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SOME TOPOLOGICAL PROPERTIES OF AN INVOLUTION PERMUTATION METRIC SPACES

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ABSTRACT. Various works have been done on defining metric on permutation spaces using different approaches. All the researches focus on the algorithmic and combinatorial properties of the involution permutation metric space. However, none of these works studied the topological properties of metric on permutation spaces. Hence, in this study, a metric is constructed on set of some permutations on S_n called involution permutations. Some topological properties of the involution permutation metric space were investigated. The study shows that every subset of the involution permutation metric space is open and the topological space generated by the involution permutation metric space is Hausdorff, disconnected and normal.

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1. INTRODUCTION

Metric on S_n have been defined by many researchers as well as its application to some areas such as coding theory, statistics, bell-ringing, computing, etc. [12], [1], [8],[9], [11].

Levenshtein, [13], introduced a metric called "edit distance" which defined distance between two distinct permutations as the minimum cost of the "edit" operation required to transform one permutation into the other. This distance is called string edit distance, where the edit operations are inserting a new character, removing an existing character, or changing a character to a different one. Levenshtein was concerned with binary strings. This distance leads to various works on defining metric on permutation space using different approach.

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Wagner and Fischer ,[19], extended the work of ,[13], to non-binary strings, introduced the ability to apply different costs to the three types of edit operations, and provided a dynamic programming algorithm for computing it, while Cicirello , [5], extended the work of [13] by defining edit distance with a single atomic edit operation, removal/insertion which removes an element and reinserts it elsewhere in the permutation. It is the minimum number of removals/insertions needed to transform permutation p_1 into p_2.

On the other hand, Caprara ,[4], introduced reversal edit distance as the minimum number of reversals needed to transform p_1 into p_2, but, it was argued that the best available approximations are insufficient for search landscape analysis by [17].

However distance/metric were also defined in relation to graph theory. Useful measures for permutation to represent set of edges, cyclic edge distance and acyclic edge distance were defined in [14]. Also, Campos, Laguna and Mart'i ,[3], defined the r-type distance as a directed edge version of acyclic edge distance, while Cicirello ,[5], defined the cyclic r-type distance which is a cyclic counterpart to r-type distance, which includes an edge between the end points. Similarly, r-type distance satisfies the metric properties, while cyclic r-type is a pseudo-metric. They were defined respectively as

$$d(p_1,p_2)=\sum_{i=1}^{n-1}\begin{cases}0, & if\ \ \exists x\ :\ p_1(i)=p_2(x)\wedge p_1(i+1)\neq p_2(x+1)),\\ 1, & otherwise.\end{cases}$$

$$d(p_1,p_2)=\sum_{i=1}^{n}\begin{cases}0, & if\ \ \exists j\exists x\exists y\ :\ i=(imodn)+1\wedge y=(xmodn)\\ +1\wedge P_1(i)=p_2(x)\wedge p_1(j)\neq p_2(y)), & \\ 1, & otherwise.\end{cases}$$

Ronald [15], extended Hamming distance to non-binary string, producing a permutation distance called "exact match distance", which is the number of positions with different elements. It is an edit distance where the only edit operation is element changes. It is also a metric defined as

$$d(p_1,p_2)=\sum_{i=1}^{n}\begin{cases}1, & if\ \ p_1(i)\neq p_2(i)),\\ 0, & otherwise.\end{cases}$$

In the same year a metric on permutation space called deviation distance was also defined by Ronald as the sum of the positional deviations of the permutation elements.

Sevaux and Sorensen [18] suggested squared deviation distance on involution permutation metric space, which is based on Spearman's rank correlation coefficient. They mistakenly stated that squared deviation distance as well as deviation distance require quadratic time as observed by Cicirello, [6].

