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ON NAYO ALGEBRAS

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ABSTRACT. In this paper, nayo algebras are introduced. Properties of homomorphisms in relation to translation maps in nayo algebras are investigated. Moreover, monics and krib maps are introduced and studied in some clases of nayo algebras.

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1. INTRODUCTION

In [4], Kim and Kim introduced the notion of BE-algebras which are algebras of type (2,0). Ahn and So, in [5] and [6] introduced the notions of ideals and upper sets in BE-algebras and investigated related properties. Several logical algebras have also been studied by some researchers. BCK-algebras were introduced in [8]. BCI-algebras were studied in [9]. In [10], Q-algebras were studied. Precommutative algebras were studied in [11]. In [12], d-algebras were introduced. Fenyves BCI-algebras were studied in [[13],[14],[15]]. In this paper, a new class of algebras called nayo algebras which generalize the aforementioned logical algebras are introduced. Their properties are investigated. Homomorphisms in relation to translation maps of nayo algebras are studied. Moreover, monics and krib maps are introduced and investigated in some classes of nayo algebras.

2. PRELIMINARIES

Definition 2.1. ([4]). An algebra $(X; *, 1)$ of type (2, 0) is called a BE-algebra if the following hold:

(1) $x * x = 1$ for all $x \in X$;
(2) $x * 1 = 1$ for all $x \in X$;
(3) $1 * x = x$ for all $x \in X$;
(4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$.

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Example 2.1. ([4]). Let $X = \{1, a, b, c, d, 0\}$ be a set with the following table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Then $(X; *, 1)$ is a BE-algebra.

Proposition 2.1. ([4]). If $(X; *, 1)$ is a BE-algebra, then $x * (y * x) = 1$ for any $x, y \in X$.

Definition 2.2. ([4]). A BE-algebra $(X; *, 1)$ is said to be self distributive if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$.

Example 2.2. ([4]). Let $X = \{1, a, b, c, d\}$ be a set with the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Then $(X; *, 1)$ is a self distributive BE-algebra.

3. NAYO ALGEBRAS

Definition 3.1. A triple $(X; *, 0)$; where X is a non-empty set, $*$ a binary operation on X, and 0 a constant element of X is called a nayo algebra if the following axioms hold for all $x, y \in X$:

(1) $x * 0 = x$
(2) $x * y = 0$ and $y * x = 0 \Rightarrow x = y$
(3) $x * x = 0$

Example 3.1. Let $X = \{0, 1, a\}$. Define a binary operation $*$ on X by the multiplication table below:
Then $(X; *, 0)$ is a nayo algebra.

$*$	0	1	a
0	0	0	a
1	1	0	0
a	a	a	0

Example 3.2. Let $\mathbb{Z}$ denote the set of integers; $-$, the operation of subtraction, and 0 the zero integer. Then $(\mathbb{Z}; -, 0)$ is a nayo algebra.

Example 3.3. Let X be a non-empty set. Let 2^X denote the collection of all subsets of X, "-" the set difference operation and ϕ the empty set. Then $(2^X; -, \phi)$ is a nayo algebra.

The notation X will be adopted for a nayo algebra $(X; *, 0)$.

Definition 3.2. A nayo algebra X is called prime if $0 * x = 0$ for all $x \in X$.

Definition 3.3. A nayo algebra X is called associative if $(x*y)*z = x*(y*z)$ for all $x, y, z \in X$.

Definition 3.4. A nayo algebra X is called medial if $(x*y)*(u*v) = (x*u)*(y*v)$ for all $x, y, u, v \in X$.

Definition 3.5. A nayo algebra X is called hyper-commutative if $x*y = y*x \Rightarrow x = y$ for all $x, y \in X$.

Definition 3.6. A subset K of a nayo algebra X is called a sub-algebra of X if $(K; *, 0)$ is also a nayo algebra.

Example 3.4. Consider the set $K = \{0, 1\}$ with binary operation $*$ defined by the multiplication table below:

$*$	0	1
0	0	0
1	1	0

Then $(K; *, 0)$ is a sub-algebra of the nayo algebra in example 3.1.

Proposition 3.1. A non-empty subset K of a nayo algebra X is a sub-algebra if and only if $x * y \in K$ for all $x, y \in K$.

Proof: Let $x \in K$. By the hypothesis, $x * x = 0 \in K$. The axioms of a nayo algebra hold in K by virtue of K being a subset of X. The converse is obvious.

