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GENERALIZED MIDDLE BOL LOOPS

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ABSTRACT. In this paper, some algebraic properties of generalized middle Bol loop are studied. Some bi-variate and j-variate mappings are suited to generalized Bol context to study some of these properties. Necessary and sufficient conditions, in terms of these mappings and otherwise, for generalized middle Bol loop to have RIP, LIP, right α-alternative property and left α-alternative property are established.

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1. INTRODUCTION

A groupoid $(Q, \cdot)$ consists of a set Q together with a binary operation $\cdot$ on Q. For $x \in Q$, define the left, respectively right, translation by x by $(y)L_x = x \cdot y$, respectively $(y)R_x = y \cdot x$, for all $y \in Q$. A quasigroup is a groupoid $(Q, \cdot)$ with a binary operation $\cdot$ such that for each $a, b \in Q$ the equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions $x, y \in Q$. A loop is a quasigroup with identity element. A loop satisfying the right Bol identity

$$((xy)z)y = x((yz)y)$$

for all $x, y, z \in Q$ is called a right Bol loop. A loop satisfying the mirror identity $(x(yx))z = x(y(xz))$ for all $x, y, z \in Q$ is called a left Bol loop, and a loop which is both left and right Bol is a Moufang loop. A good account on Bol and Moufang loop, with some interesting results, can be found in [8, 12, 14] and for historical note on loop theory, Pflugfelder treatise in [13] will be a good choice. A loop is a generalized Bol loop if it satisfies the relation

$$(xy \cdot z)y^{\alpha} = x(yz \cdot y^{\alpha})$$

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where y^{α} is the image of y under α, a single valued self-map. For some results on generalized Bol loops readers are refer to [1, 2, 3, 4, 5, 6].

A loop $Q(\cdot)$ is called generalized middle Bol loop if it satisfies the identity

$$x(y^{\alpha}z\backslash x^{\alpha}) = (x/z)(y^{\alpha}\backslash x^{\alpha}). \tag{1}$$

Generalized middle Bol loop was introduced by the authors in [1] as a consequence of a generalized Moufang loop with universal α-elasticity property where the self map α is a homomorphism. Thus, the homomorphic property of the map α will be used without necessarily stating it. When α is an identity map, the generalized middle Bol identity in (1) reduces to middle Bol identity– which was introduced in 1967 by Belousov [7] and was later studied in 1971 by Gwaramija [10].

The middle Bol identity is known to be universal under loop isotopy and this implies power associativity of the associated loops [7]. It is also a necessary and sufficient condition for the universality of the anti-automorphic inverse property [15] among several other results on middle Bol loop that have been obtained by different researchers. The question now is how much of the results obtained in middle Bol loop are transferable to generalized middle Bol loop or obtain its analogue? This study prepares ground to address this question and it is believed that the results of this study will pave ways for further studies on generalized middle Bol loops theory and beyond. This study is, however, meant to reveal some properties of generalized middle Bol loop. The bi-variate and j-variate mappings introduced by Jaiyeola et. al. in [11] is suited to generalized Bol context to study some of these properties. Necessary and sufficient conditions, in terms of these mappings, for generalized middle Bol loop to have right inverse property (RIP), left inverse property (LIP), right α-alternative property and left α-alternative property are established.

The paper is organized as follows. In Section 1, a brief account of the study and related studies while Section 2 contains definitions of different terminologies and notations that will be used throughout the paper. Section 3 comprises the main result of this study; it is found that generalized middle Bol loop is an Anti-Automorphic Inverse Property (AAIP) loop, generalized middle Bol loop has the right α-alternative property and, provided it is commutative, it has the left α-alternative property. It is also shown in this section that left and right inverse properties hold in generalized middle Bol loop among other results.

2. PRELIMINARY

This chapter gives definitions of different terminologies and notations used throughout the paper as well as some previous results.

Definition 1. *A loop $(Q, \cdot)$ is called a left inverse property loop if it satisfies the left inverse property (LIP) given by: $x^{\lambda}(xy) = y$.*

Definition 2. *A loop $(Q, \cdot)$ is called a right inverse property loop if it satisfies the right inverse property (RIP) given by: $(yx)x^{\rho} = y$.*

A loop is called an IP loop if it is both LIP-loop and RIP-loop. The Nuclei are known to coincide in an IP loop. However, the authors [1] have realized that this does not hold in an IP loop with universal α-elasticity.