Cicirello and Cernera ,[7], also defined Interchange distance as an edit distance with one edit operation, elements interchange (or swap). It is the minimum number of swaps needed to transform p_1 into p_2, and is computed efficiently by counting the number of cycles between the permutations. The interchange distance is defined by

$$d(p_1, p_2) = n - CycleCount(p_1, p_2).$$

Cameron ,[2], studied the metric and topological aspect of the symmetric group of countable degree using Baire category and Haar measure.

Deza and Huang ,[10], surveyed on distances on the symmetric group together with their applications in many contexts such as statistics, coding theory, computing, bell-ringing and so on, which were originally seen unrelated. All these researches focus on the algorithmic and combinatorial properties of the permutation metric space. However, none of these works studied the topological properties of metric on involution permutations. Investigating the topological properties of involution permutation metric space is of immense importance, since involutions play vital role in euclidean geometry (reflection of plane), projective geometry, linear algebra and so on. Hence, in this paper, we defined a metric on involution permutations using an Eulerian permutation statistics called descent and investigate its topological properties.

2. NOTATION

Definition 2.1. [16]; A permutation π on a non-empty set is called an involution if it is not the identity, but its square (π^2) is.

Definition 2.2. Let $X_n \subset S_n$ be a set of involutions permutations of length n with identity permutation. We defined a metric as a function $d : X_n \times X_n \to \mathbb{R}$ as:

$$d(\pi_i, \pi_j) = des(\pi_i \circ \pi_j)$$

where $\pi_i, \pi_j \in X_n$ and $i, j \in \mathbb{N}$.
Also,

$$Des(\pi) = \{i : a_i > a_{i+1}, a_i \in \pi\}$$

and

$$des(\pi) = |Des(\pi)|.$$

Remark 2.1. Let $\pi = (a_1 a_2 \cdots a_n) \in S_n$. A *descent* in π is an integer j with $1 \leq j \leq n$ such that $a_j > a_{j+1}$. The number of descents in π is donated by $des(\pi)$, while the set of descents is denoted by $Des(\pi)$.

Definition 2.3. Let X_n be a nonempty set of involution permutations with identity. A function $d : X_n \times X_n \to \mathbb{R}$ is said to be a metric on X_n if it satisfies the following properties. For each $\pi_i, \pi_j, \pi_k \in X_n$

(1) $d(\pi_i, \pi_j) \geq 0$
(2) $d(\pi_i, \pi_j) = 0 \Leftrightarrow \pi = \pi_j$
(3) $d(\pi_i, \pi_j) = d(\pi_j, \pi_i)$
(4) $d(\pi_i, \pi_j) \leq d(\pi_i, \pi_j) + d(\pi_j, \pi_k)$.

The pair (X_n, d) is called a involution permutation metric space.

Example 2.2. Consider the symmetric group of length 3

$$S_3 = \left\{ e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \right\}.$$

The set of involutions permutations $I_3 \subset S_3$ is

$$I_3 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \right\}$$

and X_n is defined as $X_n = \{e\} \cup I_n$. Therefore,

$$X_3 = \left\{ e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \right\}.$$

Then we defined a metric $d : X_3 \times X_3 \to \mathbb{R}$ by

$$d(\pi_i, \pi_j) = des(\pi_i \circ \pi_j), \ \forall \ \pi_i, \pi_j \in X_3$$

Now let

$$\pi_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$\pi_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$\pi_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

(1) $d(\pi_i, \pi_j) \geq 0$

$$\begin{aligned} d(\pi_1, \pi_2) &= des(\pi_1 \circ \pi_2) \\ &= des\left(\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\right) \\ &= des\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \\ &= 1 > 0 \end{aligned}$$

(2) $d(\pi_i, \pi_j) = 0 \Leftrightarrow \pi_i = \pi_j$

$$\begin{aligned} d(\pi_3, \pi_3) &= des(\pi_3 \circ \pi_3) \\ &= des\left(\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}\right) \\ &= des\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \\ &= 0 \end{aligned}$$

