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SECOND REFINEMENT OF GENERALIZED JACOBI ITERATIVE METHOD FOR SOLVING LINEAR SYSTEM OF EQUATIONS

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ABSTRACT. The Jacobi and Gauss-Seidel algorithms are among the stationary iterative methods for solving linear system of equations. In this paper, we present the new method which is called second-refinement of generalized Jacobi (SRGJ) method for solving linear system of equations. This new method is the fastest method to converge to the exact solution as compared with Jacobi (J), refinement of Jacobi (RJ), generalized of Jacobi and refinement of generalized Jacobi (RGJ) method by considering strictly diagonally dominant (SDD), symmetric positive definite (SPD) and M-matrices. It is verified by checking the number of iterations and rate of convergence.
The SRGJ method can be applied to solve ODE and PDE problems when finite difference method results system of linear equations with its coefficient matrices are strictly diagonally dominant (SDD) or symmetric positive definite matrices (SPD) or M-matrices.

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1. INTRODUCTION

In this paper, we consider second-refinement of generalized Jacobi iterative method (SRGJ). It is a refinement of refinement of generalized Jacobi iterative method (RRGJ), hence here after we call second-refinement of generalized Jacobi iterative method (SRGJ). In many application one face with the problem of large and sparse linear systems of the form

$$Ax = b \tag{1}$$

where $A = (a_{ij})$ is nonsingular real matrix of order n, b is a given n dimensional real vector and x is an n dimensional vector to be determined. Iterative methods, based on splitting A into $A = T_m - E_m - F_m$, where T_m is a banded matrix with band width 2m + 1, $a_{ii} \neq 0$ and E_m and F_m are

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strictly lower and upper triangular part of $T_m - A$ respectively, can compute successive approximations to obtain more accurate solutions to a linear system at each iteration step n. Second-refinement of generalized Jacobi (SRGJ) iterative method is used to accelerate the convergence of basic Jacobi iterative method. It has been proved that, if A is strictly diagonally dominant (SDD) or irreducibly diagonally dominant (IDD), then the associated Jacobi iteration converges for any initial guess. The Jacobi iteration (J) for first degree is

$$x^{(n+1)} = D^{-1}(L+U)x^{(n)} + D^{-1}b \tag{2}$$

2. PRELIMINARY

Let $A = (a_{ij})$ be an nxn matrix and $T_m = (t_{ij})$ be a banded matrix of bandwidth 2m + 1 defined as :

$$t_{ij} = \begin{cases} a_{ij}, & |i-j| \leq m \\ 0, & otherwise \end{cases}$$

We consider the decomposition $A = T_m - E_m - F_m$ where $-E_m$ and $-F_m$ are the strict lower and upper part of the matrix $A - T_m$, respectively.

$$T_m = \begin{pmatrix} a_{1,1} & \cdots & a_{1,m+1} & & \\ \vdots & \ddots & & \ddots & \\ a_{m+1,1} & & \ddots & & a_{n-m,n} \\ & \ddots & & \ddots & \vdots \\ & & a_{n,n-m} & \cdots & a_{n,n} \end{pmatrix},$$

$$E_m = \begin{pmatrix} & & \\ -a_{m+2,1} & & \\ \vdots & \ddots & \\ -a_{n,1} & \cdots & -a_{n-m-1,n} \end{pmatrix},$$

$$F_m = \begin{pmatrix} -a_{1,m+2} & \cdots & -a_{1,n} \\ & \ddots & \vdots \\ & & -a_{n-m-1,n} \\ & & \end{pmatrix}$$

Definition 1: (varga [10]). For n x n real matrices A, M, and N, A = M - N is a regular splitting of the matrix A if M is nonsingular with $M^{-1} \geq O$, and $N \geq O$. Similarly, A = M - N is weak regular splitting of the matrix A if M is nonsingular with $M^{-1} \geq O$ and $M^{-1}N \geq O$.

The following definitions, lemmas and theorems are important for our study used by Young [13], Varga [10], Datta [2], Hackbusch [3] and Saad [8].

Definition 2: If a matrix A is strictly diagonally dominant or irreducibly diagonally dominant, then it is nonsingular.

Definition 3: A complex matrix $A \in C^{nxn}$ is reducible if and only if there exist a permutation matrix P (i.e., P is obtained from the identity I by a permutation of the rows of I) and an integer $k \in \{1,...,n-1\}$ such that

$$PAP^T = \begin{pmatrix} A_{11} & A_{12} \\ O & A_{22} \end{pmatrix}$$

Where A_{11} is kxk and A_{22} is (n-k)x(n-k) . If A is not reducible, then A is said to be irreducible.

Definition 4: An nxn marix $A = (a_{ij})$ is said to be strictly diagonally dominant (SDD) if

$$|a_{ii}| > \sum_{j=1, j\neq 1}^{n} |a_{ij}|$$

Definition 5: If an nxn marix $A = (a_{ij})$ is said to be diagonally dominant (DD) if

$$|a_{ii}| \geq \sum_{j=1, j\neq 1}^{n} |a_{ij}|$$

Definition 6: A is irreducibly diagonally dominant (IDD) if A is irreducible and diagonally dominant, with strict inequality holding in definition 2 for at least one i.

Definition 7: An nxn matrix A = (a_{ij}) is said to be symmetric positive definite (SPD) if A is symmetric, $(A = A^T)$ and positive definite $x^T Ax > 0$ for all $x \neq 0$.