Definition 3.7. Let $(X; *, 0)$ and $(Y; \odot, 0')$ be nayo algebras. A function $f : X \to Y$ is called a homomorphism if $f(a * b) = f(a) \odot f(b)$ for all $a, b \in X$.

Example 3.5. Let $X = \{0, 1, 2\}$ and $Y = \{0', P\}$. Consider the nayo algebras given by the following multiplication tables:

$*$	0	1	2
0	0	0	2
1	1	0	2
2	2	2	0

$\odot$	$0'$	p
$0'$	$0'$	p
p	p	$0'$

The map $f : X \to Y$ given by $f(0) = 0', f(1) = 0', f(2) = p$ is a homomorphism.

Remark 3.1. Let $f : X \to Y$ be a homomorphism. The set $\{x \in X : f(x) = 0'\}$ is called the kernel of f, denoted by $ker(f)$. A homomorphism $f : X \to Y$ which is one to one is called a monomorphism. If f is onto, then it is called an epimorphism. If f is both one to one and onto, then it is called an isomorphism, and then X and Y are said to be isomorphic. The collection of all homomorphisms of a nayo algebra X is denoted by $Hom(X)$.

Proposition 3.2. Let $(X; *, 0)$ and $(Y; \odot, 0')$ be nayo algebras. Let $f : X \to Y$ be a homomorphism. Then

(1) $f(0) = 0'$
(2) $x * y = 0 \Rightarrow f(x) \odot f(y) = 0'$

Proof:

(1) Consider $f(0) = f(0 * 0) = f(0) \odot f(0) = 0'$ as required.
(2) Now, $f(x) \odot f(y) = f(x * y) = f(0) = 0'$ as required.

Remark 3.2. Let $f : X \to Y$ be a homomorphism. Define a relation $\sim_f$ by $x \sim_f y \Leftrightarrow f(x) = f(y)$. Then $\sim_f$ is an equivalence relation.

Definition 3.8. An equivalence relation $\sim$ on a nayo algebra X is called a congruence if $x \sim y$ and $u \sim v \Rightarrow (x * u) \sim (y * v)$ for all $x, y, u, v \in X$.

Lemma 3.1. Let X and Y be nayo algebras, and let $f : X \to Y$ be a homomorphism. The equivalence relation $\sim_f$ defined by $x \sim_f y \Leftrightarrow f(x) = f(y)$ is a congruence.

Proof: Now, $x \sim_f y \Rightarrow f(x) = f(y)$. Also, $u \sim_f v \Rightarrow f(u) = f(v)$. So, $f(x) \odot f(u) = f(y) \odot f(v)$. Hence, $f(x*u) = f(y*v)$. Therefore, $(x * u) \sim_f (y * v)$ as required.

Definition 3.9. Let X be a nayo algebra, let $[x]$ be the equivalence class of $x \in X$ and let $\overline{X}$ denote the collection of equivalence classes in the equivalence relation $\sim_f$. Define a binary operation $\diamond$ on $\overline{X}$ by $[x] \diamond [y] = [x * y]$.

Theorem 3.1. Let X and Y be nayo algebras and let $f : X \to Y$ be a homomorphism. Then $(\overline{X}; \diamond, [0])$ is a nayo algebra.

Proof: Straightforward.

Theorem 3.2. Let X and Y be nayo algebras, and let $f : X \to Y$ be a homomorphism. Then $f(X)$ is isomorphic to $\overline{X}$.

Proof: Consider the map $\phi : f(X) \to \overline{X}$ such that $\phi(f(x)) = [x]$. Then ϕ is a bijective homomorphism.

Theorem 3.3. Let X be a nayo algebra, and let $f : X \to X$ be a homomorphism. Then $ker(f) = \{0\}$ if and only if f is a monomorphism.

Proof: Suppose $ker(f) = \{0\}$. Let $x, y \in X$ such that $f(x) = f(y)$. Then $f(x * y) = f(x) * f(y) = 0$. Also, $f(y * x) = 0$. So, $x * y, y * x \in ker(f)$. So, $x * y = 0$ and $y * x = 0$. So, $x = y$.
Conversely, suppose f is a monomorphism. Let $x \in ker(f)$. Then $f(x) = 0 = f(0)$. So, $x = 0$.

Definition 3.10. Let X be a nayo algebra. A homomorphism $f : X \to X$ is called idempotent if $f[f(x)] = f(x)$ for all $x \in X$.

Theorem 3.4. Let X be a nayo algebra, and let $f : X \to X$ be an idempotent homomorphism. If f is a monomorphism, then f is the identity map.