(3) $d(\pi_i, \pi_j) = d(\pi_j, \pi_i)$

$$\begin{aligned} d(\pi_1, \pi_2) &= des(\pi_1 \circ \pi_2) \\ &= des\left(\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\right) \\ &= des\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \\ &= 1 \end{aligned}$$

Similarly,

$$\begin{aligned} d(\pi_2, \pi_1) &= des(\pi_2 \circ \pi_1) \\ &= des\left(\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}\right) \\ &= des\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \\ &= 1 \end{aligned}$$

(4) $d(\pi_i, \pi_k) \leq d(\pi_i, \pi_j) + d(\pi_j, \pi_k)$
$d(\pi_1, \pi_3) \leq d(\pi_1, \pi_2) + d(\pi_2, \pi_3)$

$$\begin{aligned}
d(\pi_1, \pi_3) &= des(\pi_1 \circ \pi_3) \\
&= des\left(\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}\right) \\
&= des\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \\
&= 1
\end{aligned}$$

$$\begin{aligned}
d(\pi_1, \pi_2) &= des(\pi_1 \circ \pi_2) \\
&= des\left(\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\right) \\
&= des\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \\
&= 1
\end{aligned}$$

$$\begin{aligned}
d(\pi_2, \pi_3) &= des(\pi_2 \circ \pi_3) \\
&= des\left(\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}\right) \\
&= des\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \\
&= 1
\end{aligned}$$

Hence $d(\pi_1, \pi_3) \leq d(\pi_1, \pi_2) + d(\pi_2, \pi_3)$.

Definition 2.4. Let (X_n, d) be a involution permutation metric space. Let $\pi \in X_n$ and $r > 0$. Then the subsets

$$B_r(\pi) = \{\pi_i \in X_n : d(\pi, \pi_i) < r\}$$
$$\overline{B}_r(\pi) = \{\pi_i \in X_n : d(\pi, \pi_i) \leq r\}$$
$$S_r(\pi) = \{\pi_i \in X_n : d(\pi, \pi_i) = r\}$$

are respectively called the open ball, closed ball and sphere centred at π with radius r with respect to the metric d.

Remark 2.3.

$$S_r(\pi) = \overline{B}_r(\pi) - B_r(\pi).$$

Definition 2.5. A subset O of an involution permutation metric space X_n is called open if for all $\pi \in O$ there exists $\epsilon > 0$ such that $B(\pi, \epsilon) \subseteq O$.

Definition 2.6. Let (X_n, d) be an involution permutation metric space. The diameter $\delta(A)$ of a nonempty subset $A \subseteq X_n$ is defined as

$$\delta(A) = \sup_{\pi_i, \pi_j \in A} \{d(\pi_i, \pi_j)\}.$$

Definition 2.7. Let (X_n, d) be an involution permutation metric space. A subset $A \subseteq X_n$ is said to be bound if $\delta(A) < \infty$.

Definition 2.8. For any two nonempty subsets A and B in X_n,

$$d(A, B) = \max\{d(\pi_i, \pi_j) : \pi_i \in A, \pi_j \in B\}.$$

Definition 2.9. Let (X_n, d) be an involution permutation metric space and $A \subseteq X_n$. A point $\pi \in X_n$ is called an interior point of A if there exists $\epsilon > 0$ such that $B(\pi, \epsilon) \subseteq A$.

Definition 2.10. The set of all interior points of A is called the interior of A and is denoted by $int(A)$.

Remark 2.4. Let (X_n, d) be an involution permutation metric space. A set $A \subseteq X_n$ is called open if and only if every point in A is an interior point of A.

3. CONSTRUCTION OF METRIC ON INVOLUTION PERMUTATION SPACE

Proposition 3.1. Let X_n be a permutation space and d be defined as $d(\pi_i, \pi_j) = des(\pi_i \circ \pi_j)$. Then the pair (X_n, d) is an involution permutation metric space.

