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BLOCK HYBRID METHOD USING THE OPERATIONAL MATRICES OF BERNSTEIN POLYNOMIAL FOR THE SOLUTION OF THIRD ORDER ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we have developed block method using collocation and interpolation of Bernstein Polynomial operational matrices to approximate third-order ordinary differential equations (ODEs). The solution give a system of non linear equations which is solved to give a continuous hybrid linear multistep method. The continuous hybrid linear multistep method is solved for the independent solutions to give a continuous hybrid block method which is then evaluated at some off-grid points to give a discrete block method. The basic properties of the discrete block were investigated and found to be zero stable, consistent and convergent. The derived scheme was tested on some numerical examples and was found to give better approximation than the existing methods in the literature.

Keywords and phrases: Collocation and Interpolation, Bernstein Polynomial, Block Method, Consistent and convergent, zero stable.
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1. INTRODUCTION

Most problems arising in physical sciences, engineering, biological and social sciences when formulated often lead to ordinary differential equation. Ordinary differential equation can be defined as an equation containing the derivative of the dependent variable with respect to an independent variable.

Many of the modeled problems in ordinary differential equation may not easily be solve analytically, hence the resort to seek approximation solution of the problems by numerical methods. Some of these numerical methods include the popular Runge Kuta Methods and Linear Multistep Methods (LMM). (see Lambert([21],[22]),

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Fatunla [15], Henrici [16]).
In recent years, considerable attention has been paid to the methods of solving higher order ordinary differential equations of the form

$$y^{(s)} = f(x,\ y,\ y', \dots, y^{(s-1)}),\ y^{(s)} = h_s,\ s = 0(1)n-1 \quad (1)$$

Earlier attempts have been made to solve (1) by the method of reduction of order. This method has some serious due to cost implication and wastage of computer time (Awoyemi [6], Awoyemi[7], Awoyemi and Idowu [8], Kayode [18], Majid et al [23], Awoyemi et al [9]).
Direct method of solution has been reported to be more efficient than the method of reduction to system of first order, Adesanya etal [2], Jator [17]. Implicit linear multistep methods which have better stability properties than explicit methods have been developed for solving (1) Adesanya et al [2], Adoghe and Omole [3].
Continuous implicit linear multistep method has greater benefits over the discrete method because they provides better error estimation, simplified coefficients for further analytical work at different points and allows easy appropriation of solution at all interior points of the integration interval. Continuous Linear multistep methods adopted were implemented in predictor-corrector mode, (Awoyemi [7], Olabode [24]). Although, the predictor-corrector method yielded good results, it is costly to implement (Jator[17]). To cater for this set back, scholars developed block methods.
Implicit block methods for solving higher order ODEs have been developed by the following scholar ([1], [5], [9], [24], [25]). In most of these works, power series was used as basis function to approximate the exact solution.

In this paper, the Bernstein polynomial shall be adopted to locally approximate the exact solution of (2).
Quite a number of work has been done on the application of this polynomial, see ([4], [11], [12], [13], [19], [29], [30], [31]).

Most recently, Ojo and Okoro ([27], [28]) adopted this polynomial to develop single step method with two off-step points for the direct solution of second order ordinary differential equations.
Our focus in this paper is to develop a block numerical approximation method based on Bernstein Polynomial Operational Matrices for solving the third order ordinary differential equations of the

form:

$$y''' = f(x,\ y,\ y'\ y''),\ y(x_0) = y_a,\ y'(x_0) = y_b\ y''(x_0) = y_c \qquad (2)$$

2. BERNSTEIN POLYNOMIALS AND THEIR PROPERTIES

Definition 1: The Bernstein polynomials of degree m defined on the interval [a, b] are given as [[10], [14]]

$$B_{i,m}(x) = \binom{m}{i} \frac{(x-a)^i (b-x)^{m-i}}{(b-a)^m},\ i = 0, 1, ..., m \qquad (3)$$

where the binomial coefficient is

$$\binom{m}{i} = \frac{m!}{i!(m-i)!}$$

For convenience, we usually set $B_{i,m} = 0$, if $i < 0$ or $i > m$.

Some useful properties for $B_{i,m}$, on [a, b]

(1) Recurrence formula: The Bernstein polynomial of degree $m-1$ in terms of a linear combination of Bernstein polynomials of degree m on the interval [a, b] is given as

$$(b-a)B_{i,m-1}(x) = \left(\frac{m-i}{m}\right) B_{i,m}(x) + \left(\frac{i+1}{m}\right) B_{i+1,m}(x)$$

(2) Derivative formula: The derivatives of the mth degree Bernstein polynomials are given by

$$\frac{d^n}{dx^n}(B_{i,m}(x)) = \frac{1}{(b-a)^n} \frac{m!}{(m-n)!} \sum_{k=max(0,i+n-m)}^{min(i,n)} (-1)^{k+n} \binom{n}{k} B_{i-k,m-n}(x)$$

3. MATHEMATICAL FORMULATION

In this section the basic ideas of the Bernstein polynomials method are introduced by considered Eq.(2) subject to the initial conditions.

