end{pmatrix} \quad \text{and}$$

$$B = \begin{pmatrix} \frac{275}{5184} & -\frac{5717}{120960} & \frac{10621}{272160} & -\frac{7703}{362880} & \frac{403}{60480} & -\frac{199}{217728} \\ \frac{194}{945} & -\frac{8}{81} & \frac{788}{8505} & -\frac{97}{1890} & \frac{46}{2835} & -\frac{19}{8505} \\ \frac{165}{448} & -\frac{267}{4480} & \frac{5}{32} & -\frac{363}{4480} & \frac{57}{2240} & -\frac{47}{13440} \\ \frac{1504}{2835} & -\frac{8}{945} & \frac{2624}{8505} & -\frac{8}{81} & \frac{32}{945} & -\frac{8}{1701} \\ \frac{8375}{12096} & \frac{3125}{72576} & \frac{25625}{54432} & -\frac{625}{24192} & \frac{275}{5184} & -\frac{1375}{217728} \\ \frac{6}{7} & \frac{3}{35} & \frac{68}{105} & \frac{3}{70} & \frac{6}{35} & 0 \end{pmatrix}$$

The schemes that made up the block are of uniform order $(7, 7, 7, 7, 7, 7)^T$ with error constant

$$c_{p+2} = \left(\frac{1}{382725}, \frac{1625}{714256704}, \frac{496}{279006525}, \frac{1}{765450}, \frac{233}{279006525}, \frac{6031}{17856417600} \right)^T$$

5. CONVERGENCE AND STABILITY ANALYSIS

Definition 2: (Convergence [20]) Suppose the initial values y_0, y_1, \dots, y_p of equation (1) subject to (2) and (3) satisfy

$$\eta(h) \equiv \max_{0 \leq i \leq p} |Y(x_i) - y_h(x_n)| \rightarrow 0 \text{ as } h \rightarrow 0$$

Then the solution $\{y_n\}$ is said to converge to $Y(x)$ if

$$\max_{x_0 \leq x_n \leq b} |Y(x_n) - y_h(x_n)| \rightarrow 0 \text{ as } h \rightarrow 0$$

If CHMM is convergent for all problems of type (1), then it is called a convergent numerical method.

Definition 3: (Stability domain[1]) The set

$$R = \left\{ \lambda \in \mathbb{C}; \begin{array}{l} \text{all roots of } \rho_j(\lambda) \text{ of the characteristic equation CHMM satisfies } |\rho_j(\lambda)| \leq 1 \\ \text{multiple roots satisfy } |\rho_j(\lambda)| < 1 \end{array} \right\}$$

is called the stability domain or zero-stability. We have A -stability if $R \supset \mathbb{C}^-$.

The zero-stability is concerned with the stability of the difference system

$$A^1 Y_{m+1} - A^0 Y_m \quad (17)$$

whose first characteristic polynomial $\rho(\lambda)$ is given by

$$\rho(\lambda) = \det[\lambda A^1 - A^0] = \lambda^5(\lambda - 1) = 0. \quad (18)$$

From (18), $\rho(\lambda) = 0$ satisfies $|\lambda_j| \leq 1$, $j = 1, \dots, k$ and for those roots with $|\lambda_j| = 1$, the multiplicity does not exceed 2. Therefore block method (16) is zero-stable, also consistent, as it has the order p greater than 1. Hence the convergence of the method is asserted as in [21].

Definition 4: (p -stable [3]) Method is p -stable if its interval of periodicity is $(0, \infty)$.

Stability Analysis: According to [11], the linear stability of block method can be investigated by applying the method to the test equation $y'' = \lambda y$. This leads to a recursion of the form:

$$Y_{n+2} = M(z)Y_n,$$

$$M(z) := [I - zD]^{-1}[A + zB], \quad z := \lambda h$$

M is called the amplification matrix and its eigenvalues the amplification factors. By requiring the elements of the diagonal matrix D to be positive, the matrix $I - zD$ is nonsingular for all z on the negative real axis. Therefore, in the sequel, we assume that the (diagonal) elements of D are positive. We shall use the result on the power of a matrix as