Proof: Let f be a monomorphism, and let $x \in X$. Then $f[x * f(x)] = f(x) * f[f(x)] = f(x) * f(x) = 0 = f(0)$. So, $x * f(x) = 0$. Similarly, $f(x) * x = 0$. Hence, $f(x) = x$.

Definition 3.11. Let X be a nayo algebra. Define '$\cdot$' in $Hom(X)$ by $(f \cdot g)(x) = f(x) * g(x)$ for all $f, g \in Hom(X), x \in X$.

Lemma 3.2. Let X be a medial nayo algebra. Then $f \cdot g \in Hom(X)$ for all $f, g \in Hom(X)$.

Proof: Let X be a medial nayo algebra, and let $f, g \in Hom(X)$, $x, y \in X$. Consider $(f \cdot g)(x * y) = [f(x * y)] * [g(x * y)] = [f(x) * f(y)] * [g(x) * g(y)] = [f(x) * g(x)] * [f(y) * g(y)] = (f \cdot g)(x) * (f \cdot g)(y)$ as required.

Theorem 3.5. Let X be a medial nayo algebra. Then $(Hom(X); \cdot, 0_X)$ is a nayo algebra.

Proof: By Lemma 3.2, $(Hom(X); \cdot, 0_X)$ is closed. Now, let $f, g \in Hom(X)$, and let $x \in X$. Consider $(f \cdot 0_X)(x) = f(x) * 0_X(x) = f(x) * 0 = f(x)$. So, $f \cdot 0_X = f$. Also consider $(f \cdot f)(x) = f(x) * f(x) = 0 = 0_X$. So, $f \cdot f = 0_X$. Now, suppose $f \cdot g = 0_X$ and $g \cdot f = 0_X$. Then $(f \cdot g)(x) = 0_X(x)$ and $(g \cdot f)(x) = 0_X(x)$. Hence, $f(x) * g(x) = 0$ and $g(x) * f(x) = 0$. So, $f(x) = g(x)$. Therefore, $f = g$. Hence, $(Hom(X); \cdot, 0_X)$ satisfies all the axioms of a nayo algebra.

Theorem 3.6. Let X be a hyper commutative medial nayo algebra. Then $(Hom(X); \cdot, 0_X)$ is hyper commutative.

Proof: Let X be a hyper commutative medial nayo algebra. By Theorem 3.5, $(Hom(X); \cdot, 0_X)$ is a nayo algebra.
Let $f, g \in Hom(X)$, $x \in X$ such that $(f \cdot g)(x) = (g \cdot f)(x)$. Then $f(x) * g(x) = g(x) * f(x)$. Then $f(x) = g(x)$. Hence, $f = g$ as required.

Definition 3.12. Let X be a nayo algebra; and let a be a fixed element of X. The map $L_a : X \to X$ such that $L_a(x) = a * x$ for all $x \in X$ is called a left translation of a.
The map $R_a : X \to X$ such that $R_a(x) = x * a$ for all $x \in X$ is called a right translation of a.

Proposition 3.3. Let X be a nayo algebra. Then every homomorphism $f : X \to X$ commutes with L_0 and R_0.

Proof: Let $f : X \to X$ be a homomorphism of a nayo algebra X, and let $x \in X$. Consider $(L_0 \circ f)(x) = L_0(f(x)) = 0 * f(x) = f(0) * f(x) = f(0 * x) = f(L_0(x)) = (f \circ L_0)(x)$. So, $f \circ L_0 = L_0 \circ f$. Similar argument shows that $f \circ R_0 = R_0 \circ f$.

Remark 3.4. Let X be a nayo algebra. Denote by $L^H(X)$, the set of all left translations which are homomorphisms.

Proposition 3.4. Let X be a nayo algebra such that $0 * (x * y) = (0 * x) * (0 * y)$ for all $x, y \in X$. Then $L^H(X) = \{L_0\}$.

Proof: Now, L_0 is a homomorphism. Let $0 \neq x \in X$. Suppose L_x is a homomorphism. Then $x = x * 0 = L_x(0) = L_x(0 * 0) = L_x(0) * L_x(0) = 0$; which is a contradiction.

Definition 3.13. Let $L(X)$ denote the set of all left translations of a nayo algebra X. Define $\otimes$ on $L(X)$ by $(L_a \otimes L_b)(x) = L_a(x) * L_b(x)$ for all $x \in X$.

Theorem 3.7. Let X be a prime nayo algebra such that the right cancellation law holds. Then $(L(X); \otimes, L_0)$ is a nayo algebra.