Definition 8: A matrix is said to be an M-matrix if it satisfies the following four properties:

(1) $a_{ii} > 0$ for i = 1,...,n
(2) $a_{ij} \leq 0$ for $i \neq j$, i, j = 1,...,n
(3) A is nonsingular
(4) $A^{-1} \geq O$

Alternatively, A matrix $A \in R^{n,n}$, n is said to be an M-matrix if A can be written as A = sI - B, where $B \geq O$ and $s \geq \rho(B)$.

Definition 9: The spectral radius matrix A is the largest absolute value of the eigenvalues of A: $\rho(A) := max\{|\lambda| : \lambda \in \sigma(A)\}$.

Lemma 1: The spectral radius satisfies the following rules

- $\rho(kA) = |k|\rho(A)$ for all $k \in C$ and $A \in C^{nxn}$.
- $\rho(A^k) = (\rho(A))^k$ for all $k \in N$ and $A \in C^{nxn}$.
- $\rho(A) = \rho(A^H) = \rho(A^T)$ for all $A \in C^{nxn}$.

Theorem 1: A linear iteration $\Phi(x,b) = Mx + Nb$ with the iteration matrix M = M[A] is convergent if and only if $\rho(M) < 1$.
Theorem 2: Let A = M - N be a regular splitting of the matrix A.Then $\rho(M^{-1}N) < 1$ if and only if A is nonsingular and $A^{-1} \geq O$.
Theorem 3: Let $A = (a_{ij})$, $B = (b_{ij})$ be two matrices such that $A \leq B$ and $b_{ij} \leq O$ for all $i \neq j$. Then if A is an M-matrix, so is the matrix B.

3. GENERALIZED JACOBI (GJ) ITERATIVE METHOD

The system of linear equation (1) is solved by different iterative methods. One of them is GJ iterative method. This method was first proposed by D.K. Salkuyeh [9].
If equation (1) can be written as $(T_m - E_m - F_m)x = b$
$\Rightarrow x^{(n+1)} = T_m^{-1}(E_m + F_m)x^{(n)} + T_m^{-1}b$

$$x^{(n+1)} = T_m^{-1}(E_m + F_m)x^{(n)} + T_m^{-1}b \tag{3}$$

This scheme is called Generalized Jacobi (GJ) iterative method for m = 0, 1, 2, ... If m = 0, then GJ = J.

4. REFINEMENT OF GENERALIZED JACOBI (RGJ) METHOD

Generalized Jacobi (GJ) iterative method is a few modification of Jacobi iterative method and refinement of generalized Jacobi (RGJ) iterative method is similarly a few modification of generalized Jacobi iterative method. It is a method with a few computations. This method was first introduced by V. B. Kumar Vatti and G. G. Gonfa [11]. Equation (1) with $A = T_m - E_m - F_m$ can be written as:
$\Rightarrow (T_m - E_m - F_m)x = b$
$\Rightarrow T_m x = (E_m + F_m)x + b$
$\Rightarrow T_m x = (T_m - A)x + b$, where $E_m + F_m = T_m - A$
$\Rightarrow T_m x = T_m x + b - Ax$
$\Rightarrow x = x + T_m^{-1}(b - Ax)$
$\Rightarrow x^{(n+1)} = \tilde{x}^{(n+1)} + T_m^{-1}(b - A\tilde{x}^{(n+1)})$ where $\tilde{x}^{(n+1)} = {T_m}^{-1}(E_m + F_m)x^{(n)} + {T_m}^{-1}b$
$\Rightarrow x^{(n+1)} = {T_m}^{-1}(E_m + F_m)x^{(n)} + {T_m}^{-1}b + T_m^{-1}[b - A$
$({T_m}^{-1}(E_m + F_m)x^{(n)} + {T_m}^{-1}b)]$
$\Rightarrow x^{(n+1)} = {T_m}^{-1}(E_m + F_m)x^{(n)} + {T_m}^{-1}b + T_m^{-1}[b - (T_m - E_m - F_m)({T_m}^{-1}$
$(E_m + F_m)x^{(n)} + {T_m}^{-1}b)]$ After simplification, we get:
$\Rightarrow x^{(n+1)} = [{T_m}^{-1}(E_m + F_m)]^2 x^{(n)} + (I + {T_m}^{-1}(E_m + F_m)){T_m}^{-1}b$

$$x^{(n+1)} = [{T_m}^{-1}(E_m + F_m)]^2 x^{(n)} + (I + {T_m}^{-1}(E_m + F_m)){T_m}^{-1}b \tag{4}$$

Equation (4) is called refinement of generalized Jacobi (RGJ) iterative method for m = 0, 1, 2, ... If m = 0, then RGJ = RJ.