Let the approximate solution of Eq. (2) and it's derivatives be given as a Bernstein Polynomial in the matrix form as

$$\begin{aligned}
y(x) &= \sum_{k=0}^{m=c+i-1} C_k B_{k,m}(x),\ 0 \le x \le 1,\ k = 0,1,...,m \\
&= C_0 B_{0,m}(x) + C_1 B_{1,m}(x) + \cdots + C_m B_{m,m}(x) \\
&= C\phi(x) \qquad (4)
\end{aligned}$$

$$\begin{aligned}
y'(x) &= \sum_{k=0}^{m=c+i-1} C_k B'_{k,m}(x),\ 0 \le x \le 1,\ k = 0,1,...,m \\
&= C_0 B'_{0,m}(x) + C_1 B'_{1,m}(x) + \cdots + C_m B'_{m,m}(x) \\
&= C\phi'(x) \qquad (5)
\end{aligned}$$

$$\begin{aligned}
y''(x) &= \sum_{k=0}^{m=c+i-1} C_k B''_{k,m}(x),\ 0 \le x \le 1,\ k = 0,1,...,m \\
&= C_0 B''_{0,m}(x) + C_1 B''_{1,m}(x) + \cdots + C_m B''_{m,m}(x) \\
&= C\phi''(x) \qquad (6)
\end{aligned}$$

$$\begin{aligned}
y'''(x) &= \sum_{k=0}^{m=c+i-1} C_k B'''_{k,m}(x),\ 0 \le x \le 1,\ k = 0,1,...,m \\
&= C_0 B'''_{0,m}(x) + C_1 B'''_{1,m}(x) + \cdots + C_m B'''_{m,m}(x) \\
&= C\phi'''(x) \qquad (7)
\end{aligned}$$

where c and i are number of distinct collocation and interpolation points respectively for $m \ge 1$, C is an unknown constant matrix of size $1\times(m+1)$ to be determining and $\phi(x), \cdots, \phi'''(x)$ are $1\times(m+1)$ matrices of Bernstein polynomials with it's derivatives defined as

$$\begin{aligned}
C &= [c_0, c_1, ..., c_m] \\
\phi(x) &= [B_{0,m}(x) + C_1 B_{1,m}(x) + \cdots + C_m B_{m,m}(x)]^T \\
&\ \ \vdots \\
\phi'''(x) &= [B'''_{0,m}(x) + C_1 B'''_{1,m}(x) + \cdots + C_m B'''_{m,m}(x)]^T.
\end{aligned}$$

$$f(x, y(x), y'(x), y''(x)) = \sum_{k=0}^{m=c+i-1} C\phi'''(x), C\phi''(x), C\phi'(x) \qquad (8)$$

We consider a grid point of step-length one and off step points at $x = x_{n+\frac{1}{6}}, x_{n+\frac{1}{3}}, x_{n+\frac{1}{2}}, x_{n+\frac{2}{3}}$ and $x_{n+\frac{5}{6}}$.

Collocating (8) at points $x = x_n, x_{n+\frac{1}{6}}, x_{n+\frac{1}{3}}, x_{n+\frac{1}{2}}, x_{n+\frac{2}{3}}, x_{n+\frac{5}{6}}$ and x_{n+1} and interpolating (4) at $x = x_n, x_{n+\frac{1}{3}}$ and $x_{n+\frac{2}{3}}$, give a system of non-linear equations which are solved to obtained the parameters $C_k's, j = 0, 1, ...9$. The parameters $C_k's$ obtained are then substituted back into (4) to give the continuous implicit hybrid one step method of the form;

$$\begin{aligned} y(x) = \alpha_0(x)y_n + \alpha_{\frac{1}{3}}(x)y_{n+\frac{1}{3}} + \alpha_{\frac{2}{3}}(x)y_{n+\frac{2}{3}} + h^3\Big[\beta_0(x)f_n \\ +\beta_{\frac{1}{6}}(x)f_{n+\frac{1}{6}} + \beta_{\frac{1}{3}}(x)f_{n+\frac{1}{3}} + \beta_{\frac{1}{2}}(x)f_{n+\frac{1}{2}} \\ +\beta_{\frac{2}{3}}(x)f_{n+\frac{2}{3}} + \beta_{\frac{5}{6}}(x)f_{n+\frac{5}{6}} + \beta_1(x)f_{n+1}\Big] \end{aligned} \tag{9}$$

where

$$\alpha_0(x) = \left[\frac{9}{2}x^2 - \frac{9}{2}x + 1\right]$$

$$\alpha_{\frac{1}{3}}(x) = \left[-9x^2 + 6x\right]$$

$$\alpha_{\frac{2}{3}}(x) = \left[\frac{9}{2}x^2 - \frac{3}{2}x\right]$$

$$\begin{aligned} \beta_0(x) = \Big[\frac{9}{70}x^9 - \frac{27}{40}x^8 + \frac{3}{2}x^7 - \frac{147}{80}x^6 + \frac{203}{150}x^5 - \frac{49}{80}x^4 + \frac{1}{6}x^3 \\ - \frac{571}{22680}x^2 + \frac{839}{510300}x\Big] \end{aligned}$$

$$\begin{aligned} \beta_{\frac{1}{6}}(x) = \Big[-\frac{27}{35}x^9 + \frac{27}{7}x^8 - \frac{279}{35}x^7 + \frac{87}{10}x^6 - \frac{261}{50}x^5 + \frac{3}{2}x^4 \\ - \frac{211}{1890}x^2 + \frac{119}{6075}x\Big] \end{aligned}$$

$$\begin{aligned} \beta_{\frac{1}{3}}(x) = \Big[\frac{27}{14}x^9 - \frac{513}{56}x^8 + \frac{1233}{70}x^7 - \frac{1383}{80}x^6 + \frac{351}{40}x^5 - \frac{15}{8}x^4 \\ + \frac{59}{15120}x^2 + \frac{727}{68040}x\Big] \end{aligned}$$

$$\begin{aligned} \beta_{\frac{1}{2}}(x) = \Big[-\frac{18}{7}x^9 + \frac{81}{7}x^8 - \frac{726}{35}x^7 + \frac{93}{5}x^6 - \frac{127}{15}x^5 + \frac{5}{3}x^4 \\ - \frac{4}{81}x^2 + \frac{32}{5103}x\Big] \end{aligned}$$

$$\beta_{\frac{2}{3}}(x) = \Big[\frac{27}{14}x^9 - \frac{459}{56}x^8 + \frac{963}{70}x^7 - \frac{921}{80}x^6 + \frac{99}{20}x^5 - \frac{15}{16}x^4 + \frac{163}{7560}x^2 - \frac{11}{6804}x\Big]$$

$$\beta_{\frac{5}{6}}(x) = \Big[-\frac{27}{35}x^9 + \frac{108}{35}x^8 - \frac{171}{35}x^7 + \frac{39}{10}x^6 - \frac{81}{50}x^5 + \frac{3}{10}x^4 - \frac{13}{1890}x^2 + \frac{23}{42525}x\Big]$$

$$\beta_1(x) = \Big[\frac{9}{70}x^9 - \frac{27}{56}x^8 + \frac{51}{70}x^7 - \frac{9}{6}x^6 + \frac{137}{600}x^5 - \frac{1}{24}x^4 + \frac{43}{45360}x^2 - \frac{11}{145800}x\Big]$$