Definition 3. *A loop $(Q, \cdot)$ is called an automorphic inverse property loop if it satisfies the automorphic inverse property given by: $(xy)^{-1} = x^{-1}y^{-1}$.*

Definition 4. *A loop $(Q, \cdot)$ is called an anti-automorphic inverse property (AAIP) loop if it satisfies the automorphic inverse property given by: $(xy)^{-1} = y^{-1}x^{-1}$.*

The following definition of α-elasticity, right and left α-alternative loop was introduced by the authors in [1].

Definition 5. *Let $(Q, \cdot)$ be a generalized Bol loop. $(Q, \cdot)$ is said to be α-elastic if the identity $(y \cdot z) \cdot y^{\alpha} = y \cdot (z \cdot y^{\alpha})$. holds in Q.*

Definition 6. *Let $(Q, \cdot)$ be a generalized Bol loop. $(Q, \cdot)$ is called a right α-alternative loop if it satisfies $(xy) \cdot y^{\alpha} = x \cdot (yy^{\alpha})$ and it is called left α-alternative loop if it satisfies $(y^{\alpha}y) \cdot x = y^{\alpha} \cdot (yx)$.*

Definition 7. *Let $(Q, \cdot)$ be a generalized Bol loop. $(Q, \cdot)$ is called an α-alternative loop if it is both right and left α-alternative.*

Definition 8. *A loop $(Q, \cdot)$ is called middle Bol loop if it satisfies the identity $x(yz\backslash x) = (x/z)(y\backslash x)$.*

We shall adopt the following notation from [11] throughout the study. Subsequently, we let $x\backslash y = yL_x^{-1} = y\mathcal{L}_x = x\mathcal{R}_y$ and $x/y = xR_y^{-1} = x\mathbb{R}_y = y\mathbb{L}_x$
and

$$x\backslash y = z \iff x \cdot z = y \text{ and } x/y = z \iff z \cdot y = x.$$

Thus, the definition of a loop can alternatively be given using the operations ($\backslash$) and ($/$) as follows.

Definition 9. *A loop* $(Q, \cdot, /, \backslash, e)$ *is a set* Q *together with three binary operations* $(\cdot)$, $(/)$, $(\backslash)$ *and one nullary operation* e *such that*

(i): $x \cdot (x\backslash y) = y$, $(y/x) \cdot x = y$ *for all* $x, y \in Q$,
(ii): $x\backslash(x \cdot y) = y$, $(y \cdot x)/x = y$ *for all* $x, y \in Q$ *and*
(iii): $x\backslash x = y/y$ *or* $e \cdot x = x$ *and* $x \cdot e = x$ *for all* $x, y \in Q$.

3. MAIN RESULTS

This section gives several properties of the generalized middle Bol loop. An identity that defines generalized middle Bol loop, besides equation (1), is given and the equivalence of this identity to the generalized middle Bol identity in the equation (1) is established. The following lemma contains some results that are used in the proofs of some other important results of this study.

Lemma 1. *Let* $(Q, \cdot, \backslash, /)$ *be a generalized middle Bol loop and* $\alpha : Q \longrightarrow Q$ *be a surjective homomorphism, Then*

(a) $(y^\alpha z)^\rho = z^\rho \cdot (y^\alpha)^\rho$ *and* $z^\rho = z^\lambda$ *i.e* $(Q, \cdot, \backslash, /)$ *is AAIP loop.*
(b) $(y^\alpha x)\backslash x^\alpha = x\backslash(y^\alpha\backslash x^\alpha)$.
(c) $(y^\alpha x)u = x^\alpha \iff y^\alpha(xu) = x^\alpha$, *thus* $R_u L_{y^\alpha} = L_{y^\alpha} R_u$.
(d) $x^\alpha z\backslash x^\alpha = x\backslash(x/z)$.
(e) $(x^\alpha z)u = x^\alpha$ *and* $(x^\alpha u)z = x^\alpha$. *Thus,* $R_z R_u$ *and* $R_u R_z$ *are identity map.*
(f) $x(z\backslash x^\alpha) = (x/z)x^\alpha$
(g) $(x/z)(z^\lambda\backslash x^\alpha) = (x/(y^\alpha)^\rho)(y^\alpha\backslash x^\alpha)$