5. SECOND-REFINEMENT OF GENERALIZED JACOBI (SRGJ) METHOD

In this paper we need to introduce second-refinement of generalized Jacobi (SRGJ) iterative method.
By taking equation $x^{(n+1)} = \tilde{x}^{(n+1)} + T_m^{-1}(b - A\tilde{x}^{(n+1)})$ substitute equation (4) on $\tilde{x}^{(n+1)}$. We get
$x^{(n+1)} = [T_m{}^{-1}(E_m + F_m)]^2 x^{(n)} + (I + T_m{}^{-1}(E_m + F_m))T_m{}^{-1}b + T_m^{-1}[b - (T_m - E_m - F_m)([T_m{}^{-1}(E_m + F_m)]^2 x^{(n)} + (I + T_m{}^{-1}(E_m + F_m))T_m{}^{-1}b)]$.
After simplifying we get:
$\Rightarrow x^{(n+1)} = [T_m{}^{-1}(E_m + F_m)]^3 x^{(n)} + (I + T_m^{-1}(E_m + F_m) + (T_m^{-1}(E_m + F_m))^2)T_m^{-1}b$
$\therefore x^{(n+1)} = [T_m{}^{-1}(E_m + F_m)]^3 x^{(n)} + (I + T_m^{-1}(E_m + F_m) + (T_m^{-1}(E_m + F_m))^2)T_m^{-1}b$ (5)
The above equation is called second refinement of generalized Jacobi (SRGJ) method for m = 0,1,2,.... If m = 0, then SRGJ = SRJ.

6. CONVERGENCE OF SECOND-REFINEMENT OF GENERALIZED JACOBI (SRGJ) METHOD

Theorem 4: If A is a strictly diagonally dominant or an irreducibly diagonally dominant matrix, then the associated Jacobi iterations converge for any $x^{(0)}$.

See the proof in R. S. Varga [10]

Theorem 5: If A and 2D - A are symmetric and positive definite matrices, then the Jacobi method is convergent for any initial guess.
Proof: Let A and 2D - A be SPD. We know $x^*Ax > 0$ and $x^*(2D - A)x > 0$, where $A = D - L - L^T$.
$\Rightarrow D^{-1}(L + L^T)x = \lambda x \Rightarrow (L + L^T)x = \lambda Dx \Rightarrow x^*(L + L^T)x = \lambda x^*Dx$
$\Rightarrow x^*Dx - x^*Ax = \lambda x^*Dx \Rightarrow x^*Ax = (1 - \lambda)x^*Dx \Rightarrow 1 - \lambda > 0 \Rightarrow \lambda < 1$
$\therefore \lambda < 1$..............⊛
And we consider $x^*(2D - A)x > 0 \Rightarrow 2x^*Dx - x^*Ax > 0 \Rightarrow x^*Ax < 2x^*Dx \Rightarrow (1 - \lambda)x^*Dx < 2x^*Dx \Rightarrow 1 - \lambda < 2 \Rightarrow \lambda > -1$
$\therefore \lambda > -1$..........⊛⊛
From ⊛ and ⊛⊛, we get $-1 < \lambda < 1$. where λ is the eigenvalues of $D^{-1}(L + L^T)$.
Hence, $\rho(D^{-1}(L + L^T)) < 1$.
Theorem 6: (Salkuyeh [9]): If A is an M-matrix, then the Jacobi iterative method is convergent for any initial guess x^0.

Proof: Given A is M-matrix. Let A = M - N. $\Rightarrow A = D - L - U \Rightarrow M = D$ and $N = L + U \Rightarrow A \leq M \Rightarrow$ by theorem 8 M is M-matrix. $\Rightarrow M^{-1} > 0$. On the other hand $N \geq 0$.
$\therefore A = M - N$ is a regular splitting of the matrix A. Having in mind that $A^{-1} \geq 0$ and theorem 7 we deduce that $\rho(G_J) < 1$.

Theorem 7: Let A be an SDD matrix. Then for any natural number $m < n$ the generalized Jacobi (GJ) iterative method is convergent for any initial guess $x^{(0)}$.

See the proof in D. K. Salkuyeh [9]

Proof: Let $M = (M_{ij})$ and $N = (N_{ij})$ be nxn matrices with M being SDD. Then (see Jin and etal, 2005, Lemma 1), $\rho(M^{-1}N) \leq \rho = max_i \rho_i$, ...$\circledast$, where $\rho_i = \frac{\sum_{j=1}^{n} |N_{ij}|}{|M_{ii}| - \sum_{j=1, j\neq i}^{n} |M_{ij}|}$. Now, let $M = T_m$ and $N = E_m + F_m$ in the GJ method. Obviously, in this case the matrix M is SDD. Hence M and N satisfy relation ($\circledast$). Having in mind that the matrix A is an SDD matrix, it can be easily verified that $\rho_i < 1$. Therefore $\rho(M^{-1}N) \leq \rho < 1$ and this completes the proof.

Theorem 8: If A and $2T_m - A$ are symmetric and positive definite matrices, then the Generalized Jacobi (GJ) iterative method converges for any initial guess $x^{(0)}$.

Proof: Let A and $2T_m - A$ be SPD.
We know that $x^*Ax > 0$ and $x^*(2T_m - A)x > 0$, where $A = T_m - E_m - E_m^T$ $\Rightarrow T_m^{-1}(E_m + E_m^T)x = \lambda x \Rightarrow (E_m + E_m^T)x = \lambda T_m x \Rightarrow x^*(E_m + E_m^T)x = \lambda x^* T_m x \Rightarrow x^* T_m x - x^*Ax = \lambda x^* T_m x \Rightarrow x^*Ax = (1-\lambda)x^* T_m x \Rightarrow 1 - \lambda > 0 \Rightarrow \lambda < 1$
$\therefore \lambda < 1$..............$\circledast$
And we consider $x^*(2T_m - A)x > 0 \Rightarrow 2x^* T_m x - x^*Ax > 0 \Rightarrow x^*Ax < 2x^* T_m x \Rightarrow (1-\lambda)x^* T_m x < 2x^* T_m x \Rightarrow 1 - \lambda < 2 \Rightarrow \lambda > -1$
$\therefore \lambda > -1$..........$\circledast\circledast$
From $\circledast$ and $\circledast\circledast$, we get $-1 < \lambda < 1$. where λ is the eigenvalues of $T_m^{-1}(E_m + E_m^T)$.
Hence, $\rho(T_m^{-1}(E_m + E_m^T)) < 1$.