Proof: Let $L_a, L_b \in L(X); x \in X$.
Consider $(L_a \otimes L_0)(x) =$
$L_a(x) * L_0(x) =$
$(a * x) * (0 * x) =$
$(a * x) * 0 =$
$a * x = L_a(x) \Rightarrow L_a \otimes L_0 = L_a$
Consider $(L_a \otimes L_a)(x) =$
$L_a(x) * L_a(x) =$
$(a * x) * (a * x) = 0$
$= L_0(x)$.
Now, suppose $L_a \otimes L_b = L_0$ and $L_b \otimes L_a = L_0$.
Then $(a * x) * (b * x) = 0$ and
$(b * x) * (a * x) = 0$. So, $a * x = b * x \Rightarrow xL_a = xL_b$. Hence $L_a = L_b$.
Definition 3.14. Let X be a nayo algebra. Define '$\wedge$' by $x \wedge y = y * (y * x)$ for all $x, y \in X$. A map $f : X \to X$ is called a monic if $f(x * y) = [x * f(y)] \wedge [y * f(x)]$ for all $x, y \in X$.

Example 3.6. Consider the nayo algebra X given by the multiplication table below:

$*$	0	a	b
0	0	0	b
a	a	0	b
b	b	b	0

Define $f : X \to X$ by $f(b) = 0, f(0) = b, f(a) = b$.
Then f is a monic of X.

Definition 3.15. Let X be a nayo algebra. A map $f : X \to X$ is called regular if $f(0) = 0$.

Definition 3.16. A nayo algebra X is called a wedge if $0 * x = x$ for all $x \in X$.

Proposition 3.5. Let f be a monic of a nayo algebra X. Then for all $x \in X$,

(1) $0 * f(x) * (0 * f(x) * (x * f(0))) = f(x)$
(2) X is associative $\Rightarrow 0 * [x * f(0)] = f(x)$
(3) f is regular $\Rightarrow 0 * f(x) * [(0 * f(x)) * x] = f(x)$
(4) f is regular and X is associative $\Rightarrow 0 * x = f(x)$
(5) X is prime $\Rightarrow f(x) = 0$
(6) X is a wedge $\Rightarrow f(x) * (f(x) * (x * f(0))) = f(x)$
(7) f is regular and X is a wedge $\Rightarrow f(x) * (f(x) * x) = f(x)$

Proof:

(1) $f(x) = f(x * 0) = (x * f(0)) \wedge (0 * f(x)) = 0 * f(x) * (0 * f(x) * (x * f(0)))$
(2) Apply the associative law to the left hand side of item (1).
(3) Apply regularity of f to the left hand side of item (1).
(4) Apply the associative law to the left hand side of item (3).
(5) Apply the property of X being prime to item (1).
(6) Apply the property of X being a wedge to item (1).
(7) Apply regularity of f to the left hand side of item (6).

Proposition 3.6. Let f be a monic of a nayo algebra X. Then for all $x \in X$,

(1) $x * f(x) = f(0)$
(2) f is regular $\Rightarrow x * f(x) = 0$
(3) X is associative $\Rightarrow 0 * (x * f(x)) = f(0)$
(4) f is regular and X is associative $\Rightarrow 0 * (x * f(x)) = 0$

Proof:

(1) $f(0) = f(x * x) = (x * f(x)) \wedge (x * f(x)) = x * f(x)$
(2) Apply regularity of f to the right hand side of item (1).
(3) Consider $f(0) = f(x * x) = [x * f(x)] \wedge [x * f(x)] = (x * f(x)) * [(x * f(x)) * (x * f(x))] = 0 * (x * f(x))$ as required.
(4) Apply regularity of f to the right hand side of item (3).

Proposition 3.7. Every monic of a prime nayo algebra X is regular.

Proof: Let $x \in X$. Consider $f(0) = f(0 * x) = (0 * f(x)) \wedge (x * f(0)) = (x * f(0)) * (x * f(0) * (0 * f(x))) = 0$.

Proposition 3.8. Let f be a monic on a wedge X. Then for all $x \in X$,

(1) $(x * f(0)) * (x * f(0) * f(x)) = f(x)$
(2) f is regular $\Rightarrow x * (x * f(x)) = f(x)$
(3) X is associative $\Rightarrow 0 * f(x) = f(x)$

Proof:

(1) Consider $f(x) = f(0 * x) = [0 * f(x)] \wedge [x * f(0)] = f(x) \wedge [x * f(0)] = (x * f(0)) * [(x * f(0)) * f(x)]$ as required.
(2) Apply regularity of f to the left hand side of item (1).
(3) Apply the associative law to the left hand side of item (1).