Proof. (a) If $x = e$ in the LHS of equation (1) then, $e(y^\alpha z\backslash e) = (e/z)(y^\alpha\backslash e)$.
Let $y^\alpha z\backslash e = u$, then $e = y^\alpha z \cdot u$. This implies $u = (y^\alpha z)^\rho$. Also, from the RHS, if $x = e$, we have $e/z = v$, then $e = v \cdot z$ which implies $v = z^\lambda$. Also, $y^\alpha\backslash e = w$, then $e = y^\alpha \cdot w$ which implies $w = (y^\alpha)^\rho$. Therefore, $(y^\alpha z\backslash e) = (e/z)(y^\alpha\backslash e)$ implies $(y^\alpha z)^\rho = z^\lambda \cdot (y^\alpha)^\rho$. Now, let $y^\alpha = e$, then $(ez)^\rho = z^\lambda \cdot e^\rho$ implies $z^\rho = z^\lambda$ which complete the proof.

(b) Let $z = x$ in equation (1), then $x(y^\alpha x\backslash x^\alpha) = (x/x)(y^\alpha\backslash x^\alpha) = e(y^\alpha\backslash x^\alpha)$. This implies that $x(y^\alpha x\backslash x^\alpha) = y^\alpha\backslash x^\alpha$. Thus,

$$(y^\alpha x\backslash x^\alpha) = x\backslash(y^\alpha\backslash x^\alpha)$$

(c) From (b) above, let $u = (y^\alpha x\backslash x^\alpha)$ then, $(y^\alpha x) \cdot u = x^\alpha$. Also, let $x\backslash(y^\alpha\backslash x^\alpha) = u$ then, $y^\alpha\backslash x^\alpha = xu$ which implies

$y^\alpha(xu) = x^\alpha \iff R_u L_{y^\alpha} = x^\alpha$.
Also, $(y^\alpha x) \cdot u = x^\alpha \iff L_{y^\alpha} R_u = x^\alpha$ and the result follows.

(d) Let $y = x$ in equation (1), then $x(x^\alpha z\backslash x^\alpha) = (x/z)(x^\alpha\backslash x^\alpha)$. Thus, $x(x^\alpha z\backslash x^\alpha) = (x/z)$ and $x^\alpha z\backslash x^\alpha = x\backslash(x/z)$.

(e) From (d), let $u = x^\alpha z\backslash x^\alpha$, then $(x^\alpha z) \cdot u = x^\alpha$ and let $u = x\backslash(x/z)$, then $xu = x/z \implies (xu)z = x$. Therefore, $(x^\alpha)R_z R_u = x^\alpha$, $(x)R_u R_z = x$ and the result follows.

(f) Put $y = e$ in equation (1), then $x(e^\alpha z\backslash x^\alpha) = (x/z)(e^\alpha\backslash x^\alpha) \implies x(z\backslash x^\alpha) = (x/z)x^\alpha$.

(g) Let $y^\alpha = z^\lambda$ in equation (1), then $x(z^\lambda z\backslash x^\alpha) = (x/z)(z^\lambda\backslash x^\alpha) \implies xx^\alpha = (x/z)(z^\lambda\backslash x^\alpha)$. Now, put $z = (y^\rho)^\alpha = (y^\alpha)^\rho$ in equation (1), then $xx^\alpha = (x/(y^\alpha)^\rho)(y^\alpha\backslash x^\alpha)$. Therefore, $(x/z)(z^\lambda\backslash x^\alpha) = (x/(y^\alpha)^\rho)(y^\alpha\backslash x^\alpha)$

□

Remark 1. *Part* (a) *of Lemma 1 has established anti-automorphic inverse property for generalized middle Bol loops which shows that the generalized middle Bol loops can be included in the class of loops called D-loops studied by Deriyenko and Dudek in* [9].