Theorem 9: Let A be an M-matrix. Then for a given natural number $m < n$, the GJ method is convergent for any initial guess $x^{(0)}$.

Proof: (Salkuyeh [9]). Let $M_m = T_m$ and $N_m = E_m + F_m$ in the GJ method. Obviously, in this case we have $A \leq M_m$. Hence by Theorem 3, we conclude that the matrix M_m is an M-matrix. On the other hand we have $N_m \geq O$. Therefore, $A = M_m - N_m$ is a regular splitting of the matrix A. Having in mind that $A^{-1} \geq 0$ and Theorem 2 we deduce that $\rho(B_{GJ}^{(m)}) < 1$.

Theorem 10: If A is strictly diagonally dominant matrix, then the refinement of generalized Jacobi method converges for any choice of the initial approximation $x^{(0)}$.

Proof: (Vatti and et al [11]). Assuming $\bar{x}$ is the real solution of (1), as A is SDD by Theorem 7, generalized Jacobi method is convergent. Let $x^{(n+1)} \to \bar{x}$ (exact solution). Then we have $||x^{(n+1)} - \bar{x}||_\infty \leq ||x^{(n+1)} - \bar{x}||_\infty + ||T_m^{-1}||_\infty ||(b - Ax^{(n+1)})||_\infty$. From the fact $||x^{(n+1)} - \bar{x}||_\infty \to 0$, we have $||b - Ax^{(n+1)}||_\infty \to 0$. Therefore, $||x^{(n+1)} - \bar{x}||_\infty \to 0$. Hence refinement of generalized Jacobi method is convergent.

Theorem 11: If A and $2T_m - A$ are SPD matrix, then the refinement of generalized Jacobi iterative method is convergent for any initial guess $x^{(0)}$.

Proof: Using equation (3) and Theorem 8, we have $\rho(T_m^{-1}(E_m + E_m^T)) < 1$.

Theorem 12: Let $A = (a_{ij})$ be an M-matrix. Then for a given natural number $m < n$, the refinement of generalized Jacobi method converges for any choice of initial approximation $x^{(0)}$.

Proof: It follows from Theorem 9. See Vatti and et al [11].

Theorem 13: If A is a strictly diagonally dominant or an irreducibly diagonally dominant matrix, then the second-refinement of generalized Jacobi iterations converge for any $x^{(0)}$.

Proof: Let X be the real solution of (1). Given that A is SDD, using theorem 4, 7, and 10, the J,GJ and RGJ methods are convergent and hence $x^{(n+1)} \to X$ (exact Solution). As we mentioned above by theorem 10, $\tilde{x}^{(n+1)} = [T_m{}^{-1}(E_m + F_m)]^2 x^{(n)} + (I + T_m{}^{-1}(E_m + F_m))T_m{}^{-1}b$ is convergent. So $\tilde{x}^{(n+1)} \to X$.

$x^{(n+1)} = \tilde{x}^{(n+1)} + T_m^{-1}(b - A\tilde{x}^{(n+1)})$ or $x^{(n+1)} - X = \tilde{x}^{(n+1)} - X + T_m^{-1}(b - A\tilde{x}^{(n+1)})$. Then,

$$\left\|x^{(n+1)} - X\right\| = \left\|\tilde{x}^{(n+1)} - X + T_m^{-1}(b - A\tilde{x}^{(n+1)})\right\|$$

$$\leq \left\|\tilde{x}^{(n+1)} - X\right\| + \left\|T_m^{-1}(b - A\tilde{x}^{(n+1)})\right\|$$

$$\Rightarrow \left\|x^{(n+1)} - X\right\| \leq \left\|\tilde{x}^{(n+1)} - X\right\| + \|T_m^{-1}\| \left\|(b - A\tilde{x}^{(n+1)})\right\|$$

$$\to \|X - X\| + \left\|T_m^{-1}\right\| \|b - AX\| = 0 + \left\|T_m^{-1}\right\| \|b - b\| = 0 + 0 = 0$$

Then, $x^{(n+1)} \to X$

$$\Rightarrow \rho[(T_m^{-1}(E_m + F_m))^3] = (\rho(T_m^{-1}(E_m + F_m)))^3 < 1$$

Therefore, the SRGJ iterative method is convergent.

Theorem 14: If A and $2T_m - A$ are SPD matrices, then the second-refinement of generalized Jacobi (SRGJ) iterative method is convergent for any initial guess $x^{(0)}$.