$$\|N^n\| = o(n^{q-1}[\rho(N)]^n) \text{ as } n \rightarrow \infty$$

where $\|\cdot\|$ and $\rho(N)$ are the spectra norm and the radius of N and where all diagonal sub-matrices of the Jordan normal form of N which have spectral radius $\rho(N)$ are at most $q \times q$. If the spectra radius $\rho(N) < 1$, then N is called power bounded. The region where the amplification matrix $M(z)$ is power bounded is called the stability region of the block method. If the stability region contains

the origin, then the method is called zero-stable (see definition 3). We obtained the stability polynomial as:

$$\rho = \frac{\eta (200 \eta z^6 + 8253 \eta z^5 + 10 z^6 + 44856 \eta z^4 - 252 z^5 - 949725 \eta z^3 - 42021 z^4 + 43936830 \eta z^2 + 6307875 z^3 - 1350253800 \eta z - 236830230 z^2 + 20832487200 \eta - 9065989800 z - 20832487200)}{200 z^6 + 8253 z^5 + 44856 z^4 - 949725 z^3 + 43936830 z^2 - 1350253800 z + 20832487200}$$

The stability domain of the methods is as shown in the figure below.

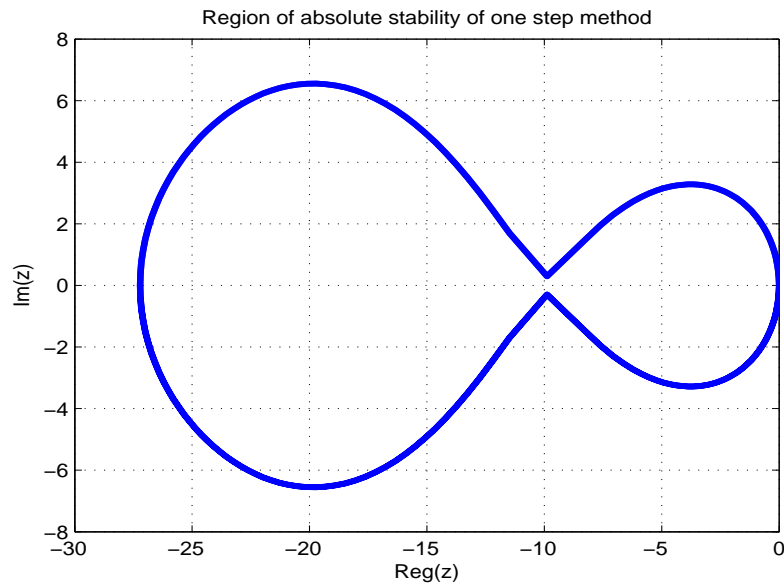


Fig. 1. Stability domain of proposed numerical integrator

4. NUMERICAL EXPERIMENTS AND RESULTS

In this section, six numerical examples are presented.

Problem 1: The first numerical example considered is the nearly periodic Stiefel and Bettis initial value problem

$$y'' + y = \frac{1}{1000} e^{ix}, \quad y(0) = 1, \quad y'(0) = 0.9995i$$

which has also appeared in [3] and [22]. The equivalent form of the problem are

$$\begin{aligned} y_1'' + y_1 &= \frac{1}{1000}\cos(x), \quad y_1(0) = 1, \quad y_1'(0) = 0 \\ y_2'' + y_2 &= \frac{1}{1000}\sin(x), \quad y_2(0) = 1, \quad y_2'(0) = 0.9995, \quad x \in [0, \pi], \\ y(x) &= y_1(x) + iy_2(x); \quad y_1, y_2 \in R, \quad D(x) = \sqrt{y_1^2(x) + y_2^2(x)}. \end{aligned}$$

The theoretical solutions are $y_1(x) = \cos(x) + \frac{1}{2000}x\sin(x)$ and $y_2(x) = \sin(x) - \frac{1}{2000}x\cos(x)$. The differential system in the problem represents motion on a perturbed circular orbit in complex plane in which the point $y(x)$ spirals slowly outward such that its distance from the origin at any given time t is $D(x)$. The exact solution of $D(x) = 1.001972$. The derived CHMM is applied on Stiefel and Bettis problem 1, in the interval which corresponds to 20 orbits of the point $y(x)$ and the integration was done with uniform grid sizes. As shown on Table 2, CHMM compared favourably with [3] and [22], as all solution values generated by the CHMM spiral outward while it was reported in [3] that the solutions generated with the Stomer-Cowell five-step scheme: spiral inward revealing its orbital instability.