The continuous method (9) is used to generate the main method. That is, we evaluate at $x = x_{n+\frac{1}{6}}, x_{n+\frac{1}{2}}, x_{n+\frac{5}{6}}$ and x_{n+1}, we obtain the methods as follows

$$y_{n+\frac{1}{6}} = \frac{3}{8}y_n + \frac{3}{4}y_{n+\frac{1}{3}} - \frac{1}{8}3y_{n+\frac{2}{3}} + h^3\Big[\frac{139}{10450944}f_n + \frac{3527}{4354560}f_{n+\frac{1}{6}} + \frac{21913}{17418240}f_{n+\frac{1}{3}} + \frac{659}{3265920}f_{n+\frac{1}{2}} + \frac{703}{17418240}f_{n+\frac{2}{3}} - \frac{43}{4354560}f_{n+\frac{5}{6}} + \frac{13}{10450944}f_{n+1}\Big]$$

$$y_{n+\frac{1}{2}} = -\frac{1}{8}y_n + \frac{3}{4}y_{n+\frac{1}{3}} + \frac{3}{8}y_{n+\frac{2}{3}} + h^3\Big[-\frac{317}{52254720}f_n - \frac{235}{870912}f_{n+\frac{1}{6}} - \frac{20443}{17418240}f_{n+\frac{1}{3}} - \frac{179}{204120}f_{n+\frac{1}{2}} + \frac{347}{17418240}f_{n+\frac{2}{3}} - \frac{41}{4354560}f_{n+\frac{5}{6}} + \frac{61}{52254720}f_{n+1}\Big] \tag{10}$$

$$y_{n+\frac{5}{6}} = \frac{3}{8}y_n - \frac{5}{4}y_{n+\frac{1}{3}} + \frac{15}{8}y_{n+\frac{2}{3}} + h^3\Big[\frac{703}{52254720}f_n + \frac{3691}{4354560}f_{n+\frac{1}{6}} + \frac{11917}{3483648}f_{n+\frac{1}{3}} + \frac{3305}{653184}f_{n+\frac{1}{2}} + \frac{7675}{3483648}f_{n+\frac{2}{3}} + \frac{121}{4354560}f_{n+\frac{5}{6}} + \frac{73}{52254720}f_{n+1}\Big]$$

$$y_{n+1} = y_n - 3y_{n+\frac{1}{3}} + 3y_{n+\frac{2}{3}} + h^3\Big[\frac{1}{25515}f_n + \frac{19}{8505}f_{n+\frac{1}{6}} + \frac{157}{17010}f_{n+\frac{1}{3}}$$
$$+\frac{358}{25515}f_{n+\frac{1}{2}} + \frac{157}{17010}f_{n+\frac{2}{3}} + \frac{19}{8505}f_{n+\frac{5}{6}} + \frac{1}{25515}f_{n+1}\Big]$$

Differentiating (9) once gives

$$y'(x) = \alpha_0'(x)y_n + \alpha_{\frac{1}{3}}'(x)y_{n+\frac{1}{3}} + \alpha_{\frac{2}{3}}'(x)y_{n+\frac{2}{3}} + h^3\Big[\beta_0'(x)f_n + \beta_{\frac{1}{6}}'(x)f_{n+\frac{1}{6}}$$
$$+\beta_{\frac{1}{3}}'(x)f_{n+\frac{1}{3}} + \beta_{\frac{1}{2}}'(x)f_{n+\frac{1}{2}} + +\beta_{\frac{2}{3}}'(x)f_{n+\frac{2}{3}} + \beta_{\frac{5}{6}}'(x)f_{n+\frac{5}{6}} + \beta_1'(x)f_{n+1}\Big] \quad (11)$$

where

$$\alpha_0'(x) = \left[9x - \frac{9}{2}\right]$$
$$\alpha_{\frac{1}{3}}'(x) = [-18x + 6]$$
$$\alpha_{\frac{2}{3}}'(x) = \left[9x - \frac{3}{2}\right]$$

$$\beta_0'(x) = \Big[\frac{81}{70}x^8 - \frac{27}{5}x^7 + \frac{21}{2}x^6 - \frac{144}{40}x^5 + \frac{203}{30}x^4 - \frac{49}{20}x^3 + \frac{1}{2}x^2$$
$$-\frac{571}{11340}x + \frac{839}{510300}\Big]$$

$$\beta_{\frac{1}{6}}'(x) = \Big[-\frac{243}{35}x^8 + \frac{216}{7}x^7 - \frac{279}{5}x^6 + \frac{261}{5}x^5 - \frac{261}{10}x^4 + 6x^3$$
$$-\frac{211}{945}x + \frac{119}{6075}\Big]$$

$$\beta_{\frac{1}{3}}'(x) = \Big[\frac{243}{14}x^8 - \frac{513}{7}x^7 + \frac{1233}{10}x^6 - \frac{4149}{40}x^5 + \frac{351}{8}x^4 - \frac{15}{2}x^3$$
$$+\frac{59}{7560}x + \frac{727}{68040}\Big]$$

$$\beta_{\frac{1}{2}}'(x) = \Big[-\frac{162}{7}x^8 + \frac{648}{7}x^7 - \frac{726}{5}x^6 + \frac{558}{5}x^5 - \frac{127}{3}x^4 + \frac{20}{3}x^3$$
$$-\frac{8}{81}x + \frac{32}{5103}\Big]$$