Proof: Using equation (3) and Theorem 8, we have $\rho(T_m^{-1}(E_m+E_m^T))<1$. Let X be the exact solution of (1). Then the generalized Jacobi iterative method can be written as
$X=[I-T_m^{-1}(E_m+E_m^T)]^{-1}T_m^{-1}b$ if $x^{(n+1)}\to X$. Using equation (5):
$x^{(n+1)}=[T_m{}^{-1}(E_m+F_m)]^3x^{(n)}+(I+T_m^{-1}(E_m+F_m)+(T_m^{-1}(E_m+F_m))^2)T_m^{-1}b$. Now using equation (4) and the exact solution X, we have:
$\Rightarrow X=[T_m{}^{-1}(E_m+F_m)]^3X+(I+T_m^{-1}(E_m+F_m)+(T_m^{-1}(E_m+F_m))^2)T_m^{-1}b$
$\Rightarrow X=(I-(T_m^{-1}(E_m+F_m))^3)^{-1}[I+T_m^{-1}(E_m+F_m)+(T_m^{-1}(E_m+F_m))^2]T_m^{-1}b$
$=[I+(T_m^{-1}(E_m+F_m))^3+(T_m^{-1}(E_m+F_m))^6+...][I+T_m^{-1}(E_m+F_m)+(T_m^{-1}(E_m+F_m))^3]T_m^{-1}b=[I+T_m^{-1}(E_m+F_m)+(T_m^{-1}(E_m+F_m))^2+(T_m^{-1}(E_m+F_m))^3+(T_m^{-1}(E_m+F_m))^4+....]T_m^{-1}b$
$=[I-T_m^{-1}(E_m+F_m)]^{-1}T_m^{-1}b$
$\therefore X=[I-T_m^{-1}(E_m+F_m)]^{-1}T_m^{-1}b$ is consistent to (1) and generalized Jacobi method. On the other hand,
$x^{(n+1)}=[T_m{}^{-1}(E_m+F_m)]^3x^{(n)}+(I+T_m^{-1}(E_m+F_m)+(T_m^{-1}(E_m+F_m))^2)T_m^{-1}b$
$=[T_m{}^{-1}(E_m+F_m)]^6x^{(n-1)}+(I+T_m^{-1}(E_m+F_m)+(T_m^{-1}(E_m+F_m))^2+(T_m^{-1}(E_m+F_m))^3+(T_m^{-1}(E_m+F_m))^4+(T_m^{-1}(E_m+F_m))^5)T_m^{-1}b$
$=[T_m{}^{-1}(E_m+F_m)]^9x^{(n-2)}+(I+T_m^{-1}(E_m+F_m)+(T_m^{-1}(E_m+F_m))^2+(T_m^{-1}(E_m+F_m))^3+...+(T_m^{-1}(E_m+F_m))^8)T_m^{-1}b$
=
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.
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$=[T_m{}^{-1}(E_m+F_m)]^{3n+3}x^{(0)}+(I+T_m^{-1}(E_m+F_m)+(T_m^{-1}(E_m+F_m))^2+(T_m^{-1}(E_m+F_m))^3+...+(T_m^{-1}(E_m+F_m))^{3n+2})T_m^{-1}b$
We are given that A is SPD then $\rho(T_m^{-1}(E_m+E_m^T))<1$.
Thus $\lim_{n\to\infty}[T_m{}^{-1}(E_m+F_m)]^{3n+3}=0$.
$\Rightarrow \lim_{n\to\infty}x^{(n+1)}=\lim_{n\to\infty}[T_m{}^{-1}(E_m+F_m)]^{3n+3}+\sum_{k=0}^{\infty}[T_m{}^{-1}(E_m+F_m)]^nT_m^{-1}b$
$=0+[I-T_m^{-1}(E_m+F_m)]^{-1}T_m^{-1}b=[I-T_m^{-1}(E_m+F_m)]^{-1}T_m^{-1}b\to X$
$\Rightarrow \rho[(T_m^{-1}(E_m+F_m))^3]=[\rho(T_m^{-1}(E_m+F_m))]^3<1$.
Therefore, the second-refinement of generalized Jacobi (SRGJ) iterative method is convergent.

Theorem 15: If A is an M-matrix, then the second-refinement of generalized Jacobi iterative method is convergent for any initial guess $x^{(0)}$.

Proof: We are given that A is an M-matrix. We want to show that SRGJ iterative method is convergent. From theorem 9 we can see that

GJ iterative method is convergent.
$\Rightarrow (T_m^{-1}(E_m+F_m) < 1$. Using theorem 1 and 12,
$\rho(B_{RGJ}) = \rho([T_m^{-1}(E_m+F_m)]^2) = [\rho(T_m^{-1}(E_m+F_m))]^2 < 1$.
$\rho(B_{RGJ}) < 1$.
Using theorem 12, $\rho(B_{SRGJ}) = \rho([T_m^{-1}(E_m+F_m)]^3) = [\rho(T_m^{-1}(E_m+F_m))]^3 < 1$.
$\therefore \rho(B_{SRGJ}) < 1$.
$\therefore$ SRGJ iterative method is convergent if A is an M-matrix.

Theorem 16: The second-refinement of generalized Jacobi method converges faster than the generalized Jacobi and refinement of generalized Jacobi method when generalized Jacobi method is convergent.

