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MISES FLOW EQUATIONS FOR GRADIENT PLASTICITY WITH ISOTROPIC AND KINEMATIC HARDENING

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ABSTRACT. This work provides a framework for which strain gradient plasticity theories can be investigated and considered admissible in the sense of thermodynamic consistency and maximum plastic dissipation in juxtaposition to the classical theory. The classical plasticity theory is studied using the maximum plastic dissipation principle on the assumption that the plastic flow is associative. Furthermore, rate-independent strain gradient plasticity theories are investigated, and it is shown that these theories mimic the classical yield criterion, the Mises flow rule, codirectionality law and in addition can predict elastic region. The simple constrained shear problem is studied with a view to demonstrating the variance between some strain gradient theories in literature. It is shown that Aifantis' flow law differs from that of the classical only in a nonlocal term accompanying the Aifantis' flow rule which involves an energetic length scale; and as this length scale approaches zero, the Aifantis model approaches the classical theory.

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1. INTRODUCTION

Gradient plasticity deals with the study of plastic distortion gradient effect on deformed bodies at varying length scales. This subject-which is over three decades old- has become a considerable aspect of plasticity with keen interest in its theory and applications mainly due to its ability to track size-effects phenomena as observed in most metals undergoing inhomogeneous plastic flow (Ashby, 1970; Fleck and Hutchinson, 1994, 1997; Hutchinson, 2000). The classical theory of plasticity does not track size-effects because length scales are missing in the constitutive relations for the stresses, and

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by implication are absent in the flow rules of plastic materials. For instance, in the classical von Mises flow rule for rate independent materials, plastic deformation does not exhibit intrinsic material length scales and hence such a theory is unaware of size-effects phenomena. The gradient theory serves to bridge this gap by including within the constitutive relations and the flow rules terms involving gradient of plastic distortion and it rate (Willis, 2019; Gudmundson and Dahlberg, 2019). This implies that the gradient plasticity is an extension or generalization of the classical theory, so that the fundamental issues that confront research in classical plasticity apply- by extension- to the gradient theory. Some of the fundamental issues are; tracking the onset of plasticity and regime of strain hardening, harmonization of the elastic and plastic regimes etc. Within the classical theory, these issues have been considered over the years with reasonable decision via established flow rules, concepts such as hardening equations, yield surface, elastic region, boundedness, codirectionality and the no flow condition (Hill, 1948; 1987; Gurtin et al., 2010; Han and Reddy, 2013; Del Piero, 2018). The aforementioned issues addressed by the classical theory are limited to the rate-independent plastic flow, whereas the rate-dependent plastic materials cannot explain concept such as the elastic region because basic to rate-dependent theory is the assumption that deviatoric stress vanishes whenever there is no plastic flow. Still, the question arise: How does the rate-independent strain gradient plasticity accommodate these fundamental issues which include; appropriate flow rules, elastic region and yield surface?

This study aims to obtain Mises flow equations for rate-independent stain gradient plasticity with kinematic and isotropic hardening. This work will discuss the classical Mises flow equations accounting for kinematic hardening using the principle of maximum plastic dissipation. Also, we will obtain the Mises flow equations equivalent to the Aifantis' strain gradient plasticity (Aifantis, 1984, 1987) and the Gurtin-Anand rate-independent strain gradient plasticity (Gurtin and Anand, 2005; 2009; Reddy et al., 2008).