$$\beta_{\frac{2}{3}}'(x) = \Big[\frac{243}{14}x^8 - \frac{459}{7}x^7 + \frac{963}{10}x^6 - \frac{2763}{40}x^5 + \frac{99}{4}x^4 - \frac{15}{4}x^3$$
$$+\frac{163}{3780}x - \frac{11}{6804}\Big]$$

$$\beta'_{\frac{5}{6}}(x) = \Big[-\frac{243}{35}x^8 + \frac{864}{35}x^7 - \frac{171}{5}x^6 + \frac{117}{5}x^5 - \frac{81}{10}x^4 + \frac{6}{5}x^3 - \frac{13}{945}x + \frac{23}{42525}\Big]$$

$$\beta'_1(x) = \Big[\frac{81}{70}x^8 - \frac{27}{7}x^7 + \frac{51}{10}x^6 - \frac{27}{8}x^5 + \frac{137}{120}x^4 - \frac{1}{6}x^3 + \frac{43}{22680}x - \frac{11}{145800}\Big]$$

Evaluating (11) at all points i.e at $x = x_n, x_{n+\frac{1}{6}}, x_{n+\frac{1}{3}}, x_{n+\frac{1}{2}}, x_{n+\frac{2}{3}}, x_{n+\frac{5}{6}}$, and x_{n+1} we obtain the following discrete methods;

$$\begin{aligned}
hy'_n &= -\frac{9}{2}y_n + 6y_{n+\frac{1}{3}} - \frac{3}{2}y_{n+\frac{2}{3}} + h^3\Big[\frac{839}{510300}f_n + \frac{119}{6075}f_{n+\frac{1}{6}} + \frac{727}{68040}f_{n+\frac{1}{3}} \\
&\quad + \frac{32}{5103}f_{n+\frac{1}{2}} - \frac{11}{6804}f_{n+\frac{2}{3}} + \frac{23}{42525}f_{n+\frac{5}{6}} - \frac{11}{145800}f_{n+1}\Big] \\
hy'_{n+\frac{1}{6}} &= -3y_n + 3y_{n+\frac{1}{3}} + h^3\Big[-\frac{12533}{65318400}f_n - \frac{47497}{10886400}f_{n+\frac{1}{6}} + \frac{739}{4354560}f_{n+\frac{1}{3}} \\
&\quad - \frac{1417}{3265920}f_{n+\frac{1}{2}} + \frac{1147}{4354560}f_{n+\frac{2}{3}} - \frac{937}{10886400}f_{n+\frac{5}{6}} + \frac{787}{65318400}f_{n+1}\Big] \\
hy'_{n+\frac{1}{3}} &= -\frac{3}{2}y_n + \frac{3}{2}y_{n+\frac{2}{3}} + h^3\Big[-\frac{47}{1020600}f_n - \frac{299}{85050}f_{n+\frac{1}{6}} - \frac{97}{8505}f_{n+\frac{1}{3}} \\
&\quad - \frac{89}{25515}f_{n+\frac{1}{2}} - \frac{1}{13608}f_{n+\frac{2}{3}} + \frac{1}{85050}f_{n+\frac{5}{6}} - \frac{1}{510300}f_{n+1}\Big] \\
hy'_{n+\frac{1}{2}} &= -3y_{n+\frac{1}{3}} + 3y_{n+\frac{2}{3}} + h^3\Big[-\frac{113}{65318400}f_n + \frac{263}{10886400}f_{n+\frac{1}{6}} - \frac{1361}{4354560}f_{n+\frac{1}{3}} \\
&\quad - \frac{2645}{653184}f_{n+\frac{1}{2}} - \frac{1361}{4354560}f_{n+\frac{2}{3}} + \frac{263}{10886400}f_{n+\frac{5}{6}} - \frac{113}{65318400}f_{n+1}\Big] \\
hy'_{n+\frac{2}{3}} &= \frac{3}{2}y_n - 6y_{n+\frac{1}{3}} + \frac{9}{2}y_{n+\frac{2}{3}} + h^3\Big[\frac{29}{510300}f_n + \frac{143}{42525}f_{n+\frac{1}{6}} + \frac{937}{68040}f_{n+\frac{1}{3}} \\
&\quad + \frac{64}{3645}f_{n+\frac{1}{2}} + \frac{83}{34020}f_{n+\frac{2}{3}} - \frac{1}{6075}f_{n+\frac{5}{6}} + \frac{13}{1020600}f_{n+1}\Big] \\
hy'_{n+\frac{5}{6}} &= 3y_n - 9y_{n+\frac{1}{3}} + 6y_{n+\frac{2}{3}} + h^3\Big[\frac{8467}{65318400}f_n + \frac{10289}{1555200}f_{n+\frac{1}{6}} + \frac{17389}{622080}f_{n+\frac{1}{3}} \\
&\quad + \frac{27211}{653184}f_{n+\frac{1}{2}} + \frac{24263}{870912}f_{n+\frac{2}{3}} + \frac{25463}{10886400}f_{n+\frac{5}{6}} - \frac{4853}{65318400}f_{n+1}\Big] \\
hy'_{n+1} &= \frac{9}{2}y_n - 12y_{n+\frac{1}{3}} + \frac{15}{2}y_{n+\frac{2}{3}} + h^3\Big[\frac{103}{1020600}f_n + \frac{901}{85050}f_{n+\frac{1}{6}} + \frac{97}{2430}f_{n+\frac{1}{3}} \\
&\quad + \frac{253}{3645}f_{n+\frac{1}{2}} + \frac{3553}{68040}f_{n+\frac{2}{3}} + \frac{2521}{85050}f_{n+\frac{5}{6}} + \frac{929}{510300}f_{n+1}\Big]
\end{aligned} \tag{12}$$