Problem 2: The two body problem given as

$$\begin{aligned} y_1'' &= \frac{-y_1}{r}, \quad y_1(0) = 1, \quad y_1'(0) = 0, \\ y_2'' &= \frac{-y_2}{r}, \quad y_2(0) = 0, \quad y_2'(0) = 1 \\ r &= \sqrt{y_1^2 + y_2^2}, \quad x \in [0, 15\pi], \quad h = 0.1. \end{aligned}$$

which has theoretical solutions $y_1(x) = \cos(x)$, and $y_2(x) = \sin(x)$. The results are shown in Table 3.

Problem 3: The nonlinear perturbed system on the range $[0, 10]$, with $\varepsilon = 10^{-3}$.

$$\begin{aligned} y_1'' + 25y_1 + \varepsilon(y_1^2 + y_2^2) &= \varepsilon\varphi_1, \quad y_1(0) = 1 \quad y_1'(0) = 0, \\ y_2'' + 25y_2 + \varepsilon(y_1^2 + y_2^2) &= \varepsilon\varphi_2, \quad y_2(0) = \varepsilon \quad y_2'(0) = 5, \end{aligned}$$

where,

$$\begin{aligned} \varphi_1 &= 1 + \varepsilon^2 + 2\sin(5x + x^2) + 2\cos(x^2) + (25 - 4x^2)\sin(x^2), \\ \varphi_2 &= 1 + \varepsilon^2 + 2\sin(5x + x^2) - 2\sin(x^2) + (25 - 4x^2)\cos(x^2). \end{aligned}$$

TABLE 1. Results for problem 1, $h = \frac{1}{320}$

$x * 10^{-2}$	y_1 -exact	y_1 -computed	Error in y_1
0.625	0.99998048834470104	0.999980488344701	4.98E-18
0.937	0.99995609895403291	0.999956098954033	3.34E-18
1.250	0.99992195414021281	0.999921954140213	9.96E-18
1.563	0.99987805423635216	0.999878054236352	1.65E-18
1.875	0.99982439967073145	0.999824399670731	1.49E-17
2.187	0.99976099096679615	0.999760990966796	6.63E-18
2.500	0.99968782874315152	0.999687828743152	1.99E-17
2.812	0.99960491371355663	0.999604913713557	1.16E-17
3.215	0.99951224668691738	0.999512246686917	2.49E-17
$x * 10^{-2}$	y_2 -exact	y_2 -computed	Error in y_2
0.625	0.00624683437101026	0.0062468343710103	1.90E-20
0.937	0.00937017537749408	0.0093701753774941	2.43E-20
1.250	0.01249342496998468	0.0124934249699847	1.10E-19
1.563	0.01561655267853829	0.0156165526785383	2.10E-20
1.875	0.01873952803440018	0.0187395280344002	2.44E-19
2.187	0.02186232057030199	0.0218623205703020	1.27E-19
2.500	0.02498489982075888	0.0249848998207589	4.49E-19
2.812	0.02810723532236683	0.0281072353223668	2.96E-19
3.215	0.03122929661409975	0.0312292966140998	7.16E-19

TABLE 2. Results for problem 1

h	[3]	[21]	CHMM	Stomer-Cowell
$\frac{\pi}{4}$	1.003067	1.002084	1.003145	0.965645
$\frac{\pi}{5}$	1.002217	1.002117	1.002312	0.993734
$\frac{\pi}{6}$	1.002047	1.002064	1.002048	0.999596
$\frac{\pi}{9}$	1.001978	1.001984	1.001982	0.001829
$\frac{\pi}{12}$	1.001973	1.001974	1.001971	0.001953

The theoretical solution are given as $y_1(x) = \cos(5x) + \varepsilon \sin(x^2)$ and $y_2(x) = \sin(5x) + \varepsilon \cos(x^2)$. The problem is solved within the interval $[0, 1]$ for 100 iterations. The results are as presented on Table 4.