Theorem 1. *In a generalized middle Bol loop, the following identities are equivalent:*

1. $(Q, /) \equiv (Q, \backslash)$
2. $x(y^\alpha x\backslash x^\alpha) = y(y^\alpha x\backslash y^\alpha)$
3. $(x/y^\alpha x)x^\alpha = y(y^\alpha x\backslash y^\alpha)$
4. $x(y^\alpha x\backslash x^\alpha) = (y/y^\alpha x)y^\alpha$
5. $(x/y^\alpha x)x^\alpha = (y/y^\alpha x)y^\alpha$

Proof. 1. $\implies$ 2.
From (b) and (d) of lemma 1, $y^\alpha\backslash x^\alpha = y(y^\alpha x\backslash y^\alpha)$ and $x/z = x(x^\alpha z\backslash x^\alpha)$. Thus, $(Q, /) \equiv (Q, \backslash)$ if and only if $x(y^\alpha x\backslash x^\alpha) = y(y^\alpha x\backslash y^\alpha)$.
2. $\implies$ 3.
Suppose

$$x(y^\alpha x\backslash x^\alpha) = y(y^\alpha x\backslash y^\alpha) \quad (2)$$

Using (f) of Lemma 1, $x(z\backslash x^\alpha) = (x/z)x^\alpha$, on putting $y^\alpha x = z$ in the LHS of equation (2), we have $x(z\backslash x^\alpha) = y(y^\alpha x\backslash y^\alpha)$ but $x(z\backslash x^\alpha) = (x/z)x^\alpha$. Therefore, $(x/y^\alpha x)x^\alpha = y(y^\alpha x\backslash y^\alpha)$.
3. $\implies$ 4.
Suppose

$$(x/y^\alpha x)x^\alpha = y(y^\alpha x\backslash y^\alpha).$$

Let $y^\alpha x = z$, then $(x/z)x^\alpha = y(z\backslash y^\alpha)$ and from (f) of Lemma 1, we have $x(y^\alpha x\backslash x^\alpha) = (y/y^\alpha x)y^\alpha$.
The remaining equivalence follows as a consequence of (f) of Lemma 1. $\square$

Theorem 2. *Let $(Q,\cdot,\backslash,/)$ be a generalized middle Bol loop. Let $f,g : Q\times Q \longrightarrow Q$ such that $f(x,y) = y^\alpha x\backslash x^\alpha$ and $g(x,y) = x^\alpha y\backslash x^\alpha$, then*

1. $f(x,y) = x\backslash(y^\alpha\backslash x^\alpha)$.
2. $g(x,y) = x\backslash(x/y)$.
3. $f(x,y) = g(x,y) \iff y^\alpha x = x^\alpha y$.
4. $f(x,y) = g(x,y) \iff (Q,(/)^*) \equiv (Q,/) \iff (Q,\backslash) \equiv (Q,(/)^*)$
5. $f(x,y) = g(x,y) \iff y^\alpha x\backslash x^\alpha = x\backslash(x/y) \iff x^\alpha y\backslash x^\alpha = x\backslash(y^\alpha\backslash x^\alpha)$.
6. $x^\alpha = y^\alpha\cdot(x/y) \iff (y^\alpha\backslash x^\alpha)\cdot y = x$

Proof.
1. It follows from lemma 1(a).
2. It follows from lemma 1(d).
3. Since $f(x,y) = (y^\alpha x\backslash x^\alpha) \iff f(x,y) = x\backslash(y^\alpha\backslash x^\alpha)$. and $g(x,y) = x^\alpha y\backslash x^\alpha \iff g(x,y) = x\backslash(x/y)$. Then, $y^\alpha x\backslash x^\alpha = x^\alpha y\backslash x^\alpha \iff y^\alpha x = x^\alpha y$. Also, $x\backslash(y^\alpha\backslash x^\alpha) = x\backslash(x/y) \iff y^\alpha\backslash x^\alpha = x/y \iff y^\alpha x = x^\alpha y$.
4. $f(x,y) = g(x,y) \iff x\backslash(y^\alpha\backslash x^\alpha) = x\backslash(x/y) \iff y^\alpha\backslash x^\alpha = x/y \iff y^\alpha\backslash x^\alpha = y(/)^*x \iff x^\alpha(\backslash)^*y^\alpha = x/y \iff (Q,(/)^*) \equiv (Q,/) \iff (Q,\backslash) \equiv (Q,(/)^*)$
5. $f(x,y) = g(x,y) \iff x^\alpha y\backslash x^\alpha = x\backslash(x/y)$ and $y^\alpha x\backslash x^\alpha = x\backslash(y^\alpha\backslash x^\alpha)$. Equation the LHS of the two equations, we have $x^\alpha y\backslash x^\alpha = y^\alpha x\backslash x^\alpha \iff x\backslash(x/y) = x\backslash(y^\alpha\backslash x^\alpha) \iff x/y = y^\alpha\backslash x^\alpha$ and the result follows.
6. It follows from the identity $y^\alpha\backslash x^\alpha = x/y$.