Proposition 3.9. Let f be a monic of an associative nayo algebra X. Then $f(x * y) = 0 * [x * f(y)]$ for all $x, y \in X$.

Proof: Consider $f(x * y) = (y * f(x)) * [(y * f(x)) * (x * f(y))] = 0 * [x * f(y)]$.

Definition 3.17. Let f be a self map of a nayo algebra X. A map $\alpha : X \to X$ is called a right krib map of X if $\alpha(x * y) = [\alpha(x) * f(y)] \wedge [f(x) * \alpha(y)]$ for all $x, y \in X$.

If $\alpha(x * y) = [f(x) * \alpha(y)] \wedge [\alpha(x) * f(y)]$ for all $x, y \in X$, then α is called a left krib map of X.
The map f is called the underlying map of X for α.

Remark 3.3. If α is both a left krib map and a right krib map of a nayo algebra X, then α is called a krib map of X.

Example 3.7. Let $X = \mathbb{N} \cup \{0\}$; where $\mathbb{N}$ is the set of natural numbers. Define $*$ on X by

$$x * y = \begin{cases} 0, & x \leq y \\ x - y, & x > y \end{cases}$$

Then X is a nayo algebra. Now, define $\alpha : X \to X$ by $\alpha(x) = 0$ for all $x \in X$. Also define $f : X \to X$ by $f(x) = 5x$ for all $x \in X$. Then α is a both a right krib map and a left krib map of X.

Definition 3.18. Let X be a nayo algebra. A map $f : X \to X$ is called krest if $f(x) = 0$ for all $x \in X$.

Proposition 3.10. Let α be a regular right krib map of a nayo algebra X. Then
$[f(x) * \alpha(x)] * [(f(x) * \alpha(x)) * (\alpha(x) * f(x))]$
$= [f(y) * \alpha(y)] * [(f(y) * \alpha(y)) * (\alpha(y) * f(y))]$ for all $x, y \in X$.

Proof: Now, $0 = \alpha(0) = \alpha(x * x)$
$= [\alpha(x) * f(x)] \wedge [f(x) * \alpha(x)]$
$= [f(x)\alpha(x)] * [(f(x)\alpha(x)) * (\alpha(x) * f(x))]$.
Similar argument gives
$[f(y)\alpha(y)] * [(f(y)\alpha(y)) * (\alpha(y) * f(y))] = 0$.
Hence the result follows.

Proposition 3.11. Let α be a regular right krib map on an associative nayo algebra X. Then $0 * [\alpha(x) * f(x)] = 0 * [\alpha(y) * f(y)]$ for all $x, y \in X$.

Proof: Consider $0 = \alpha(0) = \alpha(x * x)$
$= [\alpha(x)f(x)] \wedge [f(x)\alpha(x)]$
$= [f(x)\alpha(x)] * [f(x)\alpha(x) * \alpha(x)f(x)]$
$= 0 * [\alpha(x)f(x)]$.
Similar argument gives $0 * [\alpha(y)f(y)] = 0$.
Hence the conclusion follows.

Proposition 3.12. Let α be a right krib map of a nayo algebra X. Then for all $x \in X$, the following hold:

(1) $f(0)\alpha(x) * [f(0)\alpha(x) * \alpha(0)f(x)] = \alpha(0 * x)$
(2) X is associative $\Rightarrow 0 * [\alpha(0)f(x)] = \alpha(0 * x)$
(3) X is prime $\Rightarrow f(0)\alpha(x) * [f(0)\alpha(x) * \alpha(0)f(x)] = \alpha(0)$

Proof:

(1) Consider $\alpha(0 * x) = [\alpha(0) * f(x)] \wedge [f(0) * \alpha(x)]$
$= [f(0) * \alpha(x)] * [f(0)\alpha(x) * \alpha(0)f(x)]$.
(1) Apply the associative law to item (1).
(2) Apply the property of X being prime to the right hand side of item (1).

Corollary 3.1. Let α be a regular right krib map of a nayo algebra X. Then for all $x \in X$, the following hold:

(1) $[f(0) * \alpha(x)] * [f(0)\alpha(x) * 0 * f(x)] = \alpha(0 * x)$
(2) X is associative $\Rightarrow 0 * [0 * f(x)] = \alpha(0 * x)$
(3) X is prime $\Rightarrow f(0)\alpha(x) * [f(0)\alpha(x) * (0 * f(x))] = \alpha(0)$

Proof:

(1) Apply regularity of α to item (1) of Proposition 3.12.
(2) Apply regularity of α to item (2) of Proposition 3.12.
(3) Apply regularity of α to item (3) of Proposition 3.12.