Problem 4: The classical nonlinear Bratu’s boundary value problem in one-dimensional planar coordinates is given as

$$\left. \begin{aligned} -y''(x) &= \lambda e^y, & 0 < x < 1 \\ y(0) &= y(1) = 0. \end{aligned} \right\} \quad (19)$$

TABLE 3. Results for different values of h for problem 2.

	$h = 0.1$		$h = 0.05$		$h = 0.0025$	
x	y_1	y_2	y_1	y_2	y_1	y_2
0.2	2.27(-16)	2.59(-15)	8.80(-19)	1.01(-17)	2.2(-19)	2.0(-21)
0.4	8.89(-16)	4.98(-15)	3.44(-17)	1.94(-17)	4.4(-19)	1.0(-20)
0.6	1.93(-15)	6.99(-15)	7.51(-18)	2.73(-17)	6.1(-19)	1.0(-20)
0.8	3.28(-15)	8.48(-15)	1.27(-17)	3.31(-17)	7.6(-19)	1.0(-20)
1.0	4.80(-15)	9.30(-15)	1.87(-17)	3.63(-17)	9.2(-19)	1.0(-20)

TABLE 4. Results for problem 3, $h = 0.01$

x -value	y_1 -exact	y_1 -computed	Error in y_1
0.01	0.99500416527802576	0.9950041652780258	1.16E-16
0.02	0.98006657784124163	0.9800665778412418	2.27E-16
0.03	0.95533648912560601	0.9553364891256065	5.62E-16
0.04	0.92106099400288508	0.9210609940028859	8.89E-16
0.05	0.87758256189037271	0.8775825618903741	1.42E-15
0.06	0.82533561490967829	0.8253356149096802	1.93E-15
0.07	0.76484218728448842	0.7648421872844910	2.62E-15
0.08	0.69670670934716542	0.6967067093471686	3.28E-15
0.09	0.62160996827066445	0.62160996827066851	4.06E-15
0.1	0.54030230586813971	0.54030230586814452	4.801E-15

x -value	y_2 -exact	y_2 -computed	Error in y_2
0.01	0.09983341664682815	0.09983341664682685	1.30E-15
0.02	0.19866933079506121	0.19866933079505862	2.59E-15
0.03	0.29552020666133957	0.29552020666133577	3.80E-15
0.04	0.38941834230865049	0.38941834230864550	4.98E-15
0.05	0.47942553860420300	0.47942553860419698	6.02E-15
0.06	0.56464247339503535	0.56464247339502836	7.00E-15
0.07	0.64421768723769105	0.64421768723768327	7.77E-15
0.08	0.71735609089952276	0.71735609089951428	8.48E-15
0.09	0.78332690962748338	0.78332690962747445	8.93E-15
0.1	0.84147098480789650	0.84147098480788721	9.30E-15

The exact solution to (19) is given in [28], [27], [26], [24], [23], [25], as

$$y(x) = -2 \operatorname{Log} \left[\frac{\cosh \left(\frac{x\theta}{2} - \frac{\theta}{4} \right)}{\cosh \left(\frac{\theta}{4} \right)} \right], \quad (20)$$

where θ satisfies $\theta = \sqrt{2\lambda} \cosh \left(\frac{\theta}{4} \right)$. There are three possible solutions considering the value of λ viz:

TABLE 5. Observed absolute error for Bratu’s problem for $\lambda = 1$

x	CHMM	Non-polynomial spline [30]	Laplace [32]	Decomposition [31]	B-Spline [29]
0.1	2.54×10^{-13}	5.77×10^{-10}	1.98×10^{-6}	2.68×10^{-3}	2.98×10^{-6}
0.2	5.04×10^{-13}	2.47×10^{-10}	3.94×10^{-6}	2.02×10^{-3}	5.46×10^{-6}
0.3	7.20×10^{-13}	4.56×10^{-11}	5.85×10^{-6}	1.52×10^{-4}	7.33×10^{-6}
0.4	9.30×10^{-13}	9.64×10^{-11}	7.70×10^{-6}	2.20×10^{-3}	8.50×10^{-6}
0.5	9.34×10^{-13}	1.46×10^{-10}	9.47×10^{-6}	3.01×10^{-3}	8.89×10^{-6}
0.6	9.30×10^{-13}	9.64×10^{-11}	1.11×10^{-5}	2.20×10^{-3}	8.50×10^{-6}
0.7	7.20×10^{-13}	4.56×10^{-11}	1.26×10^{-5}	1.52×10^{-4}	7.33×10^{-6}
0.8	5.04×10^{-13}	2.47×10^{-10}	1.35×10^{-5}	2.02×10^{-4}	5.46×10^{-6}
0.9	2.54×10^{-13}	5.77×10^{-10}	1.20×10^{-5}	2.68×10^{-3}	2.98×10^{-6}

TABLE 6. Observed absolute error for Bratu’s problem for $\lambda = 2$

x	CHMM	Non-polynomial spline [30]	Laplace [32]	Decomposition [31]	B-Spline [29]
0.1	7.92×10^{-12}	9.71×10^{-9}	2.13×10^{-3}	1.52×10^{-2}	1.72×10^{-5}
0.2	1.55×10^{-11}	1.41×10^{-8}	4.21×10^{-3}	1.47×10^{-2}	3.26×10^{-5}
0.3	3.04×10^{-11}	1.98×10^{-8}	6.19×10^{-3}	5.89×10^{-3}	4.49×10^{-5}
0.4	4.46×10^{-11}	2.42×10^{-8}	8.00×10^{-3}	3.25×10^{-3}	5.28×10^{-5}
0.5	4.51×10^{-11}	2.60×10^{-8}	9.60×10^{-3}	6.98×10^{-3}	5.56×10^{-5}
0.6	4.56×10^{-11}	2.42×10^{-8}	1.09×10^{-3}	3.25×10^{-3}	5.28×10^{-5}
0.7	3.04×10^{-13}	1.98×10^{-8}	1.19×10^{-2}	5.89×10^{-3}	4.49×10^{-5}
0.8	1.55×10^{-11}	1.41×10^{-8}	1.24×10^{-2}	1.47×10^{-2}	3.26×10^{-5}
0.9	7.92×10^{-12}	9.71×10^{-9}	1.09×10^{-3}	1.52×10^{-2}	1.72×10^{-5}

- 1 If $\lambda > \lambda_c$, then the Bratu problem has zero solution,
- 2 If $\lambda = \lambda_c$, then the Bratu problem has one solution,
- 3 If $\lambda < \lambda_c$, then the Bratu problem has tow solution, where the critical value λ_c satisfies the equation

$$4 = \sqrt{2\lambda_c} \sinh \frac{\theta_c}{4}, \quad \lambda_c = 3.513830719.$$

The Bratu’s problem in (19) are solved using $h = 0.1$ for different values of $\lambda = 1, 2$ and 3.513830719 so that fair comparison can be done with [29, 30, 31, 32].The absolute errors observed are as presented in Tables 5, 6 and 7.

Problem 5: The two-point boundary value problem, Troesch’s problem, was considered.

$$\left. \begin{aligned} y''(x) &= \nu \sin(\nu y), \\ y(0) &= 0, \quad y(1) = 1, \end{aligned} \right\} \quad (21)$$

TABLE 7. Observed absolute error for Bratu's problem for $\lambda = 3.51$

x	CHMM	Non-polynomial spline [30]	B-Spline [29]
0.1	2.45×10^{-9}	6.61×10^{-6}	3.84×10^{-2}
0.2	4.74×10^{-9}	5.83×10^{-6}	7.48×10^{-2}
0.3	2.29×10^{-9}	6.19×10^{-6}	1.06×10^{-1}
0.4	1.18×10^{-10}	6.89×10^{-6}	1.27×10^{-1}
0.5	2.50×10^{-11}	7.31×10^{-6}	1.35×10^{-1}
0.6	1.18×10^{-10}	6.89×10^{-6}	1.27×10^{-1}
0.7	2.9×10^{-9}	6.19×10^{-6}	1.06×10^{-1}
0.8	4.74×10^{-9}	5.83×10^{-6}	7.48×10^{-2}
0.9	2.45×10^{-9}	6.61×10^{-6}	3.84×10^{-2}