$\square$

Remark 2. *The statement of Theorem 2 assigns values to the bivariate mappings f and g. These mappings together with the assigned values are used at different stages of this work to study some of the properties under consideration.*

Theorem 3. *Let $(Q,\cdot,\backslash,/)$ be a generalized middle Bol loop and let $f,g : Q^2 \longrightarrow Q$ and $\theta_i,\beta_i : Q^i \longrightarrow Q$ be defined as*
$f(x,y) = y^\alpha x\backslash x^\alpha$ or $f(x,y) = x\backslash(y^\alpha\backslash x^\alpha)$ and $g(x,y) = x^\alpha y\backslash x^\alpha$ or $g(x,y) = x\backslash(x/y)$
$\theta_i(x_1,x_2,\cdots,x_i) = ((((x_1x_2)x_3)x_4)\cdots x_{i-1})x_i$ and
$\beta_i(x_1,x_2,\cdots,x_i) = x_1\backslash(x_2\backslash(x_3\backslash(\cdots x_{i-2}\backslash(x_{i-1}\backslash x_i))))\,\forall\, i\in\mathbb{N}$ and

$\alpha : Q \longrightarrow Q$ preserves inverse.
Then,

1. $f(x, \theta_n(y, x^{\frac{1}{\alpha}}, x^{\frac{1}{\alpha}}, \cdots, x^{\frac{1}{\alpha}})) = \beta_n(x, x, \cdots, x, f(x, y))$.
2. $f(x, \theta_n(x, y^{\frac{1}{\alpha}}, x^{\frac{1}{\alpha}}, x^{\frac{1}{\alpha}}, \cdots, x^{\frac{1}{\alpha}})) = \beta_{n+1}(x, x, x, \cdots, x, g(x, y))$.
3. *$(Q, \cdot)$ has the right α-alternative property if and only if $f(x, y) = x^{\alpha}[(y^{\alpha}(xx^{\alpha}))\backslash x^{\alpha}]$.*
4. *$(Q, \cdot)$ has the left α-alternative property if and only if $g(x, y) = x[((xx^{\alpha})y)\backslash x^{\alpha}]$ provided Q is commutative.*

Proof. 1. By (b) of lemma 1, $y^{\alpha}x\backslash x^{\alpha} = x\backslash(y^{\alpha}\backslash x^{\alpha}) \implies R_x\mathcal{R}_{x^{\alpha}} = \mathcal{R}_{x^{\alpha}}\mathcal{L}_x$
$\implies$

$$R_x = \mathcal{R}_{x^{\alpha}}\mathcal{L}_x\mathcal{R}_{x^{\alpha}}^{-1} \tag{3}$$

By equation 3,

$$R_x^2 = \mathcal{R}_{x^{\alpha}}\mathcal{L}_x^2\mathcal{R}_{x^{\alpha}}^{-1}$$
$$R_x^3 = \mathcal{R}_{x^{\alpha}}\mathcal{L}_x^3\mathcal{R}_{x^{\alpha}}^{-1}$$

Hence,

$$R_x^n = \mathcal{R}_{x^{\alpha}}\mathcal{L}_x^n\mathcal{R}_{x^{\alpha}}^{-1} \quad n \geq 0.$$

Thus, for all $y \in Q$,

$$(((y^{\alpha}\underbrace{x \cdot x)x \cdot x)x \cdots)x}_{n-times}\backslash x^{\alpha} = (\underbrace{x\backslash \cdots (x\backslash(x\backslash}_{n-times}(y^{\alpha}\backslash x^{\alpha})))) \tag{4}$$

Equation 4 above implies

$f(x, \theta_n(y, x^{\frac{1}{\alpha}}, x^{\frac{1}{\alpha}}, \cdots, x^{\frac{1}{\alpha}})) = \beta_n(x, x \cdots, x, f(x, y))$.