Proof: We can write equation (3) by $x^{(n+1)} = Gx^{(n)} + C$, (4) by $x^{(n+1)} = G^2x^{(n)} + B$ and (5) by $x^{(n+1)} = G^3x^{(n)} + K$ where $G = T_m^{-1}(E_m+F_m)$, $C = T_m^{-1}b$, $B = [I + T_m^{-1}(E_m+F_m)]T_m^{-1}b$ and $K = [I + T_m^{-1}(E_m+F_m) + (T_m^{-1}(E_m+F_m))^2]T_m^{-1}b$. Given that, $\|G\| < 1$.
Let X be the exact solution of (1). $\Rightarrow X = GX + C$, $X = G^2X + B$ and $X = G^3X + K$.
Let us consider generalized Jacobi method:

$$\Rightarrow x^{(n+1)} = Gx^{(n)} + C \Rightarrow x^{(n+1)} - X = Gx^{(n)} - X + C$$
$$\Rightarrow x^{(n+1)} - X = G(x^{(n)} - X) + GX + C - X$$
$$\Rightarrow x^{(n+1)} - X = G(x^{(n)} - X)$$
$$\Rightarrow \left\|x^{(n+1)} - X\right\| = \left\|G(x^{(n)} - X)\right\| \le \|G\| \left\|x^{(n)} - X\right\|$$
$$\le \left\|G^2\right\| \left\|x^{(n-1)} - X\right\| \le ... \le \|G^n\| \left\|x^{(1)} - X\right\|$$
$$\Rightarrow \left\|x^{(n+1)} - X\right\| \le \|G^n\| \left\|x^{(1)} - X\right\| = \|G\|^n \left\|x^{(1)} - X\right\|$$

Now let us consider refinement of generalized Jacobi method:

$$\Rightarrow x^{(n+1)} = G^2x^{(n)} + B \Rightarrow x^{(n+1)} - X = G^2x^{(n)} - X + B$$
$$\Rightarrow x^{(n+1)} - X = G^2(x^{(n)} - X) + G^2X + B - X$$
$$\Rightarrow x^{(n+1)} - X = G^2(x^{(n)} - X)$$
$$\Rightarrow \left\|x^{(n+1)} - X\right\| = \left\|G^2(x^{(n)} - X)\right\| \le \left\|G^2\right\| \left\|x^{(n)} - X\right\|$$
$$\le \left\|G^4\right\| \left\|x^{(n-1)} - X\right\| \le ... \le \left\|G^{2n}\right\| \left\|x^{(1)} - X\right\|$$
$$\Rightarrow \left\|x^{(n+1)} - X\right\| \le \left\|G^{2n}\right\| \left\|x^{(1)} - X\right\| = \|G\|^{2n} \left\|x^{(1)} - X\right\|$$

Again let us consider second-refinement of generalized Jacobi method:

$$\Rightarrow x^{(n+1)} = G^3x^{(n)} + K \Rightarrow x^{(n+1)} - X = G^3x^{(n)} - X + K$$
$$\Rightarrow x^{(n+1)} - X = G^3(x^{(n)} - X) + G^3X + K - X$$
$$\Rightarrow x^{(n+1)} - X = G^3(x^{(n)} - X)$$
$$\Rightarrow \left\|x^{(n+1)} - X\right\| = \left\|G^3(x^{(n)} - X)\right\| \le \|G\| \left\|x^{(n)} - X\right\|$$

$\leq \|G^6\| \left\|x^{(n-1)} - X\right\| \leq ... \leq \|G^{3n}\| \left\|x^{(1)} - X\right\|$
$\Rightarrow \left\|x^{(n+1)} - X\right\| \leq \|G^{3n}\| \left\|x^{(1)} - X\right\| = \|G\|^{3n} \left\|x^{(1)} - X\right\|$
According to the coefficients of the above inequalities, we have $\|G\|^{3n} < \|G\|^{2n} < \|G\|^{n}$ since $\|G\| < 1$.
Therefore, the second-refinement of generalized Jacobi method converges faster than the generalized Jacobi method and refinement of generalized Jacobi method.

7. NUMERICAL EXAMPLES

Example 7.1: Consider the following system of linear equations whose coefficient matrix is both SDD and SPD with tolerance 0.0001.

$$\begin{cases} 6x_1 + 2x_2 + 2x_3 = 5 \\ 2x_1 + 8x_2 + 2x_3 = 6 \\ 2x_1 + 2x_2 + 10x_3 = 7 \end{cases}$$

Let us consider spectral radius and solution

TABLE 1. Spectral Radius

Method	J	GJ	RGJ	SRGJ
Spectral radius	0.5146	0.2972	0.0884	0.0263

Table 1 shows that the SRGJ method has small spectral radius than J, GJ and RGJ whereas Table 2 shows that the second- refinement of generalized Jacobi (SRGJ) iterative method is much better than generalized Jacobi (GJ) method and refinement of generalized Jacobi (RGJ) method. We think that it is almost half of faster than generalized Jacobi (GJ) method.

Example 7.2: Consider the following system of linear equations whose coefficient matrix is SDD but not SPD with tolerance 0.0001.

$$\begin{cases} 6x_1 + 4x_2 - x_3 = 9 \\ 3x_1 + 7x_2 + 2x_3 = 12 \\ -4x_1 + 3x_2 + 8x_3 = 7 \end{cases}$$

Let us consider the spectral radius and solution:

TABLE 2. Numerical results of example 7.1 and comparison between GJ, RGJ and SRGJ.