TABLE 8. Numerical solutions of Troesch's problem for the case $\nu = 0.5$

x	Exact [34]	CHMM	SincGalerkin method[34]	HPM [35]	Laplace Method[36]
0.1	0.095176902	0.095944349	0.095944347	0.095948026	0.0959444
0.2	0.190633869	0.19212875	0.192128740	0.192135797	0.1921288
0.3	0.286653403	0.28879440	0.288794409	0.288804238	0.2887944
0.4	0.383522929	0.38618485	0.386184841	0.386196642	0.3861849
0.5	0.481537385	0.48454716	0.484547165	0.4845599	0.4845472
0.6	0.581001975	0.58413325	0.584133254	0.584145785	0.5841333
0.7	0.682235133	0.68520115	0.685201142	0.685212297	0.6852012
0.8	0.785571787	0.68520115	0.788016528	0.788025104	0.7880166
0.9	0.891366988	0.89285422	0.892854218	0.892859085	0.8928542

where ν is a positive constant. The closed-form solution according to [33] in term of the Jacobian elliptic $sc(n|m)$ has been given as

$$y(x) = \frac{2}{\nu} \sinh^{-1} \left[\frac{y'(0)}{2} sc(\nu x|m) \right], \quad (22)$$

where $m = 1 - \frac{1}{4}(y'(0))^2$ and satisfies the transcendental equation

$$sc(\nu|m)(1-m)^{\frac{1}{2}} = \sinh\left(\frac{\nu}{2}\right) \quad (23)$$

The solution $y(x)$ has a singularity at a pole of the $sc(\nu|m)$ which make the problem to be very difficult to solve as n increases. The results are as reported in the Tables 8, 9 and 10 as compared with those in [34, 35, 36].

TABLE 9. Errors of Troesch's problem for the case $\nu = 0.5$

x	CHMM	SincGalerkin method[34]	HPM [35]	Laplace Method[36]
0.1	7.675×10^{-4}	0.000767445	0.000771124	7.7×10^{-4}
0.2	1.495×10^{-3}	0.001494871	0.001501928	1.5×10^{-3}
0.3	2.141×10^{-3}	0.002141006	0.002150835	2.1×10^{-3}
0.4	2.662×10^{-3}	0.002661912	0.002673713	2.7×10^{-3}
0.5	3.010×10^{-3}	0.003009780	0.003022515	3.0×10^{-3}
0.6	3.131×10^{-3}	0.003131279	0.00314381	3.1×10^{-3}
0.7	2.966×10^{-3}	0.002966009	0.002977164	3.0×10^{-3}
0.8	2.445×10^{-3}	0.002444741	0.002453317	2.4×10^{-3}
0.9	1.487×10^{-3}	0.001487230	0.001492098	1.5×10^{-3}

TABLE 10. Errors of Troesch's problem for the case $\nu = 1$

x	CHMM	SincGalerkin method[34]	HPM [35]	Laplace Method[36]
0.1	2.864×10^{-3}	0.002864253	0.003137419	2.9×10^{-3}
0.2	5.640×10^{-3}	0.005640467	0.006166675	5.9×10^{-3}
0.3	8.227×10^{-3}	0.008226572	0.008965863	8.2×10^{-3}
0.4	1.049×10^{-2}	0.010490630	0.011384418	1.0×10^{-2}
0.5	1.225×10^{-2}	0.012252675	0.013225579	1.2×10^{-2}
0.6	1.326×10^{-2}	0.013260386	0.014224205	1.3×10^{-2}
0.7	1.316×10^{-2}	0.013157446	0.01401684	1.3×10^{-2}
0.8	1.114×10^{-2}	0.011439736	0.012099173	1.1×10^{-2}
0.9	7.395×10^{-3}	0.007392506	0.007763039	7.4×10^{-3}

Problem 6: The Michaelis-Menten Oxygen diffusion problem with uptake kinetics given in [37] was also considered.

$$y''(x) + \frac{2}{x}y'(x) = \delta \frac{y(x)}{y(x) + \mu}, \quad 0 < x < 1 \tag{24}$$

with the boundary conditions

$$y'(0) = 0, \tag{25}$$

$$ay(1) + by'(1) = c, \quad a > 0, \quad b \geq 0, \quad c \geq .0, \tag{26}$$

In Fig. 2, we display the numerical results for the values $a = c = 1$, $b = 0.2$, $\delta = 10$ and μ . The results was obtained by choosing the value of x such that the singular point will be avoided.