2. NOTATIONS

In component form, second-order tensor $\mathbf{A}$ is written as A_{ij} for $i, j = 1, 2, 3$. The expression $\mathbf{Au}$ is written in component form as $A_{ij}u_j$, where the summation convention is adopted. The trace of a second-order tensor $\mathbf{A}$ is denoted as tr$\mathbf{A}$ and written as the sum of the diagonal elements for the matrix of $\mathbf{A}$. The symmetric and skew parts of a second-order tensor $\mathbf{A}$ would be denoted by sym$\mathbf{A}$ and skw$\mathbf{A}$ respectively, and are defined as $\text{sym}\mathbf{A} = \frac{1}{2}(\mathbf{A}+\mathbf{A}^T)$, $\text{skw}\mathbf{A} = \frac{1}{2}(\mathbf{A}-\mathbf{A}^T)$, where $\mathbf{A}^T$ is the transpose of $\mathbf{A}$. The magnitude of $\mathbf{A}$ is denoted as $|\mathbf{A}|$, and defined as $|\mathbf{A}| = (A_{ij}A_{ij})^{1/2}$. We define the inner-product ':' of nonzero second-order tensors $\mathbf{A}$ and $\mathbf{B}$ by $\mathbf{A} : \mathbf{B} = A_{ij}B_{ij}$. Also, the inner product '$\vdots$' of the third order tensors $\mathbb{A}$ and $\mathbb{B}$ is denoted by $\mathbb{A}\vdots\mathbb{B}$ and defined as $\mathbb{A}\vdots\mathbb{B} = A_{ijk}B_{ijk}$. The deviatoric part of a second-order tensor $\mathbf{A}$ would be denoted as $\mathbf{A}_o$ or dev$\mathbf{A}$, and defined as $\mathbf{A}_o = \text{dev}\mathbf{A} = \mathbf{A} - \frac{1}{3}(\text{tr}\mathbf{A})\mathbf{I}$, where $\mathbf{I}$ is the second-order unit tensor. Given any nonzero vector $\mathbf{a}$, we define second-order tensor $(\mathbf{a}\times)$ in component form as $(\mathbf{a}\times)_{ij} = \epsilon_{ikj}a_k$, where ϵ_{ikj} is the permutation symbol.

The partial derivative with respect to the material coordinate X_i would be denoted as $(\cdot)_{,i}$, and defined as $(\cdot)_{,i} = \frac{\partial(\cdot)}{\partial X_i}$ for $i = 1, 2, 3$. We denote the gradient of a vector field $\mathbf{a}$ and tensor field $\mathbf{A}$ as $\nabla\mathbf{a}$ and $\nabla\mathbf{A}$ respectively, and these are defined in component forms as $(\nabla\mathbf{a})_{ij} = a_{i,j}$ and $(\nabla\mathbf{A})_{ijk} = A_{ij,k}$. The divergence of $\mathbf{a}$ and $\mathbf{A}$ are denoted by div$\mathbf{a}$ and div$\mathbf{A}$ respectively, and are defined in component forms as $\text{div}\mathbf{a} = a_{k,k}$ and $(\text{div}\mathbf{A})_i = A_{ik,k}$. The curl of a tensor $\mathbf{A}$ would be written as Curl$\mathbf{A}$ or $\nabla\times\mathbf{A}$ and defined in component form as $(\text{Curl}\mathbf{A})_{ij} = \epsilon_{ipq}A_{jq,p}$.

3. KINEMATIC RELATIONS

Let $\mathbf{u}$ be the displacement vector of an arbitrary point $\mathbf{X}$ at time t in a polycrystalline occupying a region of space Ω of the Euclidean space. Within the context of small deformation, the displacement gradient $\nabla\mathbf{u}(\mathbf{X}, t)$ admits the additive decomposition

$$\nabla\mathbf{u} = \mathbf{H}^e + \mathbf{H}^p, \text{ with } \text{tr}\mathbf{H}^p = 0 \tag{1}$$

into elastic distortion $\mathbf{H}^e$ and plastic distortion $\mathbf{H}^p$. Stretch and rotation of material structure are measured by the elastic distortion $\mathbf{H}^e$, whereas the irreversible distortion in material structure resulting from motion of dislocation are typically measured by the plastic distortion $\mathbf{H}^p$.