	GJ			RGJ			SRGJ		
n	$x_1^{(n)}$	$x_2^{(n)}$	$x_3^{(n)}$	$x_1^{(n)}$	$x_2^{(n)}$	$x_3^{(n)}$	$x_1^{(n)}$	$x_2^{(n)}$	$x_3^{(n)}$
0	0	0	0	0	0	0	0	0	0
1	0.6923	0.4231	0.6154	0.4541	0.5222	0.4571	0.5166	0.4932	0.5105
2	0.4541	0.5222	0.4571	0.4958	0.5020	0.4963	0.4996	0.5002	0.4992
3	0.5166	0.4932	0.5103	0.4996	0.5002	0.4992	0.5000	0.5000	0.5000
4	0.4958	0.5020	0.4963	0.5000	0.5000	0.5000			
5	0.5014	0.4994	0.5010						
6	0.4996	0.5002	0.4997						
7	0.5001	0.4999	0.5001						
8	0.5000	0.5000	0.5000						

TABLE 3. Spectral radius

Method	J	GJ	RGJ	SRGJ
Spectral radius	0.7937	0.4844	0.2346	0.1136

Table 3 shows that the SRGJ has small spectral radius than J, GJ and RGJ whereas Table 4 shows that the second- refinement of generalized Jacobi (SRGJ) iterative method is much better than generalized Jacobi (GJ) method and refinement of generalized Jacobi (RGJ) method. We

TABLE 4. Numerical results of example 7.2 and comparison between GJ, RGJ and SRGJ.

	GJ			RGJ			SRGJ		
n	${x_1}^{(n)}$	${x_2}^{(n)}$	${x_3}^{(n)}$	${x_1}^{(n)}$	${x_2}^{(n)}$	${x_3}^{(n)}$	${x_1}^{(n)}$	${x_2}^{(n)}$	${x_3}^{(n)}$
0	0	0	0	0	0	0	0	0	0
1	0.5980	1.3529	0.3676	0.7820	1.1690	0.7357	0.9010	1.0824	0.8601
2	0.7820	0.1690	0.7357	0.9502	1.0398	0.9356	0.9884	1.0093	0.9847
3	0.9010	1.0824	0.8601	0.9884	1.0093	0.9847	0.9987	1.0011	0.9982
4	0.9502	1.0398	0.9356	0.9973	1.0022	0.9964	0.9999	1.0001	0.9998
5	0.9764	1.0193	0.9699	0.9994	1.0005	0.9992	1.0000	1.0000	1.0000
6	0.9884	1.0093	0.9847	0.9999	1.0001	0.9998			
7	0.9944	1.0045	0.9925	1.0000	1.0000	1.0000			
8	0.9973	1.0022	0.9964						
9	0.9987	1.0011	0.9982						
10	0.9994	1.0005	0.9992						
11	0.9997	1.0002	0.9996						
12	0.9999	1.0001	0.9998						
13	0.9999	1.0001	0.9999						
14	1.0000	1.0000	1.0000						

can also compare the iteration number, i.e, GJ at 14, RGJ at 7 and SRGJ at 5. So our new method is better than others.

Example 7.3: Consider the following system of linear equations whose coefficient matrix is SPD but not SDD with tolerance 0.0001.

$$\begin{cases} 6x_1 + 4x_2 + 3x_3 = 13 \\ 4x_1 + 5x_2 + 2x_3 = 11 \\ 3x_1 + 2x_2 + 2x_3 = 7 \end{cases}$$

Let us consider the spectral radius and solution:

TABLE 5. Spectral radius

Method	J	GJ	RGJ	SRGJ
Spectral radius	1.4900	12.8739	165.7364	2133.6854

The iterative solution of the above equation diverges from the exact solution. The system has no solution when we apply Generalized Jacobi method, refinement of generalized Jacobi method and second refinement of generalized Jacobi method. Since the eigenvalues of iteration matrix is greater than one. We know that the Jacobi method to be convergent the matrix should satisfy the following conditions:

(1) A must be SPD, and
(2) $2T_m - A$ must be SPD

Example 7.4: Consider the following system of linear equations whose coefficient matrix is SDD but not PD and SPD with tolerance 0.0001.

$$\begin{cases} 5x_1 + 3x_2 + x_3 = 9 \\ 4x_1 - 6x_2 + x_3 = -1 \\ 2x_1 + x_2 + 4x_3 = 7 \end{cases}$$

Let us consider the spectral radius and solution:

TABLE 6. Spectral radius

Method	J	GJ	RGJ	SRGJ
Spectral radius	0.6227	0.2939	0.0863	0.0254

Table 6 shows spectral radius of the methods whereas Table 7 shows that the Second- Refinement of Generalized Jacobi (SRGJ) iterative method is much better than Generalized Jacobi (GJ) method and Refinement of Generalized Jacobi (RGJ) method. We can also conclude that SRGJ method minimizes iteration number to half as compared to GJ method.

Example 7.5: Consider the following system whose coefficient matrix is an M-matrix (or 2-cyclic matrix), which arises from the discretization of the Poissons equation $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = f$, on the unit square as considered by Vatti and Genanew [11], Datta [2] and Dafchahi [1], with tolerance 0.00001. Now consider $AX = b$ where m = 1, $X = (x_1\ x_2\ x_3\ x_4\ x_5\ x_6)^T$ and $b = (1\ 0\ 0\ 0\ 0\ 0)^T$ or

TABLE 7. Numerical results of example 7.4 and comparison between GJ, RGJ and SRGJ.