Differentiating (9) twice gives

$$\begin{aligned}
y''(x) &= \alpha''_0(x)y_n + \alpha''_{\frac{1}{3}}(x)y_{n+\frac{1}{3}} + \alpha''_{\frac{2}{3}}(x)y_{n+\frac{2}{3}} + h^3\Big[\beta''_0(x)f_n + \beta''_{\frac{1}{6}}(x)f_{n+\frac{1}{6}} \\
&+ \beta''_{\frac{1}{3}}(x)f_{n+\frac{1}{3}} + \beta''_{\frac{1}{2}}(x)f_{n+\frac{1}{2}} + +\beta''_{\frac{2}{3}}(x)f_{n+\frac{2}{3}} + \beta''_{\frac{5}{6}}(x)f_{n+\frac{5}{6}} + \beta''_1(x)f_{n+1}\Big]
\end{aligned} \tag{13}$$

where

$$\alpha''_0(x) = 9$$

$$\alpha''_{\frac{1}{3}}(x) = -18$$

$$\alpha''_{\frac{2}{3}}(x) = 9$$

$$\beta''_0(x) = \left[\frac{324}{35}x^7 - \frac{189}{5}x^6 + 63x^5 - \frac{441}{8}x^4 + \frac{406}{15}x^3 - \frac{147}{20}x^2 + x - \frac{571}{11340}\right]$$

$$\beta''_{\frac{1}{6}}(x) = \left[-\frac{1944}{35}x^7 + 216x^6 - \frac{1674}{5}x^5 + 261x^4 - \frac{522}{5}x^3 + 18x^2 - \frac{211}{945}\right]$$

$$\beta''_{\frac{1}{3}}(x) = \left[\frac{972}{7}x^7 - 513x^6 + \frac{3699}{5}x^5 - \frac{4149}{8}x^4 + \frac{351}{2}x^3 - \frac{45}{2}x^2 + \frac{59}{7560}\right]$$

$$\beta''_{\frac{1}{2}}(x) = \left[-\frac{1296}{7}x^7 + 648x^6 - \frac{4356}{5}x^5 + 558x^4 - \frac{508}{3}x^3 + 20x^2 - \frac{8}{81}\right]$$

$$\beta''_{\frac{2}{3}}(x) = \left[\frac{972}{7}x^7 - 459x^6 + \frac{2889}{7}x^5 - \frac{2763}{8}x^4 + 99x^3 - \frac{45}{4}x^2 + \frac{163}{3780}\right]$$

$$\beta''_{\frac{5}{6}}(x) = \left[-\frac{1944}{35}x^7 + \frac{864}{5}x^6 - \frac{1026}{5}x^5 + 117x^4 - \frac{162}{5}x^3 + \frac{18}{5}x^2 - \frac{13}{945}\right]$$

$$\beta''_1(x) = \left[\frac{324}{35}x^7 - 27x^6 + \frac{153}{5}x^5 - \frac{135}{8}x^4 + \frac{137}{30}x^3 - \frac{1}{2}x^2 + \frac{43}{22680}\right]$$

Evaluating (13) at all points i.e at $x = x_n, x_{n+\frac{1}{6}}, x_{n+\frac{1}{3}}, x_{n+\frac{1}{2}}, x_{n+\frac{2}{3}}$, $x_{n+\frac{5}{6}}$, and x_{n+1} we obtain the following discrete methods;

$$
\begin{aligned}
h^2y''_n &= 9y_n - 18y_{n+\frac{1}{3}} + h^3\Big[-\frac{571}{11340}f_n - \frac{211}{945}f_{n+\frac{1}{6}} + \frac{59}{7560}f_{n+\frac{1}{3}} \\
&\quad -\frac{8}{81}f_{n+\frac{1}{2}} + \frac{163}{3780}f_{n+\frac{2}{3}} - \frac{13}{945}f_{n+\frac{5}{6}} + \frac{43}{22680}f_{n+1}\Big] \\
h^2y'_{n+\frac{1}{6}} &= 9y_n - 18y_{n+\frac{1}{3}} + 9y_{n+\frac{2}{3}} + h^3\Big[\frac{163}{72576}f_n - \frac{221}{5040}f_{n+\frac{1}{6}} - \frac{14543}{120960}f_{n+\frac{1}{3}} \\
&\quad +\frac{13}{2835}f_{n+\frac{1}{2}} - \frac{169}{13440}f_{n+\frac{2}{3}} + \frac{11}{3024}f_{n+\frac{5}{6}} - \frac{5}{10368}f_{n+1}\Big] \\
h^2y'_{n+\frac{1}{3}} &= 9y_n - 18y_{n+\frac{1}{3}} + 9y_{n+\frac{2}{3}} + h^3\Big[-\frac{1}{7560}f_n + \frac{8}{315}f_{n+\frac{1}{6}} + \frac{1}{108}f_{n+\frac{1}{3}} \\
&\quad -\frac{38}{945}f_{n+\frac{1}{2}} + \frac{19}{2520}f_{n+\frac{2}{3}} - \frac{2}{945}f_{n+\frac{5}{6}} + \frac{1}{3780}f_{n+1}\Big] \\
h^2y'_{n+\frac{1}{2}} &= 9y_n - 18y_{n+\frac{1}{3}} + 9y_{n+\frac{2}{3}} + h^3\Big[\frac{223}{362880}f_n + \frac{269}{15120}f_{n+\frac{1}{6}} + \frac{11393}{120960}f_{n+\frac{1}{3}} \\
&\quad +\frac{179}{2835}f_{n+\frac{1}{2}} - \frac{269}{24192}f_{n+\frac{2}{3}} + \frac{1}{432}f_{n+\frac{5}{6}} - \frac{19}{72576}f_{n+1}\Big] \\
h^2y'_{n+\frac{2}{3}} &= 9y_n - 18y_{n+\frac{1}{3}} + 9y_{n+\frac{2}{3}} + h^3\Big[\frac{1}{11340}f_n + \frac{1}{45}f_{n+\frac{1}{6}} + \frac{571}{7560}f_{n+\frac{1}{3}} \\
&\quad +\frac{472}{2835}f_{n+\frac{1}{2}} + \frac{31}{420}f_{n+\frac{2}{3}} - \frac{1}{189}f_{n+\frac{5}{6}} + \frac{11}{22680}f_{n+1}\Big] \\
h^2y'_{n+\frac{5}{6}} &= 9y_n - 18y_{n+\frac{1}{3}} + 9y_{n+\frac{2}{3}} + h^3\Big[\frac{101}{120960}f_n + \frac{83}{5040}f_{n+\frac{1}{6}} + \frac{11569}{120960}f_{n+\frac{1}{3}} \\
&\quad +\frac{23}{189}f_{n+\frac{1}{2}} + \frac{1171}{5760}f_{n+\frac{2}{3}} + \frac{967}{15120}f_{n+\frac{5}{6}} - \frac{229}{120960}f_{n+1}\Big] \\
h^2y'_{n+1} &= 9y_n - 18y_{n+\frac{1}{3}} + 9y_{n+\frac{2}{3}} + h^3\Big[-\frac{1}{648}f_n + \frac{32}{945}f_{n+\frac{1}{6}} + \frac{151}{3780}f_{n+\frac{1}{3}} \\
&\quad +\frac{638}{2835}f_{n+\frac{1}{2}} + \frac{569}{7560}f_{n+\frac{2}{3}} + \frac{46}{189}f_{n+\frac{5}{6}} + \frac{115}{2268}f_{n+1}\Big]
\end{aligned}
\tag{14}
$$