2. By (d) of lemma 1, $x^{\alpha}z\backslash x^{\alpha} = x\backslash(x/z) \implies L_{x^{\alpha}}\mathcal{R}_{x^{\alpha}} = \mathbb{L}_x\mathcal{L}_x$
$\implies$

$$L_{x^{\alpha}} = \mathbb{L}_x\mathcal{L}_x\mathcal{R}_{x^{\alpha}}^{-1} \tag{5}$$

By equations 3 and 5,

$$L_{x^{\alpha}}R_x = \mathbb{L}_x\mathcal{L}_x^2\mathcal{R}_{x^{\alpha}}^{-1}$$
$$L_{x^{\alpha}}R_x^2 = \mathbb{L}_x\mathcal{L}_x^3\mathcal{R}_{x^{\alpha}}^{-1}$$
$$L_{x^{\alpha}}R_x^3 = \mathbb{L}_x\mathcal{L}_x^4\mathcal{R}_{x^{\alpha}}^{-1}$$

Therefore,

$$L_{x^{\alpha}}R_x^n = \mathbb{L}_x\mathcal{L}_x^{(n+1)}\mathcal{R}_{x^{\alpha}}^{-1}, \quad n \geq 0.$$

Thus, for all $y \in Q$.

$$(((x^{\alpha}y \cdot \underbrace{x)x \cdot x)x \cdots)x}_{n-times}\backslash x^{\alpha} = (\underbrace{x\backslash \cdots (x\backslash(x\backslash}_{(n+1)-times}(x/y)))) \tag{6}$$

Equation 6 above implies

$$f(x,\theta_n(x,y^{\frac{1}{\alpha}},x^{\frac{1}{\alpha}},x^{\frac{1}{\alpha}},\cdots,x^{\frac{1}{\alpha}}))=\beta_{n+1}(x,x\cdots,x,f(x,y)).$$

3. Suppose Q has right α-alternative property, then $z(xx^\alpha)=zx\cdot x^\alpha$. Post multiplying both sides by x^α we have $z(xx^\alpha)\cdot x^\alpha=(zx\cdot x^\alpha)\cdot x^\alpha$. Take left division of both sides by $z(xx^\alpha)$ to obtain $x^\alpha=zx\cdot x^\alpha\cdot(z(xx^\alpha))\backslash x^\alpha \implies zx\backslash x^\alpha=x^\alpha\cdot[z(xx^\alpha))\backslash x^\alpha]$.
   On replacing z by y^α, $y^\alpha x\backslash x^\alpha=x^\alpha[(y^\alpha(xx^\alpha))\backslash x^\alpha]$ and the result follows.
   Conversely, let $f(x,y)=x^\alpha[(y^\alpha(xx^\alpha))\backslash x^\alpha]$ holds. The reasoning above can be reversed.

   Alternatively, Let $n=1$ in Equation 4, then $y^\alpha x\backslash x^\alpha=x\backslash(y^\alpha\backslash x^\alpha) \iff x^\alpha=(y^\alpha x)\cdot x\backslash(y^\alpha\backslash x^\alpha) \iff xx^\alpha=(y^\alpha x)\cdot(y^\alpha\backslash x^\alpha) \iff y^\alpha(xx^\alpha)=(y^\alpha x)x^\alpha \iff y^\alpha x\backslash x^\alpha=x^\alpha[(y^\alpha(xx^\alpha))\backslash x^\alpha]$ and the result follows.
4. Suppose Q has left α-alternative property, then $(xx^\alpha)z=x\cdot x^\alpha z$. Post multiplying both sides by x^α we have $(xx^\alpha)z\cdot x^\alpha=(x\cdot x^\alpha z)\cdot x^\alpha$. Taking left division of both sides by $(xx^\alpha)z$, we obtain $x^\alpha=x(x^\alpha z)\cdot((xx^\alpha)z)\backslash x^\alpha$ which implies $x^\alpha z\backslash x^\alpha=x\cdot[(xx^\alpha)z\backslash x^\alpha]$. On replacing z by y we have $x^\alpha y\backslash x^\alpha=x\cdot[(xx^\alpha)y\backslash x^\alpha]$ and the result follows.
   The converse follows easily only that commutativity is inevitable.