We will denote the elastic strain by $\mathbf{E}$ which admits additive decomposition

$$\mathbf{E} = \mathbf{E}^e + \mathbf{E}^p \tag{2}$$

into elastic strain $\mathbf{E}^e$ and plastic strain $\mathbf{E}^p$ defined as

$$\mathbf{E}^e = \text{sym}\mathbf{H}^e \text{ and } \mathbf{E}^p = \text{sym}\mathbf{H}^p, \tag{3}$$

At the outset, strain gradient plasticity theories ignore the skew-symmetric part of the plastic distortion, and the relevant basic kinematic field variables are; $\mathbf{u}, \mathbf{H}^e,$ and $\mathbf{E}^p$. Virtual power principle and interest in plastic flow allow the use of rate-like kinematic fields consistent with eq. (1) satisfying

$$\nabla\dot{\mathbf{u}} = \dot{\mathbf{H}}^e + \dot{\mathbf{E}}^p, \text{ with } \mathrm{tr}\dot{\mathbf{E}}^p = 0. \tag{4}$$

Here $\dot{\mathbf{E}}^p$ is called the plastic flow. To accommodate for the plastic strain gradient rate, we defined the generalized plastic strain E^p and its rate $\dot{\mathsf{E}}^p$ by

$$\mathsf{E}^p = (\mathbf{E}^p, l\nabla\mathbf{E}^p) \text{ and } \dot{\mathsf{E}}^p = \left(\dot{\mathbf{E}}^p, l\nabla\dot{\mathbf{E}}^p\right) \tag{5}$$

respectively. Where l is called the dissipative length scale associated with the generalized flow rate. We define the magnitude of the generalized plastic flow by

$$|\dot{\mathsf{E}}^p| = \sqrt{|\dot{\mathbf{E}}|^2 + l^2|\nabla\dot{\mathbf{E}}^p|^2}. \tag{6}$$

The most general hardening variable for rate-independent plastic flow in the absence of strain gradient is the accumulated plastic strain denoted by $e^p(\mathbf{X}, t)$ and it is define via the ordinary differential equation (Gurtin et al., 2010)

$$\dot{e}^p = |\dot{\mathbf{E}}^p| \text{ with } e^p(\mathbf{X}, 0) = 0. \tag{7}$$

Similarly, we define the accumulated generalized plastic strain $\mathfrak{e}^p(\mathbf{X}, t)$ via

$$\dot{\mathfrak{e}}^p = |\dot{\mathsf{E}}^p| \text{ with } \mathfrak{e}^p(\mathbf{X}, 0) = 0. \tag{8}$$

Define the flow direction $\mathbf{N}^p$ and generalized flow direction N^p by

$$\mathbf{N}^p = \frac{\dot{\mathbf{E}}^p}{|\dot{\mathbf{E}}^p|} \text{ for } \dot{\mathbf{E}}^p \neq \mathbf{0}, \text{ and} \tag{9}$$

$$\mathsf{N}^p = \frac{\dot{\mathsf{E}}^p}{|\dot{\mathsf{E}}^p|} \text{ for } \dot{\mathsf{E}}^p \neq \mathbf{0}. \tag{10}$$

4. THE CONVENTIONAL PLASTICITY WITH KINEMATIC HARDENING

Here, the plastic strain gradient and its rate are absent. Basic to what follows is the assumption that:

- The Cauchy stress $\mathbf{T}$ is power-conjugate to the elastic distortion rate $\dot{\mathbf{H}}^e$. This stress is also know as the macroscopic stress.

- The symmetric and deviatoric plastic stress $\mathbf{T}^p$ is power-conjugate to the plastic strain rate $\dot{\mathbf{E}}^p$. It is also known as the microscopic stress because it is associated with the dislocation in the material structure.

The use of virtual power principle establishes the local macroscopic and microscopic force balances in an arbitrary subregion Ω_1 of the body Ω following Gurtin et al. (2010) viz:

-
$$\mathrm{div}\mathbf{T} + \mathbf{b} = \mathbf{0} \text{ in } \Omega_1 \text{ and } \mathbf{Tn} = \mathbf{t}(\mathbf{n}) \text{ on } \partial\Omega_1, \tag{11}$$

-
$$\mathbf{T}_o = \mathbf{T}^p \text{ in } \Omega_1. \tag{12}$$

$\mathbf{b}$, $\mathbf{t}(\mathbf{n})$ and $\mathbf{n}$ are the body force in Ω_1, traction and outward unit normal on $\partial\Omega_1$ respectively. