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ROBUST EXTENDED TRAPEZOIDAL RULES FOR TWO-POINT STIFF AND NON-STIFF BOUNDARY VALUE PROBLEMS

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ABSTRACT. A family of boundary value methods (BVMs) referred to as robust extended trapezoidal rules (RETRs) is derived using the Taylor series expansion approach. The class of methods developed is symmetric with higher order and smaller error constants compared with the conventional extended trapezoidal rules (ETRs). The BVMs are natural candidates for the solution of boundary value problems (BVPs) and they simultaneously generate the approximate solutions to BVPs on the entire interval of integration. We applied the RETRs to standard non-stiff, stiff and/or singularly nonlinear perturbed two-point BVPs to analyze the efficiency and accuracy of the scheme and it was found to compare favorably with standard methods.

Keywords and phrases: Boundary value methods, ODEs, Boundary value problems, linear multistep formula, Stiff/Singularly Perturbed BVPs, Symmetric BVMs
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1. INTRODUCTION

Ordinary differential equations are usually classified into Initial Value Problems (IVPs)

$$y'(x) = f\left(x, y(x)\right), \quad y\left(a\right) = y_0, \quad x \in \left[a, b\right], \tag{1}$$

and Boundary Value Problems (BVPs)

$$y'\left(x\right) = f\left(x, y\left(x\right)\right), \quad x \in \left(a, b\right), \quad g\left(y\left(a\right), \ y\left(b\right)\right) = 0, \tag{2}$$

where $y, f, g \in R^n$, based on the subsidiary conditions that accompany these problems [7]. The latter class of problems is more difficult to handle, since it is a broader class of continuous problems unlike the former and they are usually solved by the Shooting

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Method (SHM) [1]. However, this SHM works by first reducing the BVP to its equivalent system of IVPs which makes it suffer from some numerical instability in the process of conversion [18]. Although, the advent of numerical schemes has done a great deal to proffering solutions to two-point boundary value problems, the number of schemes that can handle stiff and/or singularly perturbed nonlinear BVPs are not nearly plentiful. Singularly perturbed problems which are characterized by differential equations where the highest derivatives are multiplied by small parameters are an important subclass of stiff problems. These problems must often be solved numerically. However, because they typically feature boundary layers (narrow intervals where the solution varies rapidly) their accurate numerical solutions have been far from trivial [15]. Hence, the literature on singularly perturbed BVPs is not adequate [25]. Amongst other numerous methods proposed for the solution of linear and non-linear BVPs that may be stiff or non-stiff, the BVMs have been proven to be well suited ([1], [3], [4], [6], [8]-[10]). This is because the process of developing and applying these schemes makes it suitable for solving the BVPs directly without necessarily converting them to their equivalent system of IVPs [1]. Since the BVMs consist in approximating a continuous initial value problem by means of a discrete boundary value one, they are natural candidates to approximate continuous boundary value problems, with some slight modification [9]. The stability and convergence properties of the BVMs have been fully discussed and readily available in [10]. According to Brugnano and Trigiante [9], a BVM with $(k_1, k_2)-$boundary conditions can be used to approximate the solution of the BVP provided that the root within the unit circle of the characteristic polynomial associated with the difference equation is the principal root of the method whose order is ≥ 1. From their findings, it was established that the Symmetric BVMs were the most favorite natural candidate to approximate continuous BVPs (see [9],[10]).

In this paper, we have derived a class of Symmetric BVMs and given a general framework on how to use the BVMs on systems of BVPs in ODEs of the form (2). The applicability of BVMs to the approximation of BVPs was first considered by [9]. The essence of high order and relatively good stability properties to achieve considerably accurate approximate solutions to differential problems cannot be overemphasized (see, [3], [9], [13] and [29]). The methods developed herein are not only symmetric but are also of higher

order up-to $k+3$ for values of the step length k compared with the conventional ETRs which are only of order $k+1$. The boundary value technique we considered follows after the approach of ([10], [22]-[24]) such that the approximate solutions to the exact solutions of the BVP are simultaneously generated on the entire interval of integration. Hence, a reduction in global error and computational time is achieved. The proposed BVMs were developed using the Taylor series expansion technique (see,[28]).
The paper is organized as follows: in Section 2, we present the theoretical procedure which involves the general structure of BVMs and the framework for the derivation of our method. Derivation and the analysis of the RETRs are presented in Section 3. The use of the method is given in Section 4 while in Section 5, some numerical experiments are considered. In Section 6, we give the conclusion of the paper.

2. THEORETICAL PROCEDURES

The continuous initial value problem (1), which corresponds to the BVP (2) is usually solved by means of a discrete initial value problem, that is, a set of k initial conditions $y_0,\ y_1,\dots,y_{k-1}$ is associated with the linear multistep formula (LMF):

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h\sum_{j=0}^{k}\beta_j f_{n+j} \tag{3}$$

whereby approximation at x_n is only obtained when the previous x_{n-1} has been computed. That is,

$$D: a = x_0 < x_1 < x_2 < \ldots < x_N = b;\quad x_n = x_{n-1}+h, n = 1,2,...,N$$

where h is the constant step size of the partition of D, N is a partition integer and n is the grid index. If k_1 and k_2 are two integers such that $k_1+\ k_2 = k$ then one may impose the k conditions for the LMF (3) by fixing the first $k_1 \le\ k$ values of the discrete solution $y_0,\ y_1,\dots,y_{k_1-1}$ and the last $k_2\ \equiv\ k-\ k\ _1$ values $y_{N-k_2+1},\ .\ .\ .\ ,\ y_N$ so that the discrete problem becomes:

$$\sum_{j=-k_1}^{k_2}\alpha_{j+k_1}y_{n+j} = h\sum_{j=-k_1}^{k_2}\beta_{j+k_1}f_{n+j},\qquad n = k_1,\dots,N-k_2,$$

$$y_0,\ y_1,\dots,y_{k_1-1},\quad y_{N-k_2+1}\ ,\dots,\ y_N, fixed \tag{4}$$

Thus, the given continuous initial value problem is approximated by means of a discrete boundary value problem. The methods obtained in this way are called Boundary Value Methods (BVMs) with $(k_1,\ k_2)$-boundary conditions [11]. Consider that the continuous problem (1) only provides the initial solution y_0. The remaining $y_1, \ldots, y_{k_1-1}$, initial values and $y_{N-k_2+1}, \ .\ .\ .\ , y_N$ final values need to be found by introducing a set of $k-1$ additional equations which are derived by a set of $k_1 - 1$ additional initial methods.

$$\sum_{j=-k_1}^{k_2} \alpha_j^{(i)} y_j = h \sum_{j=-k_1}^{k_2} \beta_j^{(i)} f_j, \qquad i = 1, \ldots, k_1 - 1 \tag{5}$$

and k_2 final methods

$$\sum_{j=-k_1}^{k_2} \alpha_{K-j}^{(i)} y_{N-j} = h \sum_{j=-k_1}^{k_2} \beta_{k-j}^{(i)} f_{N-j}, \qquad i = N - k_2 + 1, \ldots, N \tag{6}$$

Clearly, the equations (4), (5) and (6) form a composite scheme of the same order. Observe that the ideology on which BVMs are built makes them suitable for direct solution of BVPs without first converting to IVP and their application to BVPs is straight forward (see, [1], [6], [9] and [10]). Therefore, consider the test two-point BVP:

$$y' = \lambda y, \qquad b_1 y\,(0) + b_2 y\,(T) = \eta \tag{7}$$

whose solution is $y\,(t) = e^{\lambda t} \frac{\eta}{b_1 + b_2 e^{\lambda T}}$, where $Re\,(\lambda)$ is either positive or negative due to dichotomy in general well-conditioned BVP ([5], [20]). The approximation provided by (3) is given as:

$$\sum_{j=0}^{k} (\alpha_i - q\beta_i) y_{n+i} = 0, \quad n = 0,\ \ldots\ ,\ N - k_1 - 1 \tag{8}$$

where as usual $q = h\lambda$ and the step-size is $h = T/(N + k_2 - 1), \quad k_1 + k_2 = \ k$. Then the k conditions needed by the discrete scheme are now giving by fixing the following $k - 1$ values of the discrete solution $\ y_1, \ldots, y_{k_1-1}, \quad y_N, \ldots,\ y_{N+k_2-1}$, while the remaining condition is obtained by the continuous problem $b_1 y_0 + b_2 y_{N+k_2-1} = \eta$. In analogy with the case of the methods for IVPs, the scheme (8) is used with $(k_1,\ k_2)$ $-$boundary conditions [9].

2.1 STABILITY OF BVMs

The stability of BVMs is characterized by two kinds of polynomials; $S_{k_1 k_2}$ and $N_{k_1 k_2}$.

Definition 1: A polynomial $p(\mathcal{Z})$ of degree $k = k_1 + k_2$ is an $S_{k_1 k_2}$-polynomial if its roots are such that

$$|\mathcal{Z}_1| \leq |\mathcal{Z}_2| \leq \cdots \leq |\mathcal{Z}_{k_1}| < 1 < |\mathcal{Z}_{k_1+1}| \leq \ldots \leq | \mathcal{Z}_k |,$$

whereas it is an $N_{k_1 k_2}$-polynomial if

$$|\mathcal{Z}_1| \leq |\mathcal{Z}_2| \leq \cdots \leq |\mathcal{Z}_{k_1}| \leq 1 < |\mathcal{Z}_{k_1+1}| \leq \ldots \leq | \mathcal{Z}_k |$$

with simple roots of unit modulus.
Let $\rho(\mathcal{Z})= \sum_{j=0}^{k} \alpha_j \mathcal{Z}^j$ and $\sigma(\mathcal{Z}) = \sum_{j=0}^{k} \beta_j \mathcal{Z}^j$ denote the two characteristic polynomials associated with the LMF (3). Thus $\pi(\mathcal{Z}, q) = \rho(\mathcal{Z}) - q\sigma(\mathcal{Z})$, is the stability polynomial when (3) is applied on the test problem (7). By using the usual definitions for the polynomials $\rho(\mathcal{Z})$ and $\sigma(\mathcal{Z})$ associated with a given LMF, the definitions for BVMs follows (see, [10], [11]).
Definition 2: A BVM with $(k_1,\ k_2)-$ boundary conditions is $O_{k_1 k_2}$-stable if the corresponding polynomial, $\rho(\mathcal{Z})$, is an $N_{k_1 k_2}$-polynomial.
Definition 3: A k-step BVM with k_1 initial conditions and $k_2 = k - k_1$ final conditions is said to be $(k_1,\ k_2)-$ absolutely stable for a given complex number q if the polynomial, $\pi(\mathcal{Z}, q)$, is of type $(k_1, 0,\ k_2)$, i.e. it is an $S_{k_1 k_2}-$ polynomial. Similarly, one defines the region of $(k_1,\ k_2)-$absolute stability of the method as follows:
$\mathcal{D}_{k_1 k_2}= \{\ q \in \mathbb{C} : \pi(\mathcal{Z}, q) \text{ is of type } (k_1, 0,\ k_2)\}$.
Finally, a BVM with $(k_1,\ k_2)-$ boundary conditions is said to be $A_{k_1 k_2}-$stable if $\mathbb{C}^- \subseteq \mathcal{D}_{k_1 k_2}$ and perfectly $A_{k_1 k_2}-$stable if $\mathbb{C}^- \equiv \mathcal{D}_{k_1 k_2}$.
It is observed that for BVMs there are no barriers concerning the maximum order for methods which are $0_{k_1 k_2}$-stable and /or $A_{k_1 k_2}$ $-$stable (see, [10], [21]-[24]).

3. DERIVATION AND THE ANALYSIS OF THE RETRs

Consider the two point BVP,

$$y'(x) = f(x, y(x)), \quad x \in (a, b), \tag{9}$$

$$g_a(y(a)) = y_0, \quad g_b(y(b)) = y_N$$

where, $g_a \in R^s$ and $g_b \in R^{n-s}$ for some value s, such that $1 < s < n$ and where each of the vector functions g_a *and* g_b are independent. The general k-step LMF for solving (9) is of the form (3), where $y_{n+j} \approx y(x_n + jh)$ and $f_{n+j} \equiv f(x_n + jh,\ \ y(x_n + jh))$. While, α_j and β_j are parameters to be determined. The conventional

ETRs by [9] are based on the LMF (3) and are obtained in the form;

$$y_n - y_{n-1} = +h \sum_{j=-v}^{v-1} \beta_{j+v} f_{n+j}, \qquad n = v, \ \ldots, \ N-1. \quad (10)$$

We observed that the conventional ETRs were developed considering the following choice of parameters, $\alpha_k = \alpha_{k-1} = 1$, $v \geq 1$ and $k = 2v-1$, hence, the scheme could only attain a maximum general order $k+1$ with $(v, v-1)-$boundary conditions. However, enforcing the following definition of the parameters, $\alpha_k = \alpha, \alpha_{k-1} = 1-\alpha$, $v \geq 2$ and $k = 2v-3$ on (10), we obtain a robust form of (10) with larger region of absolute stability, higher order $k+3$ and smaller error constants. The new methods are all symmetric and perfectly $A_{k_1 k_2}-$stable for all $k \equiv 2v-3$ with $(v, v-1)-$boundary conditions. Therefore, in other to get a distinctive view on the implication of the parameter choices, we rewrite (10) in terms of its conventional parameter definition and its new parameter definition for α, k and v to give the structures,

$$y_{n+v} - y_{n+v-1} = h \sum_{j=0}^{k} \beta_j f_{n+j}$$

$$v = \frac{k+1}{2} \quad , \quad k \geq 1 \ and \ k \ is \ Odd, \qquad (11)$$

$$\alpha \ y_{n+v} - (1-\alpha) y_{n+v-1} = h \sum_{j=0}^{k+2} \beta_j f_{n+j} \ ;$$

$$v = \frac{k+3}{2} \ , \ k \geq 1 \ and \ k \ is \ Odd. \qquad (12)$$

Equation (11) is referred to as the conventional ETRs by [9] and (12) is referred to as the proposed RETRs. Both schemes are offshoots of the general Adams Moulton methods:

$$y_{n+i} - y_{n+i-1} = h \sum_{j=0}^{k} \beta_j f_{n+j} \ ; \qquad k \geq 1 \ and \ i = k. \qquad (13)$$

