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A QUANDLE OF ORDER $2N$ AND THE CONCEPT OF QUANDLES ISOMORPHISM

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ABSTRACT. Quandles (non-trivial) are non-associative algebraic structures that are idempotent and distributive. The concept of quandles is still relatively new. Hence, this work is aimed at developing methods of constructing new quandles of finite even orders. The concept of quandles isomorphism is discussed. The inner automorphism structure and the centralizer of certain element(s) of some of the quandles constructed were obtained, and these were used to classify the constructed examples up to isomorphism.

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1. INTRODUCTION

The concept of quandles was introduced in 1982 independently by Joyce ([15]) and Matveev ([20]). Both associated quandles to knot invariants of a topology. Joyce, in his maiden paper, refers to these structures as knot quandles while Matveev refers to them as distributive groupoids. However, the notion was not completely new. It appeared, in part or whole, in the literature with many different names ([5]). For example, Burstin and Mayer ([3]), in 1929 presented particular classes of this notion as distributive groups, Takasaki ([23]) , in 1943 called this notion Kei. Precisely, though, Kei is an involutory quandle. In 1955, Orrin ([21]) published a paper on self distributive systems which involved, in part, the notion of quandle. Then, in 1976 Smith ([22]) published an article on distributive quasigroups that incorporated, in whole, the quandle notion. A distributive quasigroup is a latin quandle. Quandles have been investigated by topologists because of its importance in knot theory, and quandles provide several invariants of knots

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and singular knots([2, 4]). The two operations of conjugation of a group $(x, y) \mapsto y^{-1}xy$ and $(x, y) \mapsto yxy^{-1}$ describe succinctly, the conjugation quandle([16]). The symmetry at each element of a quandle is an automorphism of the quandle which fixes that element. Quandle theory is still relatively new and examples are sparse ([17]). Thus, there is need to develop methods of constructing new examples of quandles. A lot has been revealed from the construction of examples and counter examples of algebraic structures (see [9, 10, 11, 13, 14]). Hence, a method of constructing new examples of quandles of order $2n$ is developed. The method is such that if a quandle with a smaller order is used to kick start, then a higher order of such quandle (structurally) is constructed. This method is demonstrated in section 3. The problem of classifying quandles of the same order up to isomorphism, especially when they are expressed in cayley tables, is not peculiar to quandles alone. This is addressed in subsection 3.2 with Theorem 3.4 and Examples 3.4 and 3.5 as significant contributions in this paper.

2. PRELIMINARY

This section presents some definitions and results that are relevant to this work.

Definition 2.1: Let X be a set and $* : X \mapsto X$ be a binary operator. The pair $(X, *)$ is called a quandle if

(1): $\forall\ x \in X, x * x = x$
(2): $\forall\ x, y \in X,\ \exists\ z \in X : z * y = x$ and
(3): $\forall\ x, y, z \in X,\ (x * y) * z = (x * z) * (y * z)$

If $(X, *)$ is a quandle, then $*$ is called a quandle structure on X.

Specifically , a quandle can also be defined by the following:

Definition 2.2: A quandle is a set X equipped with two binary operations denoted as $x \rhd y$ and $x \rhd^{-1} y$ satisfying three identities below:

(1): $x \rhd x = x$
(2): $(x \rhd y) \rhd^{-1} y = x = (x \rhd^{-1} y) \rhd y$
(3): $(x \rhd y) \rhd z = (x \rhd z) \rhd (y \rhd z)$

for all $x, y, z \in X$

For a group quandle, $(x \rhd y) = y^{-1}xy$ and $(x \rhd^{-1} y) = yxy^{-1}$.

Definition 2.3: A trivial quandle is a quandle $(Q, \rhd)$ which satisfies only the identity:

$$(x \rhd y) = x\ \forall\ x, y \in Q$$

Definition 2.4: Let n be a positive integer. For elements $i, j \in Z_n$ (integers modulo n), define $i \star j \equiv 2j - i(mod\ n)$. Then $\star$ defines a quandle structure called the dihedral quandle.

This quandles can be gotten by considering conjugacy classes of involutions in dihedral groups. This translates into a nice formula $x \star y = 2y - x$ mod n , $x, y \in Z$ as defined above.

Definition 2.5: An Involutory quandle is a quandle $(Q, \triangleright)$ which satisfies the identity:

$$(x \triangleright y) \triangleright y = x \ \forall \ x, y \in Q$$

Definition 2.6: A quandle is said to be abelian if it satisfies the identity

$$(w \triangleright x) \triangleright (y \triangleright z) = (w \triangleright y) \triangleright (x \triangleright z)$$

Definition 2.7: Centralizer of an element $'a'$ of a quandle Q is the set of all members of Q that commute with $'a'$.