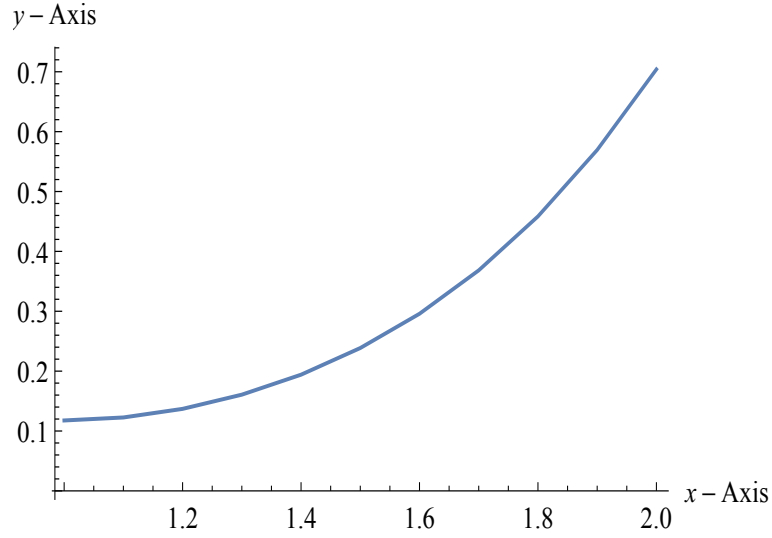


Fig. 2. CHMM solution of Michaelis-Meneten Oxygen diffusion problem

Problem 7: Van Der Pol oscillator [38]

$$y'' - 2\epsilon(1 - y^2)y' + y = 0, \quad y(0) = 0, \quad y'(0) = 0.5, \quad \epsilon = 0.025. \quad (27)$$

This solution to this problem within interval $[0,50]$ over 100 iteration is given in **Fig.3**.

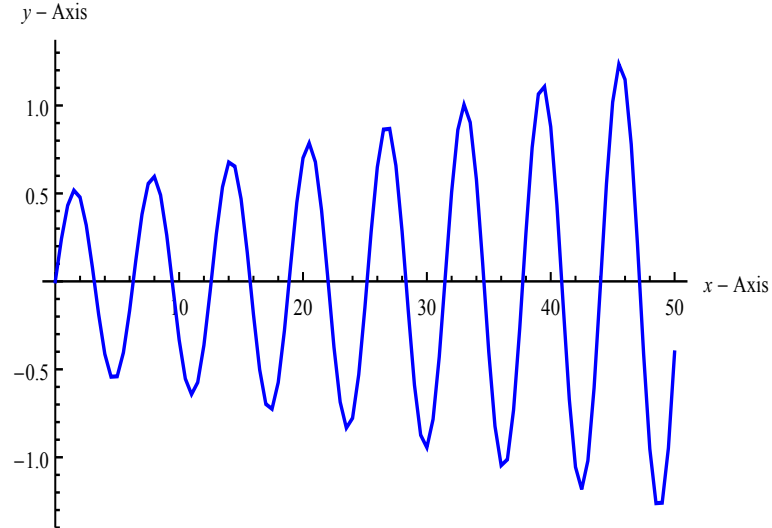


Fig. 3. CHMM solution of Van-der-Pol problem

5. CONCLUSION

An efficient and accurate CHMM had been derived for directly solving second order initial and boundary value problems in ordinary differential equations. The application of the method to classical problems in physics such as Troesch's, Bratu's and Michaelis-Mentene oxygen problem was presented. The new methods are therefore recommended for general purpose use. Finally, the stability domain of the CHMM displaced in (**Fig 1**) shows that the method is p -stable.

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