Proposition 3.13 Let α be a right krib map of a nayo algebra X with underlying map $f \in Hom(X)$. Then for all $x \in X$, the following hold: $[0 * \alpha(x)] * [(0 * \alpha(x)) * \alpha(0)f(x)] = \alpha(0 * x)$

Proof: Now, $\alpha(0 * x) = [\alpha(0)f(x)] \wedge [f(0)\alpha(x)]$
$= [0 * \alpha(x)] * [(0 * \alpha(x)) * \alpha(0)f(x)]$.

Corollary 3.2. Let α be a regular right krib map of a nayo algebra X with underlying map $f \in Hom(X)$. Then for all $x \in X$, we have:

(1) $[0 * \alpha(x)] * [(0 * \alpha(x)) * (0 * f(x))] = \alpha(0 * x)$
(2) X is associative $\Rightarrow 0 * [0 * f(x)] = \alpha(0 * x)$

Proof:

(1) Apply regularity of α to Proposition 3.13.
(2) Apply the associative law to item (1) of this corollary.

Proposition 3.14. Let α be a regular right krib map of a nayo algebra X with a krest underlying map f. Then for all $x, y \in X$, the following hold:

(1) $[0 * \alpha(x)] * [(0 * \alpha(x)) * \alpha(x)] = 0$
(2) $[0 * \alpha(x)] * [(0 * \alpha(x)) * \alpha(x)] = [0 * \alpha(y)] * [(0 * \alpha(y)) * \alpha(y)]$
(3) $\alpha(0 * x) = 0$

Proof:

(1) Consider $0 = \alpha(0) = \alpha(x * x) = [\alpha(x) * f(x)] \wedge [f(x) * \alpha(x)] = [0 * \alpha(x)] * [(0 * \alpha(x)) * \alpha(x)]$ as required.
(2) Replacing x with y in item (1), we have $0 = [0 * \alpha(y)] * [(0 * \alpha(y)) * \alpha(y)]$. Hence, the conclusion follows.
(3) Consider $\alpha(0 * x) = [\alpha(0) * f(x)] \wedge [f(0) * \alpha(x)] = 0 * \alpha(x) * [(0 * \alpha(x)) * 0] = 0$ as required.

Corollary 3.3. Let α be a regular right krib map of a nayo algebra X with a krest underlying map f. Then for all $x, y \in X$, the following hold: $\alpha(0 * x) = \alpha(0 * y)$

Proof: Replacing x with y in item (3) of Proposition 3.14, we have $\alpha(0 * y) = 0$. Hence, the conclusion follows.

Proposition 3.15. Let α be a regular right krib map of an associative nayo algebra X with krest underlying map. Then for all $x, y \in X$, the following hold:

(1) $0 * \alpha(x) = 0$
(2) $0 * \alpha(x) = 0 * \alpha(y)$

Proof:

(1) Consider $0 = \alpha(0) = \alpha(x * x) = f(x)\alpha(x) * [f(x)\alpha(x) * \alpha(x)f(x)] = 0 * \alpha(x)$
(2) Replacing x with y in item (1), we have $0 * \alpha(y) = 0$. Since $0 * \alpha(x) = 0$ and $0 * \alpha(y) = 0$, we have $0 * \alpha(x) = 0 * \alpha(y)$ as required.

Remark 3.5. One can follow the fore-going arguments to prove the following propositions for left krib maps:

Proposition 3.16. Let α be a regular left krib map of a nayo algebra X. Then
$[\alpha(x) * f(x)] * [(\alpha(x) * f(x)) * (f(x) * \alpha(x))]$
$= [\alpha(y) * f(y)] * [(\alpha(y) * f(y)) * (f(y) * \alpha(y))]$ for all $x, y \in X$.

Proof: Now, $0 = \alpha(0) = \alpha(x * x) = [f(x) * \alpha(x)] \wedge [\alpha(x) * f(x)] = [\alpha(x) * f(x)] * [(\alpha(x) * f(x)) * (f(x) * \alpha(x))]$. Similar argument gives $[\alpha(y) * f(y)] * [(\alpha(y) * f(y)) * (f(y) * \alpha(y))] = 0$. Hence the conclusion follows.