The block methods are derived by combining equation (10), (12) and (14) and solved simultaneously to obtain values for $y_{n+\frac{1}{6}}$, $y_{n+\frac{1}{3}}$, $y_{n+\frac{1}{2}}, y_{n+\frac{2}{3}}, y_{n+\frac{5}{6}}, y_{n+1}, y'_{n+\frac{1}{6}}, y'_{n+\frac{1}{3}}, y'_{n+\frac{1}{2}}, y'_{n+\frac{2}{3}}, y'_{n+\frac{5}{6}}, y'_{n+1}$, $y''_{n+\frac{1}{6}}$, $y''_{n+\frac{1}{3}}$, $y''_{n+\frac{1}{2}}$, $y''_{n+\frac{2}{3}}$, $y''_{n+\frac{5}{6}}$ and y''_{n+1} needed to implement (1).

4. MODIFIED BLOCK METHOD

In order to form the block method, we adopted the techniques by (Adoghe & Omole [3])

$$A^{(0)}Y_{m+1} = \sum_{i=0}^{k}\frac{(jh)^i}{i!}e_i y^{(i)} + h^{3-i}\Big[d_i f(y_n) + b_i F(Y_m)\Big] \tag{15}$$

were

$$Y_{m+1} = \begin{bmatrix} y_{n+1} \\ y_{n+\frac{1}{6}} \\ y_{n+\frac{1}{3}} \\ y_{n+\frac{1}{2}} \\ y_{n+\frac{2}{3}} \\ y_{n+\frac{5}{6}} \end{bmatrix}, \qquad f(Y_m) = \begin{bmatrix} f_{n+1} \\ f_{n+\frac{1}{6}} \\ f_{n+\frac{1}{3}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{2}{3}} \\ f_{n+\frac{5}{6}} \end{bmatrix},$$

$$y_n^{(i)} = \begin{bmatrix} y_{n+1}^{(i)} \\ y_{n+\frac{1}{6}}^{(i)} \\ y_{n+\frac{1}{3}}^{(i)} \\ y_{n+\frac{1}{2}}^{(i)} \\ y_{n+\frac{2}{3}}^{(i)} \\ y_{n+\frac{5}{6}}^{(i)} \end{bmatrix}, \qquad f(y_n) = \begin{bmatrix} f_{n-\frac{5}{6}} \\ f_{n-\frac{2}{3}} \\ f_{n-\frac{1}{2}} \\ f_{n-\frac{1}{3}} \\ f_{n-\frac{1}{6}} \\ f_n \end{bmatrix}$$

Thus (10), (12) and (14) reduces to the following

$$A^{(0)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

When $i = 0$

$$e_o = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad d_o = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{343801}{783820800} \\ 0 & 0 & 0 & 0 & 0 & \frac{6887}{3061800} \\ 0 & 0 & 0 & 0 & 0 & \frac{1959}{358400} \\ 0 & 0 & 0 & 0 & 0 & \frac{3863}{382725} \\ 0 & 0 & 0 & 0 & 0 & \frac{505625}{31352832} \\ 0 & 0 & 0 & 0 & 0 & \frac{33}{1400} \end{bmatrix}$$

$$d_o = \begin{bmatrix} \frac{6031}{9331200} & -\frac{32981}{52254720} & \frac{5177}{9797760} & -\frac{15107}{52254720} & \frac{5947}{65318400} & -\frac{9809}{783820800} \\ \frac{1499}{255150} & -\frac{233}{58320} & \frac{52}{15309} & -\frac{379}{204120} & \frac{149}{255150} & -\frac{491}{6123600} \\ \frac{1499}{255150} & -\frac{233}{58320} & \frac{52}{15309} & -\frac{379}{204120} & \frac{149}{255150} & -\frac{491}{6123600} \\ \frac{1599}{89600} & -\frac{537}{71680} & \frac{1}{120} & -\frac{327}{71680} & \frac{129}{89600} & -\frac{71}{358400} \\ \frac{4664}{127575} & -\frac{226}{25515} & \frac{272}{15309} & -\frac{31}{3645} & \frac{344}{127575} & -\frac{142}{382725} \\ \frac{162125}{2612736} & -\frac{85625}{10450944} & \frac{66875}{1959552} & -\frac{119375}{10450944} & \frac{1625}{373248} & -\frac{18625}{31352832} \end{bmatrix}$$