   Alternatively, Let $n=0$ in Equation 6, then $x^\alpha y\backslash x^\alpha=x\backslash(x/y) \iff x^\alpha=x^\alpha y\cdot x\backslash(x/y) \iff xx^\alpha=x^\alpha y\cdot x/y \iff (xx^\alpha)y=x^\alpha y\cdot x$, using the commutativity of Q, the last equation implies $(xx^\alpha)y=x(x^\alpha y) \iff x^\alpha y\backslash x^\alpha=x\cdot[(xx^\alpha)y\backslash x^\alpha]$ and the result follows.

□

Corollary 1. *Let $(Q,\cdot)$ be a commutative generalized middle Bol loop, then Q has the α-alternative property.*

Proof. The proof of this corollary follows from the proof of 3. and 4. of Theorem 3. □

Remark 3. *The commutativity in Corollary 1 is necessary as it is required for Q to be left α-alternative. A natural question to ask here is "can commutativity be dropped?"*

Lemma 2. *Let $(Q,\cdot,\backslash,/)$ be a generalized middle Bol loop, then the following are equivalent.*

(1) $(Q,\cdot,\backslash,/)$ is a generalized middle Bol loop.
(2) $x(y^\alpha z\backslash x^\alpha)=(x/z)(y^\alpha\backslash x^\alpha)$ for all $x,y,z\in Q$.
(3) $(x/y^\alpha z)x^\alpha=(x/z)(y^\alpha\backslash x^\alpha)$ for all $x,y,z\in Q$.

Proof. 1. $\Longrightarrow$ 2.
This is obvious from the definition of generalized middle Bol loop.
2. $\Longrightarrow$ 3.
Suppose $x(y^\alpha z\backslash x^\alpha)=(x/z)(y^\alpha\backslash x^\alpha)$ holds, then from (f) of lemma 1, $x(z\backslash x^\alpha)=(x/z)x^\alpha$. Thus, $(x/y^\alpha z)x^\alpha=(x/z)(y^\alpha\backslash x^\alpha)$.
3. $\Longrightarrow$ 1.
This is also a consequence of (f) of lemma 1. □

Theorem 4. *Let $(Q,\cdot,\backslash,/)$ be a generalized middle Bol loop and let $f_1,g_1,f_2,g_2:Q\times Q\longrightarrow Q$ be defined by:*
$f_1(x,y)=y^\alpha x\backslash x^\alpha$ or $f_1(x,y)=x\backslash(y^\alpha\backslash x^\alpha)$ and $g_1(x,y)=x^\alpha y\backslash x^\alpha$ or $g_1(x,y)=x\backslash(x/y)$ $f_2(x,y)=x^\alpha/x^\alpha y$ or $f_2(x,y)=(x/y)/x$ and $g_2(x,y)=x/y^\alpha x$ or $g_2(x,y)=(y^\alpha\backslash x^\alpha)/x^\alpha$, Then:

I. $x/y^\alpha x=(y^\alpha\backslash x^\alpha)/x^\alpha$.
II. $z(y^\alpha x)=x^\alpha \iff y^\alpha(zx^\alpha)=x^\alpha$ *and* $L_{y^\alpha}L_z=I \iff L_zL_{y^\alpha}=I$.
III. $x^\alpha/x^\alpha z=(x/z)/x$.
IV. $u(x^\alpha z)=x^\alpha \iff (ux)z=x$ *and* $L_zR_u=R_uL_z=x$.
V. $y^\alpha x\cdot z=x^\alpha \iff xz=[x/(y^\alpha x)]\,x \iff y^\alpha\cdot xz=x^\alpha$.
VI. $z\cdot y^\alpha x=x \iff x\,[y^\alpha x\backslash x^\alpha] \iff y^\alpha\cdot zx^\alpha=x^\alpha$.
VII. *$(Q,\cdot)$ is a LIPL if and only if $y^{-\alpha}x^\alpha=x\,[y^\alpha x\backslash x^\alpha]$ if and only if $y^{-\alpha}x^\alpha=x\cdot f_1(x,y)$.*
VIII. $x^\alpha y\cdot z=x^\alpha \iff xz=[x^\alpha/x^\alpha y]\,x$
IX. *$(Q,\cdot)$ is a RIPL if and only if $xy^{-1}=[x^\alpha/x^\alpha y]\,x$ if and only if $xy^{-1}=f_2(x,y)\cdot x$.*

Proof. *I.* Let $z=x$ in identity 3. of Lemma 2, then $(x/y^\alpha x)x^\alpha=(x/x)(y^\alpha\backslash x^\alpha)$.
This implies $x/y^\alpha x=(y^\alpha\backslash x^\alpha)/x^\alpha$.

II. From I. above, $z=x/y^\alpha x \iff z=(y^\alpha\backslash x^\alpha)/x^\alpha$. This implies

$$z(y^\alpha x)=x \iff y^\alpha(zx^\alpha)=x^\alpha.$$

This implies,

$$L_{y^\alpha}L_z=x \iff L_zL_{y^\alpha}=x^\alpha$$

Thus, both $L_{y^\alpha}L_z$ and $L_zL_{y^\alpha}$ are identity map.

III. Substitute $y = x$ in the identity 3. of Lemma 2, then $(x/x^\alpha z)x^\alpha = x/z$.
This implies $x/x^\alpha z = (x/z)/x^\alpha$.

IV. From III. of Theorem 4, $u = x/x^\alpha z \iff u = (x/z)/x^\alpha$. This implies

$$u(x^\alpha z) = x \iff (ux^\alpha)z = x.$$

This implies

$$R_z L_u = x \iff L_u R_z = x.$$

V. Let $z = y^\alpha x \backslash x^\alpha$, then $y^\alpha x \cdot z = x^\alpha$. From I. of Theorem 4, $y^\alpha \backslash x^\alpha = (x/y^\alpha x)x^\alpha$.
So, $xz = [x/y^\alpha x]\, x^\alpha$. Using (c) of Lemma 1, $y^\alpha x \cdot z = x^\alpha \iff xz = [x/y^\alpha x]\, x^\alpha$ and thus, $y^\alpha \cdot xz = x^\alpha$.

VI. Let $z = x/y^\alpha x$, then $z \cdot y^\alpha x = x$. Also, $z = (y^\alpha \backslash x^\alpha)/x^\alpha \iff zx^\alpha = y^\alpha \backslash x^\alpha$.
But from (b) of Lemma 1, $y^\alpha \backslash x^\alpha = x\,[y^\alpha x \backslash x^\alpha]$. Thus, $zx^\alpha = x\,[y^\alpha x \backslash x^\alpha] \iff y^\alpha \cdot zx^\alpha = x^\alpha$.

VII. This is obvious from VI. above upon substituting $z = y^{-\alpha}$.

VIII. From 2. of Theorem 2, $z = x^\alpha y \backslash x^\alpha \implies x^\alpha y \cdot z = x^\alpha \iff z = x\backslash(x/y) \implies xz = x/y$. But $x/y = [x^\alpha/x^\alpha y]\, x$. Thus $xz = [x^\alpha/x^\alpha y]\, x$.

IX. This is obvious from $VIII$. above when $z = y^{-1}$.

□

4. CONCLUDING REMARKS

As a first step towards studying generalized middle Bol loops after its introduction in [1], the authors have been able to establish some properties of generalized middle Bol loops. It has been found to satisfy right and left inverse property, the right and left α-alternative property among others.
Different identities have been provided to characterize right and left inverse property as well as right and left α-alternative property in terms of the bivariate map and/or otherwise. Some of these properties hold by imposing additional properties on generalized middle Bol loop.

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