Proposition 3.17. Let α be a regular left krib map on an associative nayo algebra X. Then $0 * [f(x) * \alpha(x)] = 0 * [f(y) * \alpha(y)]$ for all $x, y \in X$.

Proof: Consider $0 = \alpha(0) = \alpha(x*x) = [f(x)*\alpha(x)] \wedge [\alpha(x)*f(x)] = [(\alpha(x)*f(x))*(\alpha(x)*f(x))]*[f(x)*\alpha(x)] = 0*[f(x)*\alpha(x)]$. Similar argument gives $0 = 0 * [f(y) * \alpha(y)]$. Hence, the conclusion follows.

Proposition 3.18. Let α be a left krib map of a nayo algebra X. Then for all $x \in X$, the following hold:

(1) $\alpha(0)f(x) * [\alpha(0)f(x) * f(0)\alpha(x)] = \alpha(0 * x)$
(2) X is associative $\Rightarrow 0 * [f(0)\alpha(x)] = \alpha(0 * x)$
(3) X is prime $\Rightarrow \alpha(0)f(x) * [\alpha(0)f(x) * f(0)\alpha(x)] = \alpha(0)$

Proof:

(1) Consider $\alpha(0 * x) = [f(0) * \alpha(x)] \wedge [\alpha(0) * f(x)] = [\alpha(0) * f(x)] * [(\alpha(0) * f(x)) * (f(0) * \alpha(x))]$ as required.
(2) Apply the associative law to the left hand side of item (1).
(3) Apply the definition of X being prime to the right hand side of item (1).

Corollary 3.4. Let α be a regular left krib map of a nayo algebra X. Then for all $x \in X$, the following hold:

(1) $[0 * f(x)] * [0 * f(x) * (f(0) * \alpha(x))] = \alpha(0 * x)$
(2) X is associative $\Rightarrow 0 * (f(0) * \alpha(x)) = \alpha(0 * x)$

(3) X is prime $\Rightarrow 0 * f(x) * [0 * f(x) * (f(0) * \alpha(x))] = 0$

Proof:

(1) Apply the definition of regularity of α to item (1) of Proposition 3.18.
(2) Apply the associative law to the left hand side of item (1) of Proposition 3.18.
(3) Apply primeness of X and regularity of α to the right hand side of item (3) of Proposition 3.18.

Proposition 3.19. Let α be a left krib map of a nayo algebra X with underlying map $f \in Hom(X)$. Then for all $x \in X$, the following hold: $[\alpha(0) * f(x)] * [\alpha(0)f(x) * 0\alpha(x)] = \alpha(0 * x)$

Proof: Now, $\alpha(0*x) = [f(0)*\alpha(x)] \wedge [\alpha(0)*f(x)] = [\alpha(0)*f(x)] * [(\alpha(0) * f(x)) * (0 * (\alpha(x)))]$ as required.

Corollary 3.5. Let α be a regular left krib map of a nayo algebra X with underlying map $f \in Hom(X)$. Then for all $x \in X$, we have:

(1) $[0 * f(x)] * [(0 * f(x)) * (0 * \alpha(x))] = \alpha(0 * x)$
(2) X is associative $\Rightarrow 0 * [0 * \alpha(x)] = \alpha(0 * x)$

Proof:

(1) Apply regularity of α to the right hand side of the identity in Proposition 3.19.
(2) Apply the associative law to the left hand side of item (1).

Proposition 3.20. Let α be a regular left krib map of a nayo algebra X with a krest underlying map f. Then for all $x, y \in X$, the following hold:

(1) $\alpha(x) * [\alpha(x) * ((0 * \alpha(x))] = 0$
(2) $\alpha(x) * [\alpha(x) * ((0 * \alpha(x))] = \alpha(y) * [\alpha(y) * ((0 * \alpha(y))]$
(3) $0 * [0 * (0 * \alpha(x))] = \alpha(0 * x)$

Proof:

(1) Consider $0 = \alpha(0) = \alpha(x*x) = [f(x)*\alpha(x)] \wedge [\alpha(x)*f(x)] = \alpha(x) * [\alpha(x) * (0 * \alpha(x))]$ as required.
(2) Replacing x with y in item (1), we have $0 = \alpha(y) * [\alpha(y) * (0 * \alpha(y))]$. Hence the conclusion follows.
(3) Consider $\alpha(0 * x) = [f(0) * \alpha(x)] \wedge [\alpha(0) * f(x)] = 0 * [0 * (0 * \alpha(x))]$ as required.