When $i = 1$

$$e_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$d_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{28549}{4354560} \\ 0 & 0 & 0 & 0 & 0 & \frac{1027}{68040} \\ 0 & 0 & 0 & 0 & 0 & \frac{253}{10752} \\ 0 & 0 & 0 & 0 & 0 & \frac{272}{8505} \\ 0 & 0 & 0 & 0 & 0 & \frac{35225}{870912} \\ 0 & 0 & 0 & 0 & 0 & \frac{41}{840} \end{bmatrix}$$

$$b_1 = \begin{bmatrix} \frac{275}{20736} & -\frac{5717}{483840} & \frac{10621}{1088640} & -\frac{7703}{1451520} & \frac{403}{241920} & -\frac{199}{870912} \\ \frac{97}{1890} & -\frac{2}{81} & \frac{197}{8505} & -\frac{97}{7560} & \frac{23}{5670} & -\frac{19}{34020} \\ \frac{165}{1792} & -\frac{267}{17920} & \frac{5}{128} & -\frac{363}{17920} & \frac{57}{8960} & -\frac{47}{53760} \\ \frac{376}{2835} & -\frac{2}{945} & \frac{656}{8505} & -\frac{2}{81} & \frac{8}{945} & -\frac{2}{1701} \\ \frac{8375}{48384} & \frac{3125}{290304} & \frac{25625}{217728} & -\frac{625}{96768} & \frac{275}{20736} & -\frac{1375}{870912} \\ \frac{3}{14} & \frac{3}{140} & \frac{17}{105} & \frac{3}{280} & \frac{3}{70} & 0 \end{bmatrix}$$

When $i = 2$

$$e_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad d_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{19087}{362880} \\ 0 & 0 & 0 & 0 & 0 & \frac{1139}{22680} \\ 0 & 0 & 0 & 0 & 0 & \frac{137}{2688} \\ 0 & 0 & 0 & 0 & 0 & \frac{143}{2835} \\ 0 & 0 & 0 & 0 & 0 & \frac{3715}{72576} \\ 0 & 0 & 0 & 0 & 0 & \frac{41}{840} \end{bmatrix}$$

$$b_2 = \begin{bmatrix} \frac{2713}{15120} & -\frac{15487}{120960} & \frac{293}{2835} & -\frac{6737}{120960} & \frac{263}{15120} & -\frac{863}{362880} \\ \frac{47}{189} & \frac{11}{7560} & \frac{166}{2835} & -\frac{269}{7560} & \frac{11}{945} & -\frac{37}{22680} \\ \frac{725}{3024} & \frac{2125}{24192} & \frac{125}{567} & \frac{3875}{24192} & \frac{235}{3024} & -\frac{275}{72576} \\ \frac{27}{112} & \frac{387}{4480} & \frac{17}{105} & -\frac{243}{4480} & \frac{9}{560} & -\frac{29}{13440} \\ \frac{725}{3024} & \frac{2125}{24192} & \frac{125}{567} & \frac{3875}{24192} & \frac{235}{3024} & -\frac{275}{72576} \\ \frac{9}{35} & \frac{9}{280} & \frac{37}{105} & \frac{9}{280} & \frac{9}{35} & \frac{41}{840} \end{bmatrix}$$

5. ANALYSIS OF THE BASIC PROPERTIES OF THE METHOD

In this section, we analyze the derived scheme by determining the order and error constant, consistency, zero stability and region of absolute stability of the scheme.

5.1 ORDER AND ERROR CONSTANT

In order to find the order of the block, we adopted the method proposed by (Zurni & Abdelrahim [32]) to obtain the order of our scheme and error constant as follows:
the main block has order of $[7,7,7,7,7,7]^T$ together with the error constant vector

$$\begin{pmatrix} \frac{4001}{109709829734400} \\ \frac{199}{857108044800} \\ \frac{29}{50164531200} \\ \frac{29}{26784626400} \\ \frac{7625}{4388393189376} \\ \frac{1}{391910400} \end{pmatrix}$$

the block of first derivative has order of $[7,7,7,7,7,7,7]^T$ together with the following error constant vector

$$\begin{pmatrix} \frac{28937}{43883931893760} \\ \frac{641}{428554022400} \\ -\frac{257}{100329062400} \\ -\frac{1699}{26784626400} \\ -\frac{4257875}{8776786378752} \\ -\frac{97}{39191040} \end{pmatrix}$$

the block of second derivative has order of $[7,7,7,7,7,7,7]^T$ together with the following error constant vector

$$\begin{pmatrix} \frac{1481761}{219419659468800} \\ \frac{271}{214277011200} \\ -\frac{433}{4777574400} \\ -\frac{25657}{26784626400} \\ -\frac{50174975}{8776786378752} \\ -\frac{89}{3628800} \end{pmatrix}$$

5.2 Consistency of the Scheme

According to Lambert [21]. A numerical method is said to be consistent, if it has an order of convergence, say $p \geq 1$. Thus, our derived methods are consistent, since the order are all seven.

5.3 Zero Stability

In find the zero stability of the block method (15), the root of the first characteristics polynomial $\rho(R)$, must be simple or less than

one. That is $\rho(R) = det\left[\sum A^{(i)} R^{k-1}\right] = 0$ satisfy $|R| \leq 1$.
From the derived block scheme, we have

$$\rho(z) = det \left| z \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right|$$

$$= z^5(z-1) = 0$$

which gives $z = 0, 0, 0, 0, 0, 1$. This implies that our method is zero stable.

5.4 CONVERGENCE

A method is said to be convergent if and only if, it is consistent and zero stable. Thus our method is convergent.