	GJ			RJ			SRJ		
n	${x_1}^{(n)}$	${x_2}^{(n)}$	${x_3}^{(n)}$	${x_1}^{(n)}$	${x_2}^{(n)}$	${x_3}^{(n)}$	${x_1}^{(n)}$	${x_2}^{(n)}$	${x_3}^{(n)}$
0	0	0	0	0	0	0	0	0	0
1	1.1098	1.1503	1.4624	0.9370	0.9509	0.9574	1.0040	1.0076	1.0296
2	0.9370	0.9509	0.9574	0.9959	0.9970	0.9988	0.9997	0.9998	1.0000
3	1.0040	1.0076	1.0296	0.9997	0.9998	1.0000	1.0000	1.0000	1.0000
4	0.9959	0.9970	0.9988	1.0000	1.0000	1.0000			
5	1.0000	1.0004	1.0020						
6	0.9997	0.9998	1.0000						
7	1.0000	1.0000	1.0001						
8	1.0000	1.0000	1.0000						

$$\begin{pmatrix} 4 & -1 & 0 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 \\ -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 0 & -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Let us consider the spectral radius and solution:
(b).

TABLE 8. Spectral radius

Method	J	GJ	RGJ	SRGJ
Spectral radius	0.6036	0.3867	0.1496	0.0578

TABLE 9. (a) Numerical results of example 7.5 and comparison between GJ, RGJ and SRGJ.

GJ, for m = 1						
n	${x_1}^{(n)}$	${x_2}^{(n)}$	${x_3}^{(n)}$	${x_4}^{(n)}$	${x_5}^{(n)}$	${x_6}^{(n)}$
0	0	0	0	0	0	0
1	0.2679	0.0714	0.0179	0	0	0
2	0.2679	0.0714	0.0179	0.0772	0.0408	0.0147
3	0.2917	0.0897	0.0261	0.0772	0.0408	0.0147
4	0.2917	0.0897	0.0261	0.0850	0.0483	0.0186
5	0.2944	0.0926	0.0278	0.0850	0.0483	0.0186
6	0.2944	0.0926	0.0278	0.0860	0.0495	0.0193
7	0.2948	0.0931	0.0281	0.0860	0.0495	0.0193
8	0.2948	0.0931	0.0281	0.0861	0.0497	0.0194
9	0.2948	0.0932	0.0281	0.0861	0.0497	0.0194
10	0.2948	0.0932	0.0281	0.0861	0.0497	0.0195
11	0.2948	0.0932	0.0282	0.0861	0.0497	0.0195

RGJ Iterative method, for m = 1						
n	${x_1}^{(n)}$	${x_2}^{(n)}$	${x_3}^{(n)}$	${x_4}^{(n)}$	${x_5}^{(n)}$	${x_6}^{(n)}$
0	0	0	0	0	0	0
1	0.2679	0.0714	0.0179	0.0772	0.0408	0.0147
2	0.2917	0.0897	0.0261	0.0850	0.0483	0.0186
3	0.2944	0.0926	0.0278	0.0860	0.0495	0.0193
4	0.2948	0.0931	0.0281	0.0861	0.0497	0.0194
5	0.2948	0.0932	0.0281	0.0861	0.0497	0.0195
6	0.2948	0.0932	0.0282	0.0861	0.0497	0.0195

(c).

SRGJ Iterative method, for m = 1						
n	${x_1}^{(n)}$	${x_2}^{(n)}$	${x_3}^{(n)}$	${x_4}^{(n)}$	${x_5}^{(n)}$	${x_6}^{(n)}$
0	0	0	0	0	0	0
1	0.2917	0.0897	0.0261	0.0772	0.0408	0.0147
2	0.2944	0.0926	0.0278	0.0860	0.0495	0.0193
3	0.2948	0.0932	0.0281	0.0861	0.0497	0.0194
4	0.2948	0.0932	0.0282	0.0861	0.0497	0.0195

Table 8 shows spectral radius of the methods whereas Table 9(a)-(c) shows that the second- refinement of generalized Jacobi (SRGJ) iterative method is much better than generalized Jacobi (GJ) method and refinement of generalized Jacobi (RGJ) method. So our new method is better than the others.

4. CONCLUDING REMARKS

Since the rate of convergence of stationary iterative process depends on spectral radius of the iterative matrix, any reasonable modification of iterative matrix that will reduce the spectral radius and increases the rate of convergence of that method. We can give the general conclusion by using table:

TABLE 10. Summary for Example 7.1. to 7.3.

Examples						
	7.1		7.2		7.3	
Methods	Number of Iterations	Spectral Radius	Number of Iterations	Spectral Radius	Number of Iterations	Spectral Radius
J	15	0.5146	37	0.7937	-	1.4900
RJ	8	0.2649	18	0.6294	-	2.2202
SRJ	5	0.1362	12	0.5000	-	3.3082
GJ	8	0.2972	14	0.4844	-	12.874
RGJ	4	0.0884	7	0.2346	-	165.74
SRGJ	3	0.0263	5	0.1136	-	2133.7

TABLE 11. Summary for Example 7.4. to 7.5.

Examples				
	7.4		7.5	
	Number of Iterations	Spectral Radius	Number of Iterations	Spectral Radius
J	23	0.6227	19	0.6036
RJ	12	0.3879	10	0.3643
SRJ	8	0.2415	7	0.2199
GJ	8	0.2939	11	0.3867
RGJ	4	0.0863	6	0.1496
SRGJ	3	0.0254	4	0.0578

In this paper, we found for m = 1 that second-refinement of generalized Jacobi iterative method for solving linear system of equations which uses to minimize the number of iteration almost by half as compared to generalized Jacobi iterative method and the rate of convergence of second-refinement of generalized Jacobi is more than the others and it has smallest spectral radius. This means that the new method that we found is much fastest than Jacobi and refinement of Jacobi method. More over one can find for m = 2, 3, ...

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