Proposition 3.21. Let α be a regular left krib map of an associative nayo algebra X with krest underlying map. Then for all $x, y \in X$, the following hold:

(1) $0 * [0 * \alpha(x)] = 0$
(2) $0 * [0 * \alpha(x)] = 0 * [0 * \alpha(y)]$

Proof:

(1) Consider $0 = \alpha(0) = \alpha(x*x) = [f(x)*\alpha(x)] \wedge [\alpha(x)*f(x)] = [\alpha(x) * \alpha(x)] * [0 * \alpha(x)] = 0 * [0 * \alpha(x)]$ as required.
(2) Replacing x with y in item (1), we have $0 = *[0 * \alpha(y)]$. Hence the conclusion follows.

Remark 3.5. Combining the results on right and left krib maps, we have the following theorems:

Theorem 3.8. Let α be a regular krib map of a nayo algebra X. Then for all $x, y \in X$, the following hold:

(1) $[f(x) * \alpha(x)] * [(f(x) * \alpha(x)) * (\alpha(x) * f(x))]$
$= [f(y) * \alpha(y)] * [(f(y) * \alpha(y)) * (\alpha(y) * f(y))]$
(2) $[\alpha(x) * f(x)] * [(\alpha(x) * f(x)) * (f(x) * \alpha(x))]$
$= [\alpha(y) * f(y)] * [(\alpha(y) * f(y)) * (f(y) * \alpha(y))]$

Proof: Since α is a krib map, it is both a right and left krib map. The result therefore follows from the combination of Propositions 3.10 and 3.16.

Theorem 3.9. Let α be a regular krib map on an associative nayo algebra X. Then for all $x, y \in X$, the following hold:

(1) $0 * [\alpha(x) * f(x)] = 0 * [\alpha(y) * f(y)]$
(2) $0 * [f(x) * \alpha(x)] = 0 * [f(y) * \alpha(y)]$

Proof: Since α is a krib map, it is both a right and left krib map. The result therefore follows from the combination of Propositions 3.11 and 3.17.

Theorem 3.10. Let α be a krib map of a nayo algebra X. Then for all $x \in X$, the following hold:

(1) $f(0)\alpha(x) * [f(0)\alpha(x) * \alpha(0)f(x)] = \alpha(0 * x)$
(2) X is associative $\Rightarrow 0 * [\alpha(0)f(x)] = \alpha(0 * x)$
(3) X is prime $\Rightarrow f(0)\alpha(x) * [f(0)\alpha(x) * \alpha(0)f(x)] = \alpha(0)$
(4) $\alpha(0)f(x) * [\alpha(0)f(x) * f(0)\alpha(x)] = \alpha(0 * x)$
(5) X is associative $\Rightarrow 0 * [f(0)\alpha(x)] = \alpha(0 * x)$
(6) X is prime $\Rightarrow \alpha(0)f(x) * [\alpha(0)f(x) * f(0)\alpha(x)] = \alpha(0)$

Proof: Since α is a krib map, it is both a right and left krib map. The result therefore follows from the combination of Propositions 3.12 and 3.18.

Theorem 3.11. Let α be a right krib map of a nayo algebra X with underlying map $f \in Hom(X)$. Then for all $x \in X$, the following hold:

(1) $[0 * \alpha(x)] * [(0 * \alpha(x)) * \alpha(0)f(x)] = \alpha(0 * x)$
(2) $[\alpha(0) * f(x)] * [\alpha(0)f(x) * 0\alpha(x)] = \alpha(0 * x)$

Proof: Since α is a krib map, it is both a right and left krib map. The result therefore follows from the combination of Propositions 3.13 and 3.19.

Theorem 3.12. Let α be a regular krib map of a nayo algebra X with a krest underlying map f. Then for all $x, y \in X$, the following hold:

(1) $[0 * \alpha(x)] * [(0 * \alpha(x)) * \alpha(x)] = 0$
(2) $[0 * \alpha(x)] * [(0 * \alpha(x)) * \alpha(x)] = [0 * \alpha(y)] * [(0 * \alpha(y)) * \alpha(y)]$
(3) $\alpha(0 * x) = 0$
(4) $\alpha(x) * [\alpha(x) * (0 * \alpha(x))] = 0$
(5) $\alpha(y) * [\alpha(y) * (0 * \alpha(y))] = \alpha(x) * [\alpha(x) * (0 * \alpha(x))]$
(6) $0 * [0 * (0 * \alpha(x))] = \alpha(0 * x)$

Proof: Since α is a krib map, it is both a right and left krib map. The result therefore follows from the combination of Propositions 3.14 and 3.20.

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