Rewriting (12) in the form

$$\alpha \ y\,(x+vh) - (1-\alpha) y\,(x+(v-1)h) - h \sum_{j=0}^{k+2} \beta_j y'(x+jh) = 0$$

expanding in Taylor's series and applying the method of undetermined coefficient yields a system of linear equations. we obtained the coefficients of the methods as a solution of the resulting system of linear equations. In Table 1, we give the coefficients of the class of methods(12) for $k = 1(2)9$.

3.1 ORDER CONDITION OF THE RETRs

Following, [14], [17] and [19],we define the local truncation error (LTE) associated with (12) as the linear difference operator $\mathcal{L}\left[y\left(x\right);h\right]$ such that;

$$\begin{aligned}\mathcal{L}\left[y\left(x\right);h\right] = \quad & \alpha\, y\left(x+vh\right) - (1-\alpha)y\left(x+(v-1)h\right) \\ & -h\sum_{j=0}^{k+2}\beta_j y'(x+jh) \qquad (14)\end{aligned}$$

Assuming that $y(x)$ is sufficiently differentiable, we can find the Taylor series expansion of the terms in (12) about the point x to give

$$\mathcal{L}\left(y\left(x\right);h\right) = C_0 y\left(x\right) + C_1 h y'\left(x\right) + C_2 h^2 y''\left(x\right) + \cdots + C_p h^p y^p\left(x\right) + \ldots$$

where

$$C_r = \frac{1}{r!}\left(-\left(1-\alpha\right)\left(v-1\right)^r + \alpha v^r\right) - \frac{1}{(r-1)!}\sum_{j=0}^{k+2} j^{r-1}\beta_j \qquad (15)$$

Such that, $0 \leq r \leq p$
The class of methods (12) is of order p if,$C_0 = C_1 = C_2 = \cdots = C_p = 0$ and $C_{p+1} \neq 0$, where C_{p+1} is the error constant (EC) of the method (12) and $C_{p+1}h^{p+1}y^{p+1}\left(x\right)$ is the principal LTE at the point x. The numerical scheme is consistent if $p \geq 1$, [17]. In line with [10], the coefficients of (12) satisfies the following relations:

$$\alpha_i = -\alpha_{k-i} \textit{ and } \beta_i = \beta_{k-i} \textit{ such that } i = 0, \ldots, k \equiv 2v-3 \qquad (16)$$

and are therefore skew symmetric and symmetric respectively.

3.2 STABILITY OF THE RETRs

In line with ([10],[11], [16]), we analyze the stability of (12) by applying it to the usual test problem

$$y' = \lambda y$$

to obtain the characteristic equation

$$-\ (1-\alpha)\,\mathcal{Z}^{v-1}+\alpha\mathcal{Z}^{v}-\sum_{j=0}^{k+2}q\beta_j\mathcal{Z}^{j}=0,\quad q=\lambda h,\quad q\in\mathbb{C}\qquad(17)$$

where $\mathcal{Z} = e^{i\theta}$, $i = 0\,(1)\,k$, $\theta \ \epsilon\ [0, 2\pi]$. Plotting the resulting equation via the boundary loci approach [10] describes the stability domain of the class of methods (12) (see Fig. 1; the shaded portion is the stability region).