In establishing isomorphism among quandles, you can find the centralizer of each element of the quandles one after the other until you have a different set of elements for the same element in check. This is developed in subsection 3.2.

Definition 2.8: Given two quandles $(X, *)$ and $(Y, \triangleright)$, let f be a mapping from quandle $(X, *)$ to a quandle $(Y, \triangleright)$, then, f is said to be a quandle homomorphism if $f(a * b) = f(a) \triangleright f(b)$ for every $a, b \in X$

Definition 2.9: A quandle homomorphism that is bijective is called quandle isomorphism. That is to say $(X, *)$ and $(Y, \triangleright)$ are isomorphic quandles if there exists an isomorphism between them.

Definition 2.10: The automorphism group of quandle $(X, *)$, denoted as $Aut(X)$ is the group of all isomorphisms $f : X \mapsto X$.

Definition 2.11: The inner automorphism group of a quandle $(X, *)$ denoted as $Inn(X)$ is the subgroup of $Aut(X)$ generated by all R_x where $R_x(y) = y * x$, for any $x, y \in X$. The map $R_x : X \mapsto X$ that maps y to $y * x$ defines a right action of X on X, so that we obtain a map $X \mapsto inn(X)$.

Axiom (2) in definition 2.1 guarantees that to each element y of a finite quandle Q, the value at x is $x \triangleright y$. This permutation is a symmetry at y, and each symmetry is a quandle automorphism of Q. The group of automorphisms of Q generated by symmetries of its elements is called the inner automorphism group of Q denoted as $Inn(Q)$. Whenever, a quandle is expressed in or defined by a

Cayley table, it is convenient to have $Inn(Q)$ act on the right of Q. For example, consider a quandle defined by the Cayley table below:

$\cdot$	1	2	3
1	1	3	2
2	3	2	1
3	2	1	3

The inner automorphism structure is as follows

A1: $= (23)$
A2: $= (13)$
A3: $= (12)$

This quandle of order 3 has 3 elements of order 2. Thus, the inner automorphism structure of a quandle plays a significant role in the classification of the quandle.

Theorem 2.1:[6] Let $R_n = Z_n$ be the dihedral quandle with the operation $i \cdot j = 2j - i(modn)$. Then the automorphism group $Aut(R_n)$ is isomorphic to the affine group $\text{Aff}(Z_n)$.

Theorem 2.2:[6] The inner automorphism group $Inn(R_n)$ of the dihedral quandle R_n is isomorphic to the dihedral group $D_{\frac{m}{2}}$ of order m where m is the least common multiple of n and 2.

Theorem 2.3:[6] Let G be a group and let the quandle X be the group G as a set with conjugation $x \star y = yxy^{-1}$ as operation. This quandle is usually denoted by Conj(G). Then the inner automorphism group of X is isomorphic (as a group) to the quotient of G by its center $Z(G)$.

The results above help in obtaining the inner automorphism and the automorphism groups and in classifying quandles up to isomorphism. There are exactly three quandles of order 3, exactly seven quandles of order 4 and 22 quandles of order 5 up to isomorphism. These quandles were classified by Ho and Nelson ([8]). Moreover, there are 73 quandles of order 6, 298 quandles of order 7 and 1581 quandles of order 8 up to isomorphism. These quandles were classified by Elhamdadi, et al ([6, 18]). The quandles of orders 3 and 4 as classified by Ho and Nelson are shown in Table 1 and Table 2 below:

Quandle X	Disjoint cycles notation	Inn(X)	Aut(X)
Q_1	(1),(1),(1)	{ 1 }	Σ_3
Q_2	(1),(1),(12)	Z_2	Z_2
Q_3	(23),(13),(12)	Σ_3	Σ_3

TABLE 1. Quandles of order 3 with automorphism groups

Quandle X	Disjoint cycles notation	Inn(X)	Aut(X)
Q_1	(1),(1),(1),(1)	{ 1 }	Σ_4
Q_2	(1),(1),(1), (23)	Z_2	Z_2
Q_3	(1),(1),(1),(123)	Z_2	Z_3
Q_4	(1),(1),(12),(12)	Z_2	$Z_2 \times Z_2$
Q_5	(1),(34),(24),(23)	Σ_3	Σ_3
Q_6	(34),(34),(12),(12)	$Z_2 \times Z_2$	D_4
Q_7	(234),(143),(124),(132)	A_4	A_4