5.5 REGION OF ABSOLUTE STABILITY OF THE METHOD

The hybrid block method in (15) is said to be absolutely stable, if for a given h, all roots of the characteristic polynomial $\pi(z, h) = \rho(z) - \overline{h}\sigma(z)$, satisfies $|z_s| < 1$. In this article, the locus method was adopted to determine the region of absolute stability. The test equation $y''' = \lambda^3 y$ is substituted in the main method in (15) where $\overline{h} = \lambda^3 h^3$ and $\lambda = \frac{df}{dy}$. Substituting $r = \cos\theta - i\sin\theta$ and considering real part yields the equation of region stability.

$$\overline{h}(\theta) = \frac{211631616000(\cos\theta - 1)}{\cos\theta - 84} \tag{16}$$

6. NUMERICAL TEST OF THE METHOD

In this section, we present examples of third order ODEs in order to illustrate the performance and effectiveness of our method.

Problem 1: $y''' = 3\sin x, y(0) = 1, y'(0) = 0, y''(0) = -2$
Exact solution: $y(x) = 2 + 2x^2 + e^x, 0 \leq x \leq 1$
Source: Khataybeh et al (2018)[19]

Problem 2: $y''' + 5y'' + 7y' + 3y = 0, y(0) = 1, y'(0) = 0, y''(0) = -1$

Exact solution: $y(x) = e^{-x} + xe^{-x},\ 0 \leq x \leq< 1$
Source: Khataybeh et al (2018) [19]

Problem 3: $y''' = x - 4y = 0, y(0) = 0, y'(0) = 0, y''(0) = 1$
Exact solution: $y(x) = e^{-x} + xe^{-x},\ 0 \leq x \leq< 1$
Source: Adoghe & Omole (2019) [3]

Table 1. Comparison of the error of the proposed method with A Fifth-fourth Continuous Block Implicit Hybrid Method Adogeh & Omole[3] and Operational Matrices of Bernstein Polynomials Method Khataybeh et al[19]

x	Exact solution	Error in our Method	Error in Method of [3]	Error in Method of [19]
0.1	0.9900124958340772983	$7.500e-19$	$2.220e-16$	$1.050e-16$
0.2	0.9601997335237248934	$3.020e-18$	$4.441e-16$	$5.020e-16$
0.3	0.9110094673768180589	$6.860e-18$	$1.332e-15$	$1.200e-15$
0.4	0.8431829820086552484	$1.209e-17$	$3.886e-15$	$2.180e-15$
0.5	0.7577476856711181484	$1.873e-17$	$9.215e-15$	$3.430e-15$
0.6	0.6560068447290348917	$2.685e-17$	$1.899e-14$	$4.890e-15$
0.7	0.5395265618534652788	$3.614e-17$	$3.408e-14$	$6.420e-15$
0.8	0.4101201280414962628	$4.670e-17$	$5.734e-14$	$7.780e-15$
0.9	0.2698299048119933694	$5.848e-17$	$9.010e-14$	$8.630e-15$
1.0	0.1209069176044191522	$7.113e-17$	$1.368e-13$	$8.670e-15$

Table 2. Comparison of the error of the proposed method with Operational Matrices Of Bernstein Polynomial Method Khataybeh et al [19] and A Three Step Implicit Hybrid Linear Multistep Method Mohammed & Adeniyi[24]

x	Exact solution	Error in our Method	Error in Method of [19]	Error in Method of [24]
0.1	0.9953211598395555348	$1.820e-18$	$7.420e-17$	$1.000e-10$
0.2	0.9824769036935782304	$5.860e-18$	$3.010e-16$	$3.000e-10$
0.3	0.9630636868862332258	$1.053e-17$	$6.110e-16$	$7.000e-10$
0.4	0.9384480644498950210	$1.488e-17$	$9.480e-16$	$7.000e-10$
0.5	0.9097959895689501354	$1.838e-17$	$1.260e-15$	$6.000e-10$
0.6	0.8780986177504422922	$2.076e-17$	$1.420e-15$	$2.000e-10$
0.7	0.8441950164453961749	$2.202e-17$	$1.110e-15$	$9.000e-10$
0.8	0.8087921354109988645	$2.217e-17$	$5.780e-16$	$2.800e-09$
0.9	0.7724823535071383125	$2.134e-17$	$5.930e-15$	$5.400e-09$
1.0	0.7357588823428846432	$1.967e-17$	$2.020e-14$	$3.500e-09$

Table 3. Comparison of the error of the proposed method with A Fifth-fourth Continuous Block Implicit Hybrid Method Adoghe & Omole [3] and One Step Hybrid Block Method Adeniran& Omotoye [1]

x	Exact solution	Error in our Method	Error in Method of [3]	Error in Method of [1]
0.01	0.00004999875001666655	$1.722e-19$	$2.153e-13$	$2.970e-08$
0.02	0.00019998000106663619	$3.284e-19$	$8.505e-12$	$1.988e-07$
0.03	0.00044989876214921896	$2.651e-18$	$6.844e-11$	$6.508e-07$
0.04	0.00079968006825886532	$2.213e-18$	$2.942e-10$	$1.548e-06$
0.05	0.00124921901037016886	$5.395e-18$	$8.999e-10$	$3.062e-06$
0.06	0.00179838077740007770	$1.412e-17$	$2.222e-09$	$5.363e-06$
0.07	0.00244700071013053027	$4.447e-17$	$4.728e-09$	$8.607e-06$
0.08	0.00319488436706994706	$9.974e-17$	$7.442e-13$	$1.293e-05$
0.09	0.00404180760222723476	$3.672e-17$	$1.110e-12$	$1.842e-05$
0.10	0.00498751665476719416	$3.661e-17$	$1.593e-12$	$2.513e-05$

7. CONCLUSION

In this paper, we have developed a Block Bernstein Operational Matrices Method of order $p = 7$ for direct solution of general third order ordinary differential equations. The new method was applied to third order initial value problems, the results generated show a better performance over the existing methods.

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