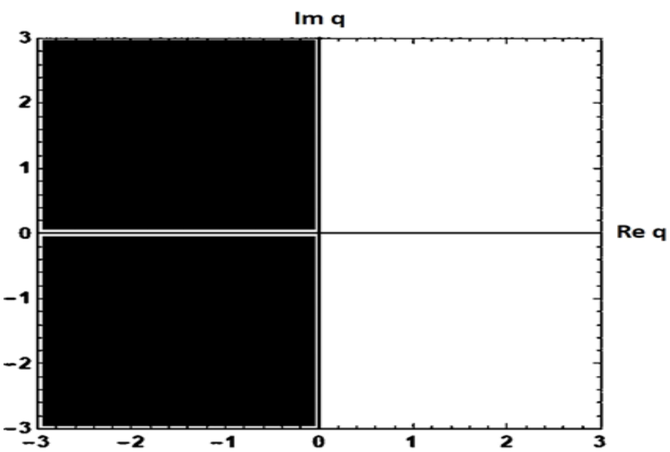

Fig. 1. Boundary Loci of the RETRs

In accordance with [9], the natural candidates to approximate continuous BVPs are symmetric BVMs. We observed that the boundary loci of the RETRs coincide with the imaginary axis and are therefore perfectly $A_{k_1k_2}-$stable (where, $k_1 = v$; representing the number of roots inside the unit circle and $k_2 = v-1$; representing the number of roots outside the unit circle) for all $k \equiv 2v-3$ and thus must be used with $(v, v-1)-$ boundary conditions. In what follows we give the coefficient list of the RETRs for the first five values of k in Table 1, where α is obtained throughout as $1/2$ while in Table 2, we compared the new scheme (RETRs) with the conventional one (ETRs) in terms of their order p and error constants (EC).

Table 1. Coefficient List of the RETRs for $k = 1(2)9$

k	**v**	β_0	β_1	β_2	β_3	β_4	β_5
1	2	$-\frac{1}{48}$	$\frac{13}{48}$				
3	3	$\frac{11}{2880}$	$-\frac{31}{960}$	$\frac{401}{1440}$			
5	4	$-\frac{191}{241920}$	$\frac{1879}{241920}$	$-\frac{353}{8960}$	$\frac{68323}{241920}$		
7	5	$\frac{2497}{14515200}$	$\frac{-28939}{14515200}$	$\frac{581}{51840}$	$-\frac{40111}{907200}$	$\frac{2067169}{7257600}$	
9	6	$\frac{-14797}{383201280}$	$\frac{331823}{638668800}$	$\frac{-6409423}{1916006400}$	$\frac{600503}{42577920}$	$\frac{-15228847}{319334400}$	$\frac{91473331}{319334400}$

Table 2. Comparison of ETRs and RETRs in terms of Order p and error constant (EC)

ETR					**RETR**			
k	**v**	**p**	**E***C*		**k**	**v**	**p**	**E***C*
1	1	2	$-\frac{1}{12}$		1	2	4	$\frac{11}{1440}$
3	2	4	$\frac{11}{720}$		3	3	6	$-\frac{191}{120960}$
5	3	6	$-\frac{191}{60480}$		5	4	8	$\frac{2497}{7257600}$
7	4	8	$\frac{2497}{3628800}$		7	5	10	$-\frac{14797}{191600640}$
9	5	10	$-\frac{14797}{95800320}$		9	6	12	$\frac{92427157}{5230697472000}$

4. USE OF METHOD

In line with the theoretical procedures, the main methods, RETRs (12) are used with the following set of additional methods,
Initial methods;

$$\alpha\ y_i - (1-\alpha)\, y_{i-1} = h \sum_{j=0}^{k+2} {\beta_j}^{(i)} f_j\ ; \qquad i = 1, \ldots, v-1 \tag{18}$$

and final methods;

$$\alpha\, y_{N+i} - (1-\alpha)\, y_{N+i-1} = h \sum_{j=0}^{k+2} {\beta_j}^{(i)} f_{N+j}\ ; \quad i = v+1, \ldots, N \tag{19}$$

Clearly, the main of methods (12) together with the complementary methods (18) and (19) invariably complete the boundary value methods of the same order $k+3$. In each case ($k = 1, 3, \ldots$), the class of methods (12) must be coupled with $v-1$ number of initial methods and $v-1$ number of final methods inline with the theoretical procedures (Section 2). Thus, we have the following examples,

Example 1: The fourth Order RETR (RETR1) given as:

$$\frac{1}{2}(y_{n+2} - y_{n+1}) = \frac{h}{48}\left(-f_n + 13f_{n+1} + 13f_{n+2} - f_{n+3}\right) \tag{20}$$

is used with the following additional initial method,

$$\frac{1}{2}(y_1 - y_0) = \frac{h}{48}(9f_0 + 19f_1 - 5f_2 + f_3)$$

and final method,

$$\frac{1}{2}(y_{N+3} - y_{N+2}) = \frac{1}{48}(f_N - 5f_{N+1} + 19f_{N+2} + 9f_{N+3})$$

Example 2: The Sixth Order RETR (RETR3) given as:

$$\begin{aligned}\frac{1}{2}(y_{n+3} - y_{n+2}) = \frac{h}{2880}(&11f_n - 93f_{n+1} + 802f_{n+2}\\ &+802f_{n+3} - 93f_{n+4} + 11f_{n+5})\end{aligned} \tag{21}$$

is used with the following additional initial methods,

$$\begin{aligned}\frac{1}{2}(y_1 - y_0) = \frac{h}{2880}(&475f_0 + 1427f_1 - 798f_2 + 482f_3\\ &-173f_4 + 27f_5),\end{aligned}$$

$$\begin{aligned}\frac{1}{2}(y_2 - y_1) = \frac{h}{2880}(&-27f_0 + 637f_1 + 1022f_2\\ &-258f_3 + 77f_4 - 11f_5)\end{aligned}$$

and final methods,

$$\begin{aligned}\frac{1}{2}(y_{N+4} - y_{N+3}) = \frac{h}{2880}(&-11f_N + 77f_{N+1} - 258f_{N+1}\\ &+1022f_{N+3} + 637f_{N+4} - 27f_{N+5})\end{aligned}$$

$$\begin{aligned}\frac{1}{2}(y_{N+5} - y_{N+4}) = \frac{h}{2880}(&27f_N - 173f_{N+1} + 482f_{N+2}\\ &-798f_{N+3} + 1427f_{N+4} + 475f_{N+5})\end{aligned}$$

The methods (20) and (21) are implemented as BVMs efficiently by combining them with their respective additional methods as simultaneous numerical integrators for the solution of the specified BVP. Specifically, the main methods and the additional methods are combined as BVMs to give a single matrix of finite difference equations which simultaneously provides the values of the solutions.

5. NUMERICAL EXPERIMENTS

In this section, the performance of the RETRs is examined on some non-linear stiff/singularly perturbed and non-stiff two-point

BVPs. Comparisons are made with other standard methods in terms of number of integration points ($N = \frac{b-a}{h}$), rate of convergence ($ROC = log_2 \left(\frac{e^{2h}}{e^h}\right)$, where e^h is the maximum absolute error for h) and maximum absolute error (Max$||y_i - y\left(x_i\right)||$). Higher-order ODEs are converted to the first-order form (2) given any scalar differential equation,

$$u^m = f\left(x, u, u', \ldots, u^{m-1}\right), \qquad a < x < b \tag{22}$$

let $y\left(x\right) = \left(y_1\left(x\right), \ y_2\left(x\right), \ \ldots, \ y_m\left(x\right)\right)^T$ be defined by

$$\begin{aligned} y_1\left(x\right) &= u(x) \\ y_2\left(x\right) &= u'(x) \\ &\vdots \\ y_m\left(x\right) &= u^{(m-1)}(x) \end{aligned} \tag{23}$$

then the ODE can be converted to the equivalent first-order form

$$\begin{aligned} y_1 &= y \\ y_1' &= y_2 \\ &\vdots \\ y_{m-1}' &= y_m \\ y_m' &= f(x, y_1, y_2, \ldots, y_m) \end{aligned} \tag{24}$$

Problem 1: Consider the non-linear stiff BVP due to Troesch ([2], [9] and [27])

$$\begin{aligned} y'' &= \lambda \text{ Sinh}\left(\lambda \text{y}\right), \qquad && 0 < x < 1 \\ y\left(0\right) &= 0, && y\left(1\right) = 1 \end{aligned}$$

The solution of the problem has a boundary layer near $x = 1$. The linearized exact solution provided for the problem by [27] is given as $y\left(x\right) = \frac{\sinh(\lambda x)}{\sinh(\lambda)}$.
We solve this problem using the RETR of order $p = 4$ (RETR1). In Table 3, our results are compared with those in [9]. It is observed that the fourth order RETR performs better than the conventional ETR and TOM of orders 4 and 6 respectively in terms of accuracy. From the computed ROC, we observe that the fourth order RETR experiences a high rate of convergence. We also note that the problem becomes strongly stiff and more difficult to solve as λ

increases ([2], [9] and [27]). In Fig. 2, we give the solution plot of the problem for $\lambda = 5,\ 7,\ 9\ and\ 20$.

Table 3. Comparison of results for Problem 1, $\lambda = 5$, in terms of maximum absolute error.

h	ETR $k=3,\ p=4$	ROC	TOM $k=2,\ p=6$	ROC	RETR1 $k=1,\ p=4$	ROC
0.1	$1.8051e-01$		$9.0884e-02$		$1.1791e-03$	
0.05	$3.2912e-02$	2.46	$1.4653e-02$	2.63	$6.7284e-06$	7.45
0.025	$5.3195e-03$	2.63	$1.7345e-03$	3.08	$1.3588e-07$	5.63
0.0125	$6.8539e-04$	2.96	$1.3131e-04$	3.72	$3.8089e-09$	5.16
0.00625	$6.9570e-05$	3.30	$6.1618e-06$	4.41	$1.1584e-10$	5.04
0.003125	$5.8186e-06$	3.58	$1.9127e-07$	5.01	$3.5958e-12$	5.01
0.0015625	$4.2736e-07$	3.77	$4.4208e-09$	5.44	$1.1218e-13$	5.00

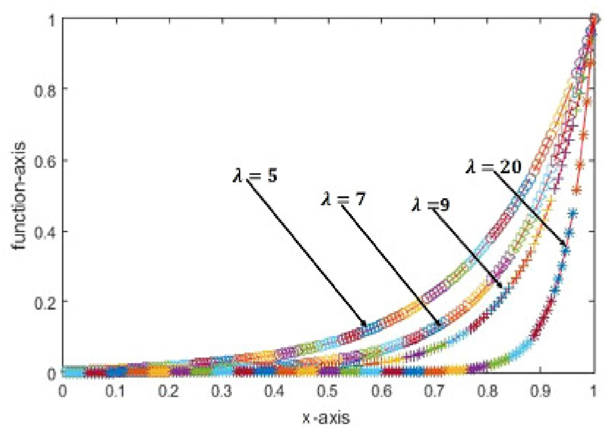

Fig. 2. Solution of Problem 1, using RETR order 4 with $N = 150$

Problem 2: Consider the non-linear perturbed BVP giving in ([12]: Test Problem 20).

$$\lambda \frac{d^2y}{dx^2} = -\left(\frac{dy}{dx}\right)^2 + 1, \qquad 0 < x < 1$$

$$y(0) = 1 + \lambda \log\left(\cosh\left(-\frac{0.745}{\lambda}\right)\right),$$

$$y(1) = 1 + \lambda \log\left(\cosh\left(\frac{0.255}{\lambda}\right)\right)$$

$$Exact : y(x) = 1 + \lambda \log\left(\cosh\left(x - \frac{0.745}{\lambda}\right)\right).$$

The solution of the problem has a corner layer at $x = 0.745$. The maximum absolute error is obtained for Problem 2 using RETR3 within the range of integration. In Table 4 and Table 5, we compare our solution with those of the ETR of order 6 given in [9] and the BVM4 of order 6 given in [6] for $\lambda = 1$ *and* 0.1 respectively. We observe from the numerical results in Table 4 and Table 5 that the RETR of order 6 is more accurate compared with the ETR and BVM4. In Fig. 3, we give the plot of the solution to Problem 2 using RETR3 with $N = 100$, $\lambda = 1,\ 0.5,\ 0.1,\ 0.05$.

Table 4. Comparison of results for Problem 2, $\lambda = 1$, in terms of maximum absolute error.

N	ETR	ROC	BVM4	ROC	RETR3	ROC
20	$2.235e$-09		$1.664e$-09		$9.673e$-10	
40	$3.370e$-11	5.96	$2.823e$-11	5.88	$3.419e$-12	8.14
80	$5.607e$-13	5.99	$4.370e$-13	6.01	$1.310e$-14	8.03
160	$9.104e$-15	5.94	$6.883e$-15	5.99	0.0000	∞

Table 5. Comparison of results for Problem 2, $\lambda = 0.1$, in terms of maximum absolute error.

N	ETR	ROC	BVM4	ROC	RETR3	ROC
20	$7.808e$-05		$2.000e$-04		$1.712e$-06	
40	$1.838e$-06	5.41	$4.090e$-06	5.61	$2.245e$-11	16.2
80	$3.163e$-08	5.86	$5.784e$-08	6.14	$2.742e$-14	9.68
160	$5.176e$-10	5.93	$7.066e$-10	6.36	$6.661e$-16	5.36

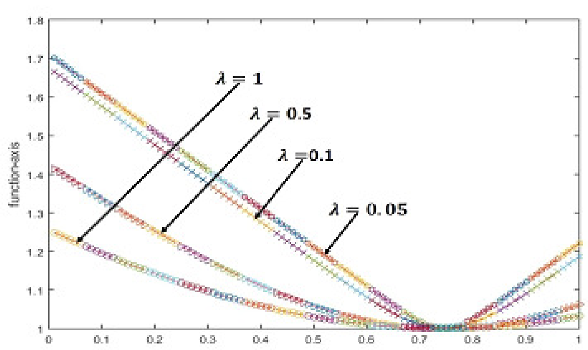

Fig. 3. Solution of Problem 2 with RETR of order 6 with $N = 100$

Problem 3: Consider the non-linear BVP given in [26].

$$y'' = \frac{(y')^2 + y^2}{2e^x}, \qquad 0 < x < 1$$
$$y(0) - y'(0) = 0, y(1) + y'(1) = 2e$$
$$Exact : y(x) = e^x$$

We solve this problem using RETR3. In Table 6, we compare our solution in terms of the maximum absolute error with those of the ETR, BVM4 and HyBVM of the same order 6 given in [9], [6] and [1] respectively. The solution plot of the problem is obtained at $N = 40$ using RETR3 and reported in Fig. 4. It can be observed from Table 6 that the RETR3 performs better than the other standard schemes considered.

Table 6. Comparison in terms of maximum absolute error for Problem 3.

N	ETR	ROC	BVM4	ROC	HyBVM	ROC	RETR3	ROC
20	2.476e-10		1.505e-10		1.592e-11		2.805e-11	
40	6.019e-12	5.36	2.347e-12	6.00	2.495e-13	6.00	1.850e-13	7.24
80	6.402e-14	6.55	3.642e-14	6.01	4.082e-15	5.93	1.332e-15	7.12
160	1.010e-15	5.99	6.661e-16	5.77	6.280e-16	2.70	0.000	∞

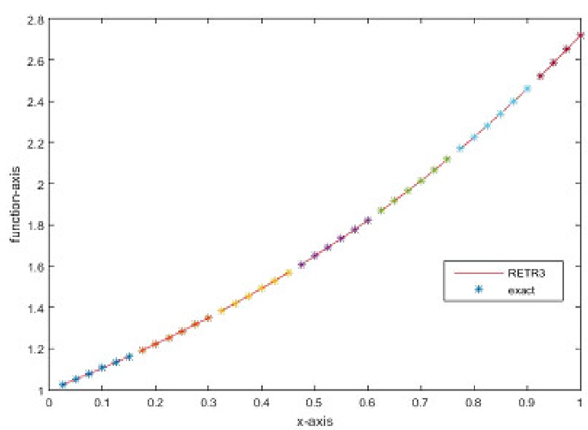

Fig. 4. Solution of Problem 3, using RETR order 6 with $N = 40$

Problem 4: Consider the non-linear perturbed BVP giving in ([12]: Test Problem 21).

$$\epsilon y'' = y + y^2 - e^{\left(-\frac{2x}{\sqrt{\epsilon}}\right)}$$

$$y\left(0\right)=1,\quad y\left(1\right)=e^{\left(-1/\sqrt{\epsilon}\right)}$$

$$Exact : y\left(x\right)=e^{\left(-\frac{x}{\sqrt{\epsilon}}\right)}$$

The solution has a boundary layer of width $O\left(\sqrt{\epsilon}\right)$ at $x=0$. Problem 4 is solved using RETR1 and RETR3 for $\epsilon=0.2$, 0.1, 0.01 and 0.005. The maximum absolute error obtained using the methods, RETR1 and RETR3 are computed for $\epsilon=0.1$ and reported in Table 7 to show the consistency of the class of methods as k increases. The solutions of the problem for $\epsilon=0.2$, 0.1, 0.01 and 0.005 are given in Fig. 5 using RETR3 with $N=150$.

Table 7. Comparison of methods for Problem 4, $\epsilon=0.1$, in terms of maximum absolute error.

N	RETR1	ROC	RETR3	ROC
4	$8.312e-02$		$9.642e-03$	
8	$4.151e-04$	7.65	$3.637e-05$	8.05
16	$2.208e-06$	7.55	$1.866e-07$	7.61
32	$2.039e-08$	6.76	$1.579e-09$	6.88
64	$1.641e-09$	3.64	$1.449e-11$	6.77

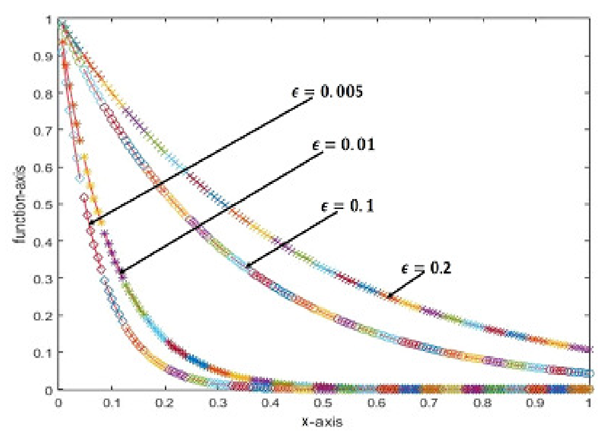

Fig. 5. Solution of Problem 4 with RETR order 6 with $N=150$

6. CONCLUDING REMARKS

Although a great deal has been done to proffer numerical solutions to two-point BVPs, the schemes that can handle stiff and/or singularly perturbed nonlinear BVPs are not nearly plentiful. The applicability and suitability of BVMs on BVPs cannot be overemphasized. The RETRs which are modified extension of the conventional ETRs have not only shown good theoretical potentials over the conventional ETRs, being of higher order with smaller error constants but have also shown practical superiority on the problems considered. The efficiency and accuracy of the RETRs on standard stiff and/or singularly perturbed nonlinear BVPs have proven very promising and well placed amongst standard existing methods for the solution of two-point BVPs judging by the comparison made with standard methods and the solution plots. Hence, we would like to extend its application to multipoint BVPs in the future.

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