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OPTIMIZED HYBRID BLOCK INTEGRATOR FOR NUMERICAL SOLUTION OF GENERAL THIRD ORDER ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. A hybrid method for the numerical approximation of the solution of general third order initial and boundary value problems is derived via the collocation technique. This method considers three intra-step points which are adequately selected so as to optimize the local truncation errors of the main formulas. The new method is zero-stable, consistent and convergent. Numerical examples from literature shows the efficiency of this method in terms of the global errors obtained.

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1. Introduction

Third order ordinary differential equations are of great interest in the fields of engineering and sciences. This can be seen in biological sciences and control theory. For instance, draining and coating flow problems [1], laminar boundary layer and sandwich beam problems [2], in fluid dynamics [3] and many others in literature. In this work, the numerical approximation for general 3rd-order problems of the type:

$$y''' = f(x, y, y', y''), \quad a \le x \le b \tag{1}$$

with the initial conditions

$$y(a) = \alpha_0, y'(a) = \alpha_1, y''(a) = \alpha_2 \tag{2}$$

and the boundary conditions

$$y(a) = \alpha_0, y'(a) = \alpha_1, y(b) = \beta_0 \tag{3}$$

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is considered, where α_i $i = 0, 1, 2,$ β_1, a, b are real constants. $x \in [a; b]$ and $f \in C^3[a, b]$. Other forms of boundary conditions can also be considered, say, $y(a) = \alpha_0$, $y'(a) = \alpha_1$, $y'(b) = \beta_1$ The theoretical existence and uniqueness of solution of (1) was discussed in [4, 5]. Thus in this work, the existence and uniqueness of solution of (1) are assumed. Furthermore, it is also assumed that (1) is well posed and the numerical solution is the focus. Accurate numerical methods for solving third-order initial and boundary value problems are available in literature. Some of the methods available are but not limited to Non-polynomial splines [6], quartic B-splines, [7], Quartic Splines [8], collocation method [9, 10]. Block methods [11, 12]. From all these methods, collocation/interpolation approach, which is the technique employed in this work, produces block method which is more efficient in that: it approximate the solution of (1) at several intra-points, it contains several linear multistep methods which constitute the main and additional methods required for direct solution of (1) such that it overcomes the overlapping of pieces of solutions, it is self starting, that is, it does not require any starting value from other methods. In this paper a three-step continuous hybrid block method with three intra-step points is developed via collocation approach, for the direct solution of (1).

2. Derivation of the Methods

Here, the optimized hybrid block method is derived. A non-optimized version of the method given in this paper can be found in [10], where the three off-step points in the interval $0 < x_{n+j_1} < x_{n+j_2} < x_{n+j_3} < 1$ are given such that $(j_1; j_2; j_3) = (\frac{3}{8}, \frac{5}{8}, \frac{7}{8})$. Optimized two-step block method with three hybrid points for solving third order initial value problems can be found in [13]. In their method, the off-step points r, s, g are such that $0 < r, s, g < 2$. In the case of this paper, the off-step points are $(j_1, j_2, j_3) = (s, u, v)$, where s, u and v are to be determined from the local truncation error of the main methods such that $0 < s < 1 < u < 2 < v < 3$. Also, the solution $y(x)$ of (1) is sort in the interval $[a, b]$ such that $a = x_0 < x_1 < \cdots < x_N = b$ with step-size $h = x_{j+1} - x_j$, $j = 0, 1, \ldots, N-1$.

Consider the approximation $q(x)$ of $y(x)$ given by the polynomial of the form

$$y(x) \approx q(x) = \sum_{n=0}^{r_1+r_2-1} \rho_n x^n \tag{4}$$

with the third derivative given by

$$y'''(x) \approx q'''(x) = \sum_{n=0}^{r_1+r_2-1} n(n-1)(n-2)\rho_n x^{n-3} \tag{5}$$

where $x \in [a,b]$, ρ_n's are real coefficient to be determined, r_1 and r_2 are numbers of interpolation and collocation points. Setting $r_1 = 3$ and $r_2 = 7$, the intra-step points are such that $0 < s < 1$, $1 < u < 2$, and $2 < v < 3$ which are used for the approximation of the three step method in the interval $[x_0, x_3]$. Here, $x_s = x_n + sh$, $x_u = x_n + uh$, $x_v = x_n + vh$. Now, Interpolating the approximation in (4) at the points x_{n+i}, $i = 0, 1, 2$ and collocating the approximation in (5) at the points x_{n+j} , $j = 0, s, 1, u, 2, v, 3$, the following system of 10 algebraic equations is arrived at

$$\begin{gathered} q(x_n) = y_n, \quad q(x_{n+1}) = y_{n+1}, \quad q(x_{n+2}) = y_{n+2}, \\ q'''(x_n) = f_n, \quad q'''(x_{n+s}) = f_{n+s}, \quad q'''(x_{n+1}) = f_{n+1}, \\ q'''(x_{n+u}) = f_{n+u}, \quad q'''(x_{n+2}) = f_{n+2}, \quad q'''(x_{n+v}) = f_{n+v}, \\ q'''(x_{n+3}) = f_{n+3}. \end{gathered}$$

where the approximation of $y_{n+j} = y(x_{n+j})$, and $f_{n+j} = y'''(x_{n+j})$ denote the approximation to $f(x_{n+j}, y(x_{n+j}), y'(x_{n+j}), y''(x_{n+j}))$. Solving the above system and obtaining the expressions for ρ_i, $i = 0(1)9$ and substituting them into the approximation (4). Then, carrying out the substitution for the variable $x = x_{n+i} + th$, the approximation in (4) takes the form

$$q(x_n + th) = \sum_{i=0}^{2} \alpha_i y_{n+i} + h^3 \left(\sum_{i=0}^{3} \beta_i f_{n+i} + \sum_{i=1}^{3} \hat{\beta}_i f_{n+w_i} \right) \tag{6}$$

where $w_1 = s$; $w_2 = u$; $w_3 = v$, and α_i, $i = 0, 1, 2$, β_i, $i = 0, 1, 2, 3$, $\hat{\beta}_i$, $i = 1, 2, 3$ are continuous coefficients.
Evaluating $q(x)$, $q'(x)$ and $q''(x)$ in (6) respectively at the point x_{n+3}, the following main formulae are obtained:

$$
\begin{aligned}
y_{n+3} = \ & y_n - 3y_{n+1} + 3y_{n+2} + \frac{h^3(47-63u-63v+42uv+21s(-3+2u+2v))}{2520suv} f_n \\
& + \frac{h^3(-572+525u+525v-462uv+21s(25-22u-22v+20uv))}{840(s-1)(u-1)(v-1)} f_{n+1} \\
& + \frac{Ah^3(-9057+4599s+4599u-2394su+4599v-2394sv-2394uv+1260ruv)}{2520} f_{n+2} \\
& + \frac{Bh^3(-614+189s+189u-42su+189v-42sv-42uv)}{2520} f_{n+3} \\
& + \frac{Ch^3(282-378u-378v+252uv)}{2520} f_{n+s} + \frac{Dh^3(-282+378s+378v-252sv)}{2520} f_{n+u} \\
& + \frac{Kh^3(-282+378s+378u-252su)}{2520} f_{n+v}
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
hy'_{n+3} = \ & \frac{3y_n}{2} - 4y_{n+1} + \frac{5y_{n+2}}{2} + \frac{h^3(-2003+360v+30u(12+v)+6s(60+5u+5v+21uv))}{15120suv} f_n \\
& + \frac{h^3(-1279+3282v-6u(-547+607v)+6s(547-607v+u(-607+602v)))}{5040(s-1)(u-1)(v-1)} f_{n+1} \\
& + \frac{Ah^3(-133809+63900s+63900u-31410su+63900v-31410sv-31410uv+15750suv)}{15120} f_{n+2} \\
& + \frac{Bh^3(-24403+8802s+8802u-3054su+8802v-3054sv-3054uv+1008suv)}{15120} f_{n+3} \\
& + \frac{Ch^3(-12018+2160u+2160v+180uv)}{15120} f_{n+s} + \frac{Dh^3(12018-2160s-2160v-180sv)}{15120} f_{n+u} \\
& + \frac{h^3K(12018-2160s-2160u-180su)}{15120} 15120 f_{n+v}
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
h^2y''_{n+3} = \ & y_n - 2y_{n+1} + y_{n+2} + \frac{h^3(-2896+999u+999v-336uv+s(999-336u-336v+154uv))}{5040suv} f_n \\
& + \frac{h^3(3615-719u-719v-280uv+s(-719-280v+56u(-5+11v)))}{1680(s-1)(u-1)(v-1)} f_{n+1} \\
& + \frac{Ah^3(-69450+30381s+30381u-13692su+30381v-13692sv-13692uv+6342suv)}{5040} f_{n+2} \\
& + \frac{Bh^3(-37955+13617s+13617u-4872su+13617v-4872sv-4872uv+1736suv)}{5040} f_{n+3} \\
& + \frac{Ch^3(-17376+5994u+5994v-2016uv)}{5040} f_{n+s} + \frac{Dh^3(17376-5994s-5994v+2016sv)}{5040} f_{n+u} \\
& + \frac{Kh^3(17376-5994s-5994u+2016su)}{5040} f_{n+v}
\end{aligned} \tag{9}
$$

where

$$
\left.
\begin{aligned}
& A = \frac{1}{(s-2)(u-2)(v-2)}; \quad B = \frac{1}{(s-3)(u-3)(v-3)}; \quad C = \frac{1}{s(-6+11s-6s^2+s^3)(s-u)(s-v)} \\
& D = \frac{1}{u(s-u)(-6+11u-6u^2+s^3)(u-v)}; \quad K = \frac{1}{v(s-v)(v-u)(-6+11v-6v^2+v^3)}
\end{aligned}
\right\} \tag{10}
$$

Now, the values of s, u and v can be determined by considering the Local Truncation Errors (LTEs) of (7)-(9) each, so that one more order for each of the formula can be gained by setting their principal truncation term to zero. In this sense, the values of s, u and v would have been optimized. The LTEs for each of (7)-(9) are:

$$
\begin{aligned}
\mathcal{L}(y(x_{n+3}); h) = \ & \frac{27(4suv-su-sv-uv)y(10)(x_n)h^{10}}{313600} + \frac{1}{27596800}(44s2uv - 11s2u - 11s2v + 44su2v - 11su2 + 44suv2 \\
& +242suv - 66su - 11sv2 - 66sv - 11u2v - 11uv2 - 66uv - 18)y(11)(xn)h11 + O(h)^{12}
\end{aligned} \tag{11}
$$

$$\begin{aligned}\mathcal{L}(hy'(x_{n+3});h) = \;& \frac{3(u(9-6v)+9(v-3)+s(9-6v+2u(7v-3)))y(10)(x_n)h^{10}}{156800} \\ & +\frac{1}{1254400}(42s^2uv - 18s^2u - 18s^2v + 27s^2 + 42su^2v - 18su^2 \\ & +42suv^2 + 216suv - 81su - 18sv^2 - 81sv + 162s - 18u^2v + 27u^2 \\ & -18uv^2 - 81uv + 162u + 27v^2 + 162v - 729)y(11)(x_n)h^{11} + O(h)^{12}\end{aligned} \tag{12}$$

$$\begin{aligned}\mathcal{L}(h^2y''(x_{n+3});h) = \;& \frac{(28suv-42su-42sv+108s-42uv+108u+108v-297)y(10)(x_n)h^{10}}{156800} \\ & +\frac{1}{627200}(14s^2sv - 21s^2u - 21s^2v + 54s^2 + 14su^2v..21su^2 + 14suv^2 \\ & +42suv - 72su - 21sv^2 - 72sv + 324s - 21u^2v + 54u^2 - 21uv^2 - 72uv \\ & +324u + 54v^2 + 324v - 1296)y(11)(x_n)h^{11} + O(h)^{12}\end{aligned} \tag{13}$$

Equating the principal terms in (11)-(13) to zero

$$\begin{cases} 4suv - su - sv - uv = 0 \\ u(9-6v) + 9(v-3) + s(9-6v+2u(7v-3)) = 0 \\ u(9-6v) + 9(v-3) + s(9-6v+2u(7v-3)) = 0 \end{cases} \tag{14}$$

This yields the solution $(s,u,v) = \left(\frac{3}{10}(5-\sqrt{15}), \frac{3}{2}, \frac{3}{10}(5-\sqrt{15})\right)$.

Substituting these values into (7)-(9), the following are the main formulae for the solution $y(x)$ of (1);

$$\begin{aligned} y_{n+3} = \;& y_n - 3y_{n+1} + 3y_{n+2} - h^3\left(\frac{97f_n}{34020} - \frac{1373f_{n+1}}{4620} - \frac{1373f_{n+2}}{4620} + \frac{97f_{n+3}}{34020} - \frac{2375f_{n+s}}{56133} - \frac{8336f_{n+u}}{25515} - \frac{2375f_{n+v}}{56133}\right) \\ hy'_{n+3} = \;& \frac{3y_n}{2} - 4y_{n+1} + 5y_{n+2} - h^3\left(\frac{509f_n}{204120} - \frac{929f_{n+1}}{1980} - \frac{3593f_{n+2}}{5544} + \frac{94f_{n+3}}{25515} - \frac{(855500+121500\sqrt{15})f_{n+s}}{6735960}\right. \\ & \left. - \frac{1024f_{n+u}}{2187} - \frac{(855500+121500\sqrt{15})f_{n+v}}{6735960}\right) \\ h^2y''_{n+3} = \;& y_n - 2y_{n+1} + y_{n+2} - h^3\left(\frac{f_n}{972} - \frac{1003f_{n+1}}{3080} - \frac{3149f_{n+2}}{4620} - \frac{6437f_{n+3}}{68040} - \left(\frac{16375}{56133} + \frac{25}{126}\sqrt{5/3}\right)f_{n+s}\right. \\ & \left. - \frac{8056f_{n+u}}{25515} - \left(\frac{16375}{56133} - \frac{25}{126}\sqrt{5/3}\right)f_{n+v}\right) \end{aligned} \tag{15}$$

To obtain the additional methods, evaluating $q(x)$, at the points $x_{n+s}, x_{n+u}, x_{n+v}$ then, evaluating $q'(x)$ and $q''(x)$ at the points x_{n+j}, $j = 0, s, 1, u, 2, v$, the following are obtained;

$$
\begin{aligned}
y_{n+s} = \;& \tfrac{(2245320000-449064000\sqrt{15})y_n}{816480000(5+\sqrt{15})} + \tfrac{(1224720000+734832000\sqrt{15})y_{n+1}}{816480000(5+\sqrt{15})} + \tfrac{(612360000-367416000\sqrt{15})y_{n+2}}{816480000(5+\sqrt{15})} \\
& -h^3\Big(\tfrac{11(178093-31689\sqrt{15})f_n}{272160000(5+\sqrt{15})} - \tfrac{(962417-93933\sqrt{15})f_{n+1}}{3360000(5+\sqrt{15})} - \tfrac{(74194461+9108855\sqrt{15})f_{n+2}}{816480000(5+\sqrt{15})} \\
& -\tfrac{(1420221-413721\sqrt{15})f_{n+3}}{816480000(5+\sqrt{15})} - \tfrac{(72392500-17797500\sqrt{15})f_{n+s}}{816480000(5+\sqrt{15})} - \tfrac{(67013056-26566848\sqrt{15})f_{n+u}}{816480000(5+\sqrt{15})} \\
& +\tfrac{(6053500+4315500\sqrt{15})f_{n+v}}{816480000(5+\sqrt{15})}\Big) \\
y_{n+u} = \;& -\tfrac{y_n}{8} + \tfrac{3y_{n+1}}{4} + \tfrac{3y_{n+2}}{8} + h^3\Big(\tfrac{14713f_n}{34836480} - \tfrac{170717f_{n+1}}{4730880} - \tfrac{457f_{n+2}}{430080} - \tfrac{2297f_{n+3}}{34836480} \\
& +\tfrac{(-3040000-850500\sqrt{15})f_{n+s}}{1149603840} - \tfrac{521f_{n+u}}{25515} + \tfrac{(-3040000+850500\sqrt{15})f_{n+v}}{1149603840}\Big) \\
y_{n+v} = \;& -\tfrac{(2245320000-449064000\sqrt{15})y_n}{816480000(\sqrt{15}-5)} - \tfrac{(1224720000-734832000\sqrt{15})y_{n+1}}{816480000(\sqrt{15}-5)} - \tfrac{(612360000+367416000\sqrt{15})y_{n+2}}{816480000(\sqrt{15}-5)} \\
& -h^3\Big(\tfrac{11(-178093+31689\sqrt{15})f_n}{272160000(\sqrt{15}-5)} - \tfrac{(-962417+93933\sqrt{15})f_{n+1}}{3360000(\sqrt{15}-5)} + \tfrac{(74194461+9108855\sqrt{15})f_{n+2}}{816480000(\sqrt{15}-5)} \\
& +\tfrac{(1420221-413721\sqrt{15})f_{n+3}}{816480000(\sqrt{15}-5)} + \tfrac{(6053500+4315500\sqrt{15})f_{n+s}}{816480000(\sqrt{15}-5)} + \tfrac{(67013056+26566848\sqrt{15})f_{n+u}}{816480000(\sqrt{15}-5)} \\
& +\tfrac{(72392500-17797500\sqrt{15})f_{n+v}}{816480000(\sqrt{15}-5)}\Big) \\
hy'_n = \;& -\tfrac{3y_n}{2} + 2y_{n+1} - \tfrac{y_{n+2}}{2} + h^3\Big(\tfrac{121f_n}{204120} + \tfrac{701f_{n+1}}{3465} + \tfrac{59f_{n+2}}{2520} + \tfrac{13f_{n+3}}{7290} + \tfrac{(428000+121500\sqrt{15})f_{n+s}}{6735960} \\
& -\tfrac{1672f_{n+u}}{76545} + \tfrac{(428000-121500\sqrt{15})f_{n+v}}{6735960}\Big) \\
hy'_{n+s} = \;& -\tfrac{3}{2}\sqrt{\tfrac{3}{5}}y_n + \tfrac{(-6735960000+4041576000\sqrt{15})y_{n+1}}{6735960000} + \tfrac{(6735960000-2020788000\sqrt{15})y_{n+2}}{6735960000} \\
& +h^3\Big(\tfrac{(-526424+17073\sqrt{15})f_n}{204120000} - \tfrac{(-5606596+1061289\sqrt{15})f_{n+1}}{27720000} + \tfrac{(1362402828-342656973\sqrt{15})f_{n+2}}{6735960000} \\
& +\tfrac{(-17371992+5198391\sqrt{15})f_{n+3}}{6735960000} + \tfrac{(150039500-42750000\sqrt{15})f_{n+s}}{6735960000} + \tfrac{(1270135328-330105600\sqrt{15})f_{n+u}}{6735960000} \\
& +\tfrac{(155871500-42750000\sqrt{15})f_{n+v}}{6735960000}\Big) \\
hy'_{n+1} = \;& -\tfrac{y_n}{2} + \tfrac{y_{n+2}}{2} + h^3\Big(\tfrac{439f_n}{204120} - \tfrac{421f_{n+1}}{3465} - \tfrac{191f_{n+2}}{27720} - \tfrac{11f_{n+3}}{20412} + \tfrac{(-74500-22500\sqrt{15})f_{n+s}}{6735960} - \tfrac{272f_{n+u}}{15309} \\
& +\tfrac{(-74500+22500\sqrt{15})f_{n+v}}{6735960}\Big) \\
hy'_{n+u} = \;& -y_{n+1} + y_{n+2} - h^3\Big(\tfrac{155f_n}{5225472} + \tfrac{9367f_{n+1}}{3548160} + \tfrac{9367f_{n+2}}{3548160} + \tfrac{155f_{n+3}}{5225472} - \tfrac{6575f_{n+s}}{43110144} + \tfrac{89729f_{n+u}}{2449440} - \tfrac{6575f_{n+v}}{43110144}\Big) \\
hy'_{n+2} = \;& \tfrac{y_n}{2} - 2y_{n+1} + \tfrac{3y_{n+2}}{2} - h^3\Big(\tfrac{401f_n}{204120} - \tfrac{491f_{n+1}}{3465} - \tfrac{751f_{n+2}}{27720} - \tfrac{37f_{n+3}}{51030} - \tfrac{(68000+22500\sqrt{15})f_{n+s}}{6735960} \\
& -\tfrac{1592f_{n+u}}{10935} - \tfrac{(68000-22500\sqrt{15})f_{n+v}}{6735960}\Big) \\
hy'_{n+v} = \;& \tfrac{3}{2}\sqrt{\tfrac{3}{5}}y_n + \tfrac{(-6735960000-4041576000\sqrt{15})y_{n+1}}{6735960000} + \tfrac{(6735960000+2020788000\sqrt{15})y_{n+2}}{673596000} \\
& -h^3\Big(\tfrac{(526424+17073\sqrt{15})f_n}{204120000} - \tfrac{(5606596+1061289\sqrt{15})f_{n+1}}{27720000} + \tfrac{(1362402828+342656973\sqrt{15})f_{n+2}}{6735960000} \\
& -\tfrac{(-17371992-5198391\sqrt{15})f_{n+3}}{6735960000} - \tfrac{(155871500+42750000\sqrt{15})f_{n+s}}{6735960000} + \tfrac{(1270135328+330105600\sqrt{15})f_{n+u}}{6735960000} \\
& -\tfrac{(150039500+42750000\sqrt{15})f_{n+v}}{6735960000}\Big) \\
h^2y''_n = \;& y_n - 2y_{n+1} + y_{n+2} - h^3\Big(\tfrac{6631f_n}{68040} + \tfrac{148f_{n+1}}{385} + \tfrac{263f_{n+2}}{9240} + \tfrac{31f_{n+3}}{17010} - \Big(\tfrac{-2000}{8019} - \tfrac{25}{126}\sqrt{\tfrac{5}{3}}\Big)f_{n+s} - \tfrac{8f_{n+u}}{729} \\
& \Big(\tfrac{2000}{8019} - \tfrac{25}{126}\sqrt{\tfrac{5}{3}}\Big)f_{n+v}\Big) \\
h^2y''_{n+s} = \;& y_n - 2y_{n+1} + y_{n+2} + h^3\Big(\tfrac{(493525+100602\sqrt{15})f_n}{34020000} + \tfrac{(-87775-170586\sqrt{15})f_{n+1}}{1540000} + \Big(\tfrac{65453}{184800} - \tfrac{85293\sqrt{\frac{3}{5}}}{154000}f_{n+2}\Big) \\
& +\Big(-\tfrac{-23621}{1360800} + \tfrac{207\sqrt{\frac{3}{5}}}{14000}\Big)f_{n+3} + \Big(\tfrac{2375}{112266} - \tfrac{1429}{1386\sqrt{15}}\Big)f_{n+s} + \Big(\tfrac{4168}{25515} - \tfrac{68}{875}\sqrt{\tfrac{3}{5}}\Big)f_{n+u} \\
& +\Big(\tfrac{2375}{112266} - \tfrac{1}{693\sqrt{15}}\Big)f_{n+v}\Big) \\
h^2y''_{n+1} = \;& y_n - 2y_{n+1} + y_{n+2} - h^3\Big(\tfrac{227f_n}{34020} + \tfrac{19f_{n+1}}{264} + \tfrac{157f_{n+2}}{4620} + \tfrac{173f_{n+3}}{68040} + \Big(\tfrac{125}{5103} + \tfrac{325\sqrt{\frac{5}{3}}}{12474}\Big)f_{n+s} - \tfrac{3848}{25515}f_{n+u} \\
& +\Big(\tfrac{125}{5103} - \tfrac{325\sqrt{\frac{5}{3}}}{12474}\Big)f_{n+v}\Big) \\
h^2y''_{n+u} = \;& y_n - 2y_{n+1} + y_{n+2} - h^3\Big(\tfrac{1289f_n}{544320} - \tfrac{7781f_{n+1}}{24640} + \tfrac{25f_{n+2}}{1344} + \tfrac{263f_{n+3}}{544320} - \Big(\tfrac{2375}{112266} + \tfrac{325\sqrt{\frac{5}{3}}}{22176}\Big)f_{n+s} \\
& -\tfrac{4168}{25515}f_{n+u} - \Big(\tfrac{2375}{112266} - \tfrac{325\sqrt{\frac{5}{3}}}{22176}\Big)f_{n+v}\Big)
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
h^2 y''_{n+2} = \;& y_n - 2y_{n+1} + y_{n+2} - h^3 \left(\frac{367 f_n}{68040} - \frac{304 f_{n+1}}{1155} - \frac{2081 f_{n+2}}{9240} - \frac{13 f_{n+3}}{3402} - \left(\frac{1000}{56133} + \frac{325\sqrt{\frac{5}{3}}}{12474} \right) f_{n+s} \right. \\
& \left. - \frac{12184}{25515} f_{n+u} - \left(\frac{1000}{56133} - \frac{325\sqrt{\frac{5}{3}}}{12474} \right) f_{n+v} \right) \\
h^2 y''_{n+v} = \;& y_n - 2y_{n+1} + y_{n+2} + h^3 \left(\frac{(493525-100602\sqrt{15}) f_n}{34020000} + \frac{(-87775+170586\sqrt{15}) f_{n+1}}{1540000} \right. \\
& \left. + \left(\frac{65453}{184800} + \frac{85293\sqrt{\frac{3}{5}}}{154000} \right) f_{n+2} + \left(\frac{-23621}{1360800} - \frac{207\sqrt{\frac{3}{5}}}{14000} \right) f_{n+3} + \left(\frac{2375}{112266} + \frac{1}{693\sqrt{15}} \right) f_{n+s} \right)
\end{aligned} \tag{17}
$$

3. Order and Local Truncation Errors (LTEs) of the method

The linear differential operator $\mathcal{L}[y(x);h]$ for the methods (7)-(9) is defined as

$$
\mathcal{L}[z(x);h] = hy(x+ih) - \sum_{i=0}^{2} \alpha_i z(x+ih) - h^3 \left(\sum_{i=0}^{3} \beta_i z'''(x+ih) + \sum_{i=0}^{3} \hat{\beta}_i z'''(x+w_i h) \right) \tag{18}
$$

Expanding (18) in Taylor series, we obtain

$$
\mathcal{L}[z(x);h] = C_0 z(x) + C_1 h z'(x) + C_2 h^2 z''(x) + \cdots + C_p h^p z^{(p)}(x) + O(h)^{p+1} \tag{19}
$$

where C_j are constants such that

$$
C_0 = C_1 = C_2 = \cdots = C_{p+3} = 0, \text{ and } C_{p+3} \neq 0
$$

so that

$$
\mathcal{L}[z(x);h] = C_{p+3} h^{p+3} z^{(p)}(x) + O(h)^{p+4} \tag{20}
$$

Substituting the values of $s,\ u,\ v$ into (11)-(13), then the order and error for the main methods in (7)-(9) are obtained as;

$$
\begin{aligned}
\mathcal{L}(y(x_{n+3});h) &= -\frac{243}{78848000} y(11)(x_n) h^{11} + O(h)^{12} \\
\mathcal{L}(hy'(x_{n+3});h) &= -\frac{27}{39424000} y(11)(x_n) h^{11} + O(h)^{12} \\
\mathcal{L}(h^2 y''(x_{n+3});h) &= -\frac{27}{394240000} y(11)(x_n) h^{11} + O(h)^{12}
\end{aligned} \tag{21}
$$

Thus, appearing in the form of (20). In this case, C_{p+3} is the error constant (see Lambert [14]). The linear difference operator $\mathcal{L}[y(x);h]$ is said to be consistent if the order $p > 1$. Hence the order of the main methods (7)-(9) is $p = 8$. Following the same approach, for the additional methods, order $p = 7$ is obtained and their error constants are given in the table below;

TABLE 1. Error constants for the additional methods (16)-(17), with $p = 7$.

y	C_{p+3}	y	C_{p+3}	y	C_{p+3}
y_{n+s}	-1.60444E-6	hy'_{n+1}	2.20459E-6	$h^2y''_{n+s}$	2.30922E-5
y_{n+u}	3.36662E-7	hy'_{n+u}	1.30680E-8	$h^2y''_{n+1}$	2.30922E-5
y_{n+v}	-1.60444E-6	hy'_{n+2}	-2.20459E-6	$h^2y''_{n+u}$	-8.81834E-6
hy'_n	6.61376E-6	hy'_{n+v}	1.62175E-6	$h^2y''_{n+2}$	-2.99255E-7
hy'_{n+s}	-1.62175E-6	$h^2y''_n$	4.40917E-6	$h^2y''_{n+v}$	8.81834E-6

3.1. Zero-stability

The main and additional methods forms a system of difference formulae which can be written compactly as

$$A_1Y = hA_2Y' + h^2A_3Y'' + h^3BF \tag{22}$$

where

$$\begin{aligned} Y &= (y_n, y_{n+s}, y_{n+1}, y_{n+u}, y_{n+2}, y_{n+v}, y_{n+3})^T;\\ Y' &= (y'_n, y'_{n+s}, y'_{n+1}, y'_{n+u}, y'_{n+2}, y'_{n+v}, y'_{n+3})^T;\\ Y'' &= (y''_n, y''_{n+s}, y''_{n+1}, y''_{n+u}, y''_{n+2}, y''_{n+v}, y''_{n+3})^T;\\ f &= (f_n, f_{n+s}, f_{n+1}, f_{n+u}, f_{n+2}, f_{n+v}, f_{n+3})^T; \end{aligned} \tag{23}$$

A_1, A_2, A_3 and B are matrices of coefficients. Zero-stability implies the stability of the difference system as $h \to 0$. So, if $h \to 0$ in , then the difference system may be rearranged in a more convenient matrix form as

$$\bar{A}_1Y_\tau - \bar{A}_0Y_{\tau-1} = 0 \tag{24}$$

where

$$Y_\tau = (y_{n+3}, y_{n+2}, y_{n+1}, y_{n+s}, y_{n+u}, y_{n+v})^T$$
$$Y_{\tau-1} = (y_n, y_{n-1}; y_{n-2}; y_{s-1}, y_{u-1}, y_{v-1})^T$$

$$\bar{A}_1 = \begin{pmatrix} 1 & 3 & -3 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & \frac{3}{20}(-7+2\sqrt{15}) & \frac{-3}{10}(-2+\sqrt{15}) & 1 & 0 & 0\\ 0 & -\frac{3}{8} & -\frac{3}{4} & 0 & 1 & 0\\ 0 & -\frac{3}{20}(4+\sqrt{15}) & \frac{9}{20}(3+\sqrt{15}) & 0 & 0 & 1 \end{pmatrix}$$

$$\bar{A}_0 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{11}{20} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{8} & 0 & 0 & 0 & 0 & 0 \\ -\frac{11}{20} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The block method which constitutes the main and additional methods is said to be zero-stable if the roots λ_i of the first characteristic polynomial $\rho(\lambda)$ defined by $\rho(\lambda) = det|\bar{A}_1\lambda - \bar{A}_0|$ satisfy $|\lambda_j| \leq 1$ and for $|\lambda_j| = 1$ the multiplicity does not exceed the order of the differential equation (see [15]), in this case 3. Here$\rho(\lambda) = \lambda^5(\lambda - 1)$. Consequently, since the block method (22) is consistent and zero-stable, therefore by [14], it is convergent.

3.2. Linear stability analysis

The study of zero-stability concerns the behavior of the numerical method for $h \to 0$. The concept of linear stability for any given $h > 0$ is concern with the behaviour of the underlined problem in question and not just the numerical method. Here, the stability properties for the intending numerical method is analyzed by considering the linear test equation for $\lambda > 0$ of the form

$$y'''(x) = -\lambda^3 y(x) \tag{25}$$

This linear test equation (25) was used in [10] and [13]. The method derived in this work is intended for solving general third order problem. Therefore, motivated by [16], the linear test equation for $\lambda > 0$ of the form

$$y'''(x) = -3\lambda y''(x) - 3\lambda^2 y'(x) - \lambda^3 y(x) \tag{26}$$

is adopted, since (25) does not have any derivative of y in the right side. It can also be verified that (26) has a bounded solution for $\lambda \geq 0$ as $x \to \infty$. Adopting the procedures in [16] and [13], the region in which the numerical method gives the behavior of the true solutions is determined. The derived method has eighteen equations in which there are fourteen different terms of derivatives and three intermediate terms y_{n+s}, y_{n+u}, y_{n+v}. These terms are eliminated from the system of equation so that the following recurrence equation with the term y_n, y_{n+1}, y_{n+2} and y_{n+3}

are derived

$$W_0(Z)y_n + W_1(Z)y_{n+1} + W_2(Z)y_{n+2} + W_3(Z)y_{n+3} = 0 \qquad (27)$$

where $Z = \lambda h$,

$$\begin{aligned}
W_0 &= -3870720000 + 1661680\lambda^6 + 1800\lambda^9 \\
W_1 &= 11612160000 - 1935360000\lambda^3 + 11142960\lambda^6 + 797040\lambda^9 \\
&- 6237\lambda^{12} \\
W_2 &= -11612160000 - 1935360000\lambda^3 - 11142960\lambda^6 + 797040\lambda^9 \\
&+ 6237\lambda^{12} \\
W_3 &= 3870720000 - 1661680\lambda^6 + 1800\lambda^9
\end{aligned}$$

Obtaining from the recurrence equation (27), the following characteristic polynomial is derived

$$W_0(Z) + W_1(Z)\tau + W_2(Z)\tau^2 + W_3(Z)\tau^3 = 0 \qquad (28)$$

from which the roots $\tau_{1,2,3}$ are obtained. These roots (containing the real and imaginary parts) are plotted and Figure 1 shows the region of stability for the method derived and the stability interval is $(0, 13.258)$. This region is the complex Z-plane where the roots of the characteristic equation (28) are bounded in modulus by unity.

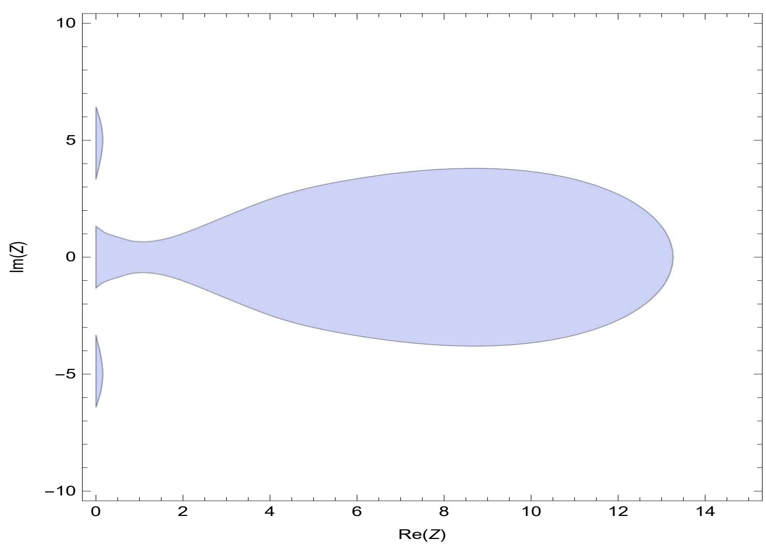

FIGURE 1. Stability region

4. Numerical Examples

To show the efficiency of this method, some examples from literature are used so as to compare errors and maximum errors where applicable.
Example 1. Consider the IVP discussed in [13].

$$\begin{aligned} &y''' = 3\sin(x), \quad x \in [0,1] \\ &y(0) = 1,\ y'(0) = 0, \quad y''(0) = -2 \end{aligned} \tag{29}$$

whose exact solution is $y(x) = 3\cos(x) + \frac{x^2}{2} - 2$.

Table 2. Comparison of maximum Errors (ME) for Example 1 for $h = 0.1$

x	ME in new method	ME in [13]
0.1	1.3107×10^{-15}	4.1078×10^{-15}
0.2	1.3157×10^{-15}	1.6875×10^{-14}
0.3	1.0758×10^{-17}	5.0848×10^{-14}
0.4	1.2101×10^{-15}	1.1779×10^{-13}
0.5	1.2246×10^{-15}	2.4081×10^{-13}
0.6	4.2042×10^{-17}	4.3709×10^{-13}
0.7	1.0229×10^{-15}	7.3708×10^{-13}
0.8	1.0456×10^{-15}	1.1662×10^{-12}
0.9	9.0975×10^{-17}	1.7587×10^{-12}

Table 2 shows the maximum error obtained for the grid points x = 0.1(0.1)0.9 using the optimized method and thus compared with the method in [13] using fixed step-size $h = 0.1$. The new method perform better.

Example 2. Consider the BVP discussed in [10].

$$\begin{aligned} &y''' + y = (x-4)\sin(x) + (1-x)\cos(x), \quad x \in [0,1] \\ &y(0) = 0, \quad y'(0) = 1, \quad y'(1) = -\sin 1 \end{aligned} \tag{30}$$

whose exact solution is $y(x) = (x-1)\sin(x)$.

Table 3. Comparison of maximum Errors (ME) for Example 2

h	ME in new method	ME in [10]
$\frac{1}{16}$	4.02552×10^{-17}	1.03179×10^{-11}
$\frac{1}{32}$	3.92786×10^{-20}	3.24907×10^{-13}
$\frac{1}{64}$	3.83607×10^{-23}	1.02789×10^{-14}

Table 3 shows the maximum error obtained using different step-sizes and compared with the method in [10] and the optimized method in this work. It clearly shows that the method presented is more superior to the cited literature.

Example 3. Consider the BVP discussed in [13].

$$\begin{aligned} &y''' + y' = 0, \quad x \in [0,1] \\ &y(0) = 0, \quad y'(0) = 1, \quad y'(1) = \cos(1) + 2\sin(1) \end{aligned} \tag{31}$$

whose exact solution is $y(x) = 2(1 - \cos(x)) + \sin(x)$.

Table 4. Comparison of maximum Errors (ME) for Example 3

x	ME in new method	ME in [13]
0.1	8.1583×10^{-16}	1.9345×10^{-14}
0.2	8.2542×10^{-16}	7.1831×10^{-14}
0.3	1.6722×10^{-17}	1.8218×10^{-13}
0.4	6.1786×10^{-16}	3.6803×10^{-13}
0.5	6.2027×10^{-16}	6.5725×10^{-13}
0.6	7.4789×10^{-18}	1.0689×10^{-12}
0.7	3.4270×10^{-16}	1.6320×10^{-12}
0.8	3.4040×10^{-16}	2.3652×10^{-12}
0.9	1.8886×10^{-17}	3.2969×10^{-12}

Table 4 shows the maximum error obtained for the grid points $x = 0.1(0.1)0.9$ using the optimized method and compared with the method in [13] using fixed step-size $h = 0.1$. The new method competes favourably with that of [13].

Example 4. Consider the nonlinear BVP discussed in [9].

$$\begin{aligned} &y''' + 2e^{-3y} = 4(1+x)^{-3}, \quad x \in [0,1] \\ &y(0) = 0, \quad y'(0) = 1, \quad y'(1) = 1/2 \end{aligned} \tag{32}$$

whose exact solution is $y(x) = \ln(1 + x)$.

Table 5. Comparison of maximum Errors (ME) for Example 4

N	ME in new method	FDM in [9]
7	2.41×10^{-9}	5.24×10^{-9}
14	4.13×10^{-12}	2.39×10^{-11}
28	5.72×10^{-15}	9.50×10^{-14}
56	5.89×10^{-18}	3.62×10^{-16}
112	6.07×10^{-21}	2.27×10^{-17}

From Table 5 different values of N has been used to obtain the maximum error for example 4. It is clear the new method perform better than the FDM in [9] despite that the later is of order $p = 8$, the method in this work is of order $p = 7$.

Example 5. Consider the BVP discussed in [10].

$$\begin{aligned} &y''' - xy = (x^3 - 2x^2 - 5x - 3)e^x, \quad x \in [0, 1] \\ &y(0) = 0, \; y'(0) = 1, \quad y'(1) = 5e \end{aligned} \tag{33}$$

whose exact solution is $y(x) = xe^x(1 + x)$.

Table 6. Comparison of maximum Errors (ME) for Example 5

h	ME in new method	ME in [10]	N	ME in new Method	FDM in [9]
$\frac{1}{16}$	7.9255×10^{-16}	1.3647×10^{-10}	7	2.48539×10^{-12}	4.12×10^{-12}
$\frac{1}{32}$	8.1312×10^{-19}	4.2086×10^{-12}	14	2.71078×10^{-15}	1.56×10^{-14}
$\frac{1}{64}$	8.5976×10^{-22}	1.2939×10^{-13}	28	3.16591×10^{-18}	6.08×10^{-17}
			56	3.18084×10^{-21}	2.37×10^{-19}
			112	3.25143×10^{-24}	9.27×10^{-22}

Table 7. Comparison of values at truncated boundary for Example 6

	New method	$(N = 15)$	FDM	$(N = 20)$	BT	$(N = 21)$
x_∞	$y''(0)$	$y''(x_\infty)$	$y''(0)$	$y''(x_\infty)$	$y''(0)$	$y''(x_\infty)$
6.67984	0.332151134	4.960205269	0.332151	4.96021	0.332151	4.96021
8.18467	0.332057669	6.463942546	0.332058	6. 4639	0.332055	6.46388
9.38665	0.332057124	7.665842125	0.332057	7.66586	0.332045	7.66582
10.57641	0.332057335	8.855622131	0.332057336	8.85562	0.332057	8.85562
11.68904	0.332057336	9.968279523	0.332057336	9.96825	0.332057	9.96825

Table 6 compares the method in this work and those of [9] and [10]. This example has a misprint in [10] but the correct version is in [9]. In both case, the method in this work performs better for different step-sizes (h) and for different number of subinterval (N), which shows its superiority over both.

Example 6. Consider the IVP, (nonlinear Blassius equation) in fluid dynamics given discussed in [9, 12].

$$\begin{aligned} &2y''' - yy'' = 0, \quad x \in [0,1] \\ &y(0) = y'(0) = 0, \quad y'(\infty) = 1 \end{aligned} \tag{34}$$

The solution of (34) does not have a closed form. For comparison of the numerical results obtained, the approach in [9] and [12] are followed.

Comparing the results obtained in the new method and those of FDM in [9] and BT in [12], it agrees with values obtained for $y''(0)$ at 6 decimal places for which that of FDM in [9] and the values obtained for $y''(x_\infty)$ at the truncated boundary x_∞ at 5 decimal places with that of FDM in [9]. The Number of steps needed in the new method was only 15 to get the required values at the truncated boundary, for [9], the number of steps needed was 20 and for [12], the number of steps needed was 21.

5. Conclusion

An optimized three-step with three intra-points hybrid block method is developed in this work and applied directly to solve third order initial and boundary value problems in ordinary differential equations. The characteristics of the new method viz-a-viz, zero stability, consistency and convergence were established. Standard numerical examples in literature were used to show the efficiency in terms of the techniques, accuracy in terms of the errors obtained, of the derived method when compared to other methods. From the results obtained as seen in the tables, it can be concluded that the proposed method compare favourably and superior to existing methods in the literature cited.

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