TABLE 2. Quandles of order 4 with automorphism groups

For quandles of orders 6, 7 and 8 (see [6, 18])

3. EVEN QUANDLES AND THE CONCEPT OF ISOMORPHISM

This section presents methods of constructing new quandles of finite even orders. These quandles are of order $2n$, and they are called in this work even quandles. Constructing quandles this way, requires that a lower-order quandle be used as a starting point. For example

$Q_2 =$

$\cdot$	1	2
1	1	1
2	2	2

is an even quandle. This is the smallest even quandle, and it is a trivial quandle. Using this as starting point, another quandle of order 4 will be constructed. This quandle will again be trivial since a trivial quandle is used as a start. Taking a different quandle of order 3, a quandle of order 6 will be constructed. The process can go on and on. Therefore, Theorem 3.1 below gives an abstract construction of an even quandle that captures the foregoing description. In the literature, it is common to describe a quandle of order n by the permutations of the n elements where the permutations are product of disjoint cycles. The disjoint cycles are the columns in the Cayley table of the quandles ([6, 18]). This representation is adopted henceforth in this section.

3.1 A Quandle of Order 2n

Theorem 3.1: Let $(Q_n, \cdot)$ be any quandle of order n, and Z_2 a residue group of order 2. Then $(Q, \rhd) = Q_n \times Z_2$ such that

$$(x,a) \rhd (y,b) = \begin{cases} (xy, a+b+1) & \text{if } a \neq 0 \text{ and } b \neq 0 \text{ or } a = 0 \text{ and } b \neq 0 \\ (xy, a+b) & \text{if otherwise} \end{cases}$$

is an even quandle of order $2n$, where n is a positive integer.

Proof:

We need to show that $(Q, \rhd)$ satisfies Definition 2.1

First, idempotent law:

Let $x', y', z' \in Q$ such that $x' = (x,a)$, $y' = (y,b)$ and $z' = (z,d) \forall\ x, y, z \in Q_n$.

Consider $x' \rhd x' = (x,a) \rhd (x,a) = (x \cdot x, a+a+1) = (x,a) = x'$ where $a \neq 0$

Second, $z' \rhd y' = x'$ implies that $[(z,d) \rhd (y,b)] = (z \cdot y, d+b+1) = (x,a) = x'$ for $d, b \neq 0$ and $a \neq 0$.

Now we show uniqueness. Suppose there exists z', z'' where $z' = (z,d)$ and $z'' = (z^*, d)$ such that $z' \rhd y' = x'$ and $z'' \rhd y' = x'$. This implies that

$$(z \cdot y, d+b+1) = (z^* \cdot y, d+b+1) \Rightarrow z = z^* \Leftrightarrow z' = z''.$$

Thus, z' is unique. Finally, we show that $(x' \rhd y') \rhd z' = (x \cdot y, a + b + 1) \rhd (z,d) = (xz \cdot yz, \alpha)$ where $\alpha \neq 0$

Next consider: $(x' \rhd z') \rhd (y' \rhd z') = [(x,a) \rhd (z,d)] \rhd [(y,b) \rhd (z,d) = (xz \cdot yz, \alpha)$ where $\alpha \neq 0$. Thus, $(Q, \rhd)$ as defined above is a quandle of order 2n.

Remark 3.1: xy is a juxtaposition of $x \cdot y$ in the construction above.

For the purpose of illustration, the three quandles of order 3 in Table 1 and two quandles of order 4 in Table 2 will be used as starting points in the examples below.

Example 3.1: Quandles of order 6

Quandle X	order 3 cycles	The cycles of the constructed quandles of order 6	Inn(X)	Aut(X)
Q_1	(1),(1),(1)	(1),(1),(1),(1),(1),(1)	{ 1 }	S_6
Q_2	(1),(1),(12)	(1),(1),(1),(1),(13)(24),(13)(24)	$A_4 \times Z_2$	$A_4 \times Z_2$
Q_3	(23),(13),(12)	(35)(46),(35)(46),(15)(26), (15)(26),(13)(24),(13)(24)	D_3	$D_3 \times Z_2$

TABLE 3. The quandle of order 2n, n=3

Example 3.2: Quandles of order 8

Quandle X	order 4 cycles	The cycles of the constructed quandles of order 8
Q_1'	(1),(1),(1),(23)	(1),(1),(1),(1),(1),(1),(35)(46),(35)(46)
Q_2'	(34),(34),(12), (12)	(57)(68),(57)(68),(57)(68),(57)(68), (13)(24),(13)(24),(13)(24),(13)(24)

TABLE 4. The quandle of order 2n, n=4

Remark 3.2: The constructed quandles in Examples 3.1 and 3.2 are appropriately isomorphic to certain quandles in the list of quandles of orders 6 and 8 respectively, presented in [6, 18].

Theorem 3.2: Let $(Q_n, \triangleright)$ be a group quandle with the conjugate operation, and Z_2 a residue group of integer (mod 2). Then, $(Q_n \times Z_2, \bullet)$ defined as

$$(x,a)\bullet(y,b) = \begin{cases} (xy, a+b+1) & \text{if } a \neq 0 \ \& \ b \neq 0 \text{ or } a = 0 \ \& \ b \neq 0 \\ (xy, a+b) & \text{if otherwise} \end{cases}$$

is a group quandle of order $2n$.

Proof:

We only need to show that the definition of a group quandle presented in Definition 2.2 is satisfied. Let $x, y, z \in Q_n$ such that $x \triangleright y = y^{-1}xy$ and $x \triangleright^{-1} y = yxy^{-1}$ hold.

Then, consider: $(x,a) \bullet (x,a) = (x,a)$

$$(x,a) \bullet (x,a) = (x^{-1}xx, a+a+1) = (x,a)$$

where $a \neq 0$.

Next consider:

$$[(x,a) \bullet (y,b)] \bullet^{-1} (y,b) = (yxy^{-1}, a+b+1) \bullet (y,b) = (x,a)$$

where $a, b \neq 0$.
Similarly

$$[(x,a) \bullet^{-1} (y,b)] \bullet (y,b) = (y^{-1}xy, a+b+1) \bullet (y,b) = (x,a)$$

Finally, we show that

$$[(x,a) \bullet (y,b)] \bullet (z,d) = [(x,a) \bullet (z,d)] \bullet [(y,b) \bullet (z,d)]$$

First,

$$[(x,a) \bullet (y,b)] \bullet (z,d) = (y^{-1}xy, b+d+1) \bullet (z,d) = (z^{-1}(y^{-1}xy)z, \beta)$$

where $\beta \neq 0$. Then,

$$\begin{aligned}[(x,a) \bullet (z,d)] \bullet [(y,b) \bullet (z,d)] &= (zxz^{-1}, a+d+1) \bullet (zyz^{-1}, b+d+1) \\ &= (z^{-1}(y^{-1}xy)z, \beta)\end{aligned}$$

where $\beta \neq 0$. Thus $(Q_n \times Z_2, \bullet)$ is a group quandle.

Corollary 3.1: Let $(Q_n, \triangleright)$ be an abelian group quandle with the conjugate operation, and Z_2 a residue group of integer (mod 2). Then, $(Q_n \times Z_2, \bullet)$ defined above is an abelian group quandle of order $2n$

Proof:

An abelian group quandle satisfies Definition 2.6. Then the result follows from Theorem 3.2

Theorem 3.3: Let $(Q_n, \cdot)$ be an involutory (Kei) quandle, and Z_2 a residue group of integer (mod 2). Then, $(Q, \star) = Q_n \times Z_2$, defined as in Theorem 3.1 is an involutory quandle of order $2n$.

Proof:

We only need to show that $(Q, \star)$ satisfies Definition 2.5 . That is

$$(x' \star y') \star y' = x' \ \forall \ x', y' \in Q$$

Let $x' = (x,a), y' = (y,b) \ \forall \ x, y \in Q_n$ then,

$$(x,a) \star (y,b) = (x \cdot y, a+b+1) \star (y,b) = [(x \cdot y) \cdot y, a] = (x,a)$$

where $a, b \neq 0$. Thus, $(Q, \star)$ is an involutory quandle

Remark 3.3: The starting quandle mostly determines the type of quandle the constructed quandle becomes. That is, if Q_n is an A-type quandle, mostly then $Q = Q_n \times Z_2$ as constructed in this work, will also be an A-type quandle. The following example will illustrate Remark 3.3

Example 3.3: A Kei quandle of order 6 constructed from a Kei quandle of order 3.

3.2 The Concept of Quandles Isomorphism

The problem of establishing isomorphism between two algebraic structures, especially when they are expressed in Cayley tables, is common to most areas of algebra. Quandles are not an exception. To some binary systems, considering their order structures is enough to establish isomorphism between them but to others, it may not be enough ([12, 14]). In quandles, the order structure corresponds to the inner automorphism. Therefore, the inner automorphism structure can distinguish two quandles of the same order up to isomorphism. Thus, two quandles of the same order shall be considered non-isomorphic if they contain different number of element(s) of the same order in their inner automorphism structures. However, whenever there is a tie, we shall go further to consider their commutative pattern by obtaining the centralizer of the same element(s) in both quandles.