Proof. Let X_n be a set of involution permutations with identity and d be a distance function ($d : X_n \times X_n \to \mathbb{R}$) defined as $d(\pi_i, \pi_j) = des(\pi_i \circ \pi_j)$. Then for each $\pi_i, \pi_j, \pi_k \in X_n$, we have

(1) $des(\pi_i \circ \pi_j) \geq 0$. This follows obviously since descent of any permutation π is non-negative.

(2) If $d(\pi_i, \pi_j) = 0$ this implies that $des(\pi_i \circ \pi_j) = 0 \Rightarrow \pi_i \circ \pi_j = e$ hence $\pi_i = \pi_j$.
Conversely, suppose $\pi_i = \pi_j$, then

$$d(\pi_i, \pi_j) = d(\pi_i, \pi_i) = des(\pi_i \circ \pi_i) = des(e) = 0.$$

(3) Similarly, $des(\pi_i \circ \pi_j) = d(\pi_i, \pi_j) = d(\pi_j, \pi_i) = des(\pi_j \circ \pi_i)$.

(4) Also,

$$\begin{aligned} des(\pi_i \circ \pi_k) &= d(\pi_i, \pi_k) \\ &\leq d(\pi_i, \pi_j) + d(\pi_j, \pi_k) \\ &= des(\pi_i \circ \pi_j) + des(\pi_j \circ \pi_k) \end{aligned}$$

Hence (X_n, d) is a metric space, called *involution permutation metric space.*

□

Remark 3.2. From Definition 2.2, we can observe that for any permutation π of length n, $des(\pi) \leq n-1$.

Lemma 3.3. Let $\pi_0 = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}$ and
$\pi_* = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n & n-1 \end{pmatrix}$, then $\pi_0, \pi_* \in X_n$

Proof. It is obvious that neither π_0 nor π_* is the identity. Also,

$$\begin{aligned} \pi_0^2 = \pi_0 \circ \pi_0 &= \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix} \end{aligned}$$

Similarly,

$$\begin{aligned} \pi_*^2 = \pi_* \circ \pi_* &= \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n & n-1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n & n-1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix} \end{aligned}$$

Therefore, π_0, π_* are an involutions. Hence, $\pi_0, \pi_* \in X_n$. □

Proposition 3.4. For any X_n, $\exists\ \pi_i, \pi_j \in X_n$, such that $d(\pi_i, \pi_j) = n-1$ and it is the supremum distance in X_n. That is,

$$\sup\{d(\pi_i, \pi_j) : \pi_i, \pi_j \in X_n\} = n-1$$

Proof. For any X_n, $\exists\ e \in X_n$, and $\pi_0 \in X_n$ from Lemma 3.3. Such that

$$d(\pi_0, e) = des(\pi_0 \circ e) = des(\pi_0) = n-1.$$

Hence from Remark 3.2 $\sup\{d(\pi_i, \pi_j) : \pi_i, \pi_j \in X_n\} = n-1$. □

Proposition 3.5. For any X_n, $\exists\ \pi_i, \pi_j \in X_n$, and $\pi_i \neq \pi_j$ such that $d(\pi_i, \pi_j) = 1$ and it is the infimum distance in X_n. That is,

$$\inf\{d(\pi_i, \pi_j) : \pi_i \neq \pi_j\} = 1$$

Proof. Let $\pi_i, \pi_j \in X_n$. Case I: If $\pi_i = \pi_j$, the result follows from the second condition i.e

$$d(\pi_i, \pi_j) = des(\pi_i \circ \pi_j) = des(e) = 0$$

Case II: if $\pi_i \neq \pi_j$, then $d(\pi_i, \pi_j) \neq 0$. Also, for any X_n, $\exists\, e \in X_n$, and $\pi_* \in X_n$ from Lemma 3.3, such that

$$d(\pi_*, e) = des(\pi_* \circ e) = des(\pi_*) = 1.$$

Hence the $\min\{d(\pi_i, \pi_j) : \pi_i \neq \pi_j\} = 1$ □

Theorem 3.6. The metric on any involution permutation metric space (X_n, d) is bounded. That is,

$$0 \leq d(\pi_i, \pi_j) \leq n - 1, \ \forall\, \pi_i, \pi_j \in X_n.$$

Proof. This follows immediately from Proposition 3.4 and Proposition 3.5. □

Proposition 3.7. Let A be a non empty subset of the involution permutation metric space (X_n, d). Then

$$\delta(A) = \sup_{\pi_i, \pi_j \in A} \{d(\pi_i, \pi_j)\} \leq n - 1.$$

Proof. Let (X_n, d) be an involution permutation metric space. Suppose A is a non empty subset of X_n. Then

$$\delta(A) = \sup_{\pi_i, \pi_j \in A} \{d(\pi_i, \pi_j)\}$$

This implies that

$$\delta(A) = \sup_{\pi_i, \pi_j \in A} \{des(\pi_i \circ \pi_j)\} \leq n - 1$$

since, $0 \leq des(\pi) \leq n - 1$, for any permutation of length n. □

Proposition 3.8. Let A and B be two non empty subsets of an involution permutation metric space (X_n, d), then

$$d(A, B) = \begin{cases} 0, & if \ \ A \cap B \neq \emptyset, \\ i, & if \ \ A \cap B = \emptyset, \ \ 1 \leq i \leq n - 1. \end{cases}$$

Proof. Given that A and B are nonempty subsets of X_n. If $A \cap B \neq \emptyset$, $\exists\ \pi_i \in A \cap B$ such that $\pi_i \in A$ and $\pi_i \in B$ this implies that $d(\pi_i, \pi_i) = 0$.
If $A \cap B = \emptyset$, then for any permutation $\pi_i \in A\ \exists\ \pi_j \in B$, such that $\pi_i \neq \pi_j$. Therefore, $d(\pi_i, \pi_j) > 0$. Hence, $0 < d(A, B) \leq n - 1$. □

Proposition 3.9. Let (X_n, d) be an involution permutation metric space, where X_n is the set of all involution of length n. Then the diameter is given as

$$\delta(X_n) = \sup\{d(\pi_i, \pi_j) : \pi_i, \pi_j \in X_n\} = n - 1.$$

Proof. This follows from Proposition 3.4. □

Proposition 3.10. If $r = 1$, then $B(\pi, r) = \{\pi\}$.

Proof. Let X_n be the set of involution permutations. Let $B(\pi, r)$ be an open ball of an arbitrary permutation $\pi \in X_n$ with radius 1, then

$$B(\pi, 1) = \{\pi_i \in X_n \ : \ d(\pi, \pi_i) < 1\}$$

Since $d(\pi, \pi_i)$ is non-negative integer then $B(\pi, 1) = \{\pi_i \in X_n \ : d(\pi, \pi_i) = 0\}$.
But, $d(\pi, \pi_i) = 0$ if and only if $\pi = \pi_i$. Hence $B(\pi, 1) = \{\pi\}$. □

Proposition 3.11. Let (X_n, d) be an involution permutation metric space and $\pi \in X_n$. Then

$$S_{n-1}(\pi) = \left\{\pi_0 \ : \ \pi_0 = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}\right\}$$