Theorem 3.4: The constructed quandles in Table 3 and Table 4 are non - isomorphic quandles of orders 6 and 8 respectively.
Proof:
First we shall consider their inner automorphism structures:
Q_1 is an identity quandle of order 6 (trivial)
Q_2 contains two elements of order 2
Q_3 contains six elements of order 2
Obviously, Q_1, Q_2 and Q_3 are non-isomorphic quandles of order 6. Similarly, the quandles of order 8 in Table 4 are also non-isomorphic since Q'_1 contains two elements of order 2 and Q'_2 contains eight elements of order 2.

Example 3.4: Show that $Q_1 = (1), (1), (1), (12), (12), (45)$ and $Q_2 = (1), (1), (1), (56), (46), (45)$ are non-isomorphic.
Solution:
Q_1 and Q_2 contain 3 elements of order 2 only in their order structures.

Quandle X	Kei quandle of order 3	The constructed Kei quandle of order 6
Q	(23),(13),(12)	(35)(46),(35)(46),(15)(26),(15)(26),(13)(24),(13)(24)

TABLE 5. A Kei quandle of order 2n, n=3

But the centralizer of $4 = \{4\}$ in Q_1 and the centralizer of $4 = \{4, 5, 6\}$ in Q_2.

Though Q_1 and Q_2 are having similar inner automorphism structures, but the centralizers of element 4 in both quandles are not the same. Thus, Q_1 and Q_2 are non-isomorphic quandles of order 6.

Example 3.5: Show that $q_1 = (1), (1), (1), (12)(34), (12)(45), (45)$ and $q_2 = (1), (1), (1), (12)(56), (13)(46), (23)(45)$ are non-isomorphic.

Solution:

q_1 and q_2 have similar order structures since both contain 3 elements of order 2 only

However, q_1 has a centralizer of $6 = \{6\}$, and q_2 has a centralizer of $6 = \{4, 5, 6\}$.

Thus, q_1 and q_2 are having similar inner automorphism structures (order structures), but their centralizers of element 6 are not the same. Therefore, q_1 and q_2 are non-isomorphic quandles of order 6.

Behold, quandles in Examples 3.4 and 3.5 above have similar order structures since they have the same number of elements of order 2 only. This presents a challenge when classifying them based on their order structures. In this case, using the order structure is not enough to establish isomorphism between these quandles. We have to go further to consider their commutative pattern, by obtaining the centralizer of certain element(s) of the quandles. On the other hand, if two quandles have the same order structure, and their commutative pattern is the same, such quandles are possibly isomorphic. To illustrate, consider the following example.

Example 3.6: Show that $Q_1 = (1), (1), (1), (1), (12), (12)$ and $Q_2 = (1), (1), (1), (1), (12), (34)$ are isomorphic.

Solution:

Q_1 contains 2 elements of order 2 only while Q_2 also contains 2 elements of order 2 only as their inner automorphism structures. Moreover, the two quandles have the same commutative pattern as the centralizer of each element in both quandles are the same.

Therefore, Q_1 and Q_2 are isomorphic quandles of order 6 ($Q_1 \cong Q_2$).

Remark 3.4: All the quandles used to kick-start the examples 3.4-3.6 were taking from the list of quandles of order 6 presented in [6, 18].

4. CONCLUDING REMARKS

Theorem 3.1 presents methods of constructing new quandles of even orders, called even quandles. These methods give an algebraic replication of any quandle used as a start. The new quandles are mainly dihedral quandles. These are used to demonstrate the concept of quandles isomorphism introduced in subsection 3.2. This concept helps to establish isomorphism between any given quandles of the same order that are expressed in Cayley tables. The order structure of a quandle is summarized by the disjoint cycles representation of the quandle which also corresponds to its inner automorphisms. Whenever we find a tie, the commutative pattern of the quandle is considered by obtaining the centralizer of each element in both quandles until you establish a difference. These are used in classifying the quandles up to isomorphism. Most of the results were verified with the help of Mapple software [24]. Noteworthily, this article shows that the order structure or the inner automorphism structure of quandles is not sufficient to establish isomorphism between them. However, the classification of quandles of order 2n ($n \geq 5$) up to isomorphism is still open for future research.

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