Proof. Let (X_n, d) be an involution permutation metric space of length n. Let S_{n-1} be the sphere of a permutation $\pi \in X_n$ with radius $n - 1$ and

$$S_{n-1}(\pi) = \{\pi_i : d(\pi, \pi_i) = n - 1\}$$

Then

$$S_{n-1}(\pi) = \left\{\pi_0 \ : \ \pi_0 = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}\right\}$$

since $d(e, \pi_0) = d(\pi_0, e) = des(\pi_0) = n - 1$. □

Proposition 3.12. Let (X_n, d) be an involution permutation metric space and $\pi \in X_n$ then

$$\overline{B}_{n-1}(\pi) = X_n$$

Proof. Let (X_n, d) be an involution permutation metric space. Let $\overline{B}_{n-1}(\pi)$ be a closed ball and $\pi \in X_n$ with radius $n - 1$ defined by

$$\overline{B}_{n-1}(\pi) = \{\pi_i : d(\pi, \pi_i) \leq n - 1\}$$

then $\overline{B}_{n-1} = X_n$ since the maximum distance between two permutations is $n - 1$. □

Proposition 3.13. Let (X_n, d) be an involution permutation metric space and $\pi \in X_n$, then

$$B_{n-1}(\pi) = \{X_n \setminus \pi_0\}$$

Proof.

$$\begin{aligned} B_{n-1}(\pi) &= \overline{B}_{n-1}(\pi) - S_{n-1}(\pi) \\ &= X_n - \{\pi_0\} \end{aligned}$$

as required. □

Proposition 3.14. Let A be any nonempty subset of X_n, then

(1) $int(A) = A$
(2) $int(int(A)) = int(A)$

Proof. (1) Let X_n be a nonempty set of permutation. Suppose $A \subseteq X_n$, and $\pi \in A$, then there exist $r > 0$ such that $\pi \in B_r(\pi) \subseteq A$. Choose $r = 1$. Then $B_1(\pi) = \{\pi\} \subseteq A$. Therefore $Int(A) = A$.

(2) from (i) $int(A) = A$. Hence, $Int(Int(A)) = Int(A) = A$. □

Proposition 3.15. For any $A \subseteq (X_n, d)$, A is open.

Proof. Let (X_n, d) be an involution permutation metric space. Let A be a nonempty subset of X_n. A is open if for any $\pi \in A$, there exist $\epsilon > 0$ such that

$$B_\epsilon(\pi) \subseteq A.$$

Thus holds since $Int(A) = A$ from Proposition 3.14 □

Theorem 3.16. For any $A, B \subseteq X_n$ then

(1) $A \cup B$ is open
(2) $A \cap B$ is open
(3) $A \setminus B$ is open
(4) $A \triangle B$ is open

Proof. Let (X_n, d) be an involution permutation metric space. Let A and B be nonempty subsets of X_n then

(1) $A \cup B \subseteq X_n$
(2) $A \cap B \subseteq X_n$
(3) $A \setminus B \subseteq X_n$
(4) $A \triangle B \subseteq X_n$

By Proposition 3.15 any subset of X_n is open. Hence (i), (ii), (iii) and (iv) are open. □

Proposition 3.17. The topology defined on involution permutation metric space is discrete

Proof. The discrete topology on X_n is the collection of all open sets in X_n. By Proposition 3.15 any subset of an involution permutation metric space X_n is open. Hence the topology generated on (X_n, d) is discrete. □

Proposition 3.18. Let (X_n, τ) be topological space generated from involution permutation metric space, where τ is the discrete topology. Then

(1) X_n is Hausdorff
(2) X_n is disconnected
(3) X_n is T_4-space (Normal Space)

Proof. (1) X_n is discrete, therefore for any two points $\pi_i, \pi_j \in X_n$, there exist open sets $O_i, O_j \in \tau$, such that $\pi_i \in O_i$, $\pi_j \in O_j$ and $O_i \cap O_j = \emptyset$.

(2) A topological space is disconnected if there exist a clopen set O in τ such that $O \neq \emptyset$ and $O = X_n$. But X_n is discrete therefore any set in τ is clopen.

(3) A topological space is Normal if X_n is Hausdorff and given two disjoint closed sets F F', there exist two disjoint open sets O, O' such that $F \subseteq O$ and $F' \subseteq O'$, since τ contains every subset of X_n. Hence X_n is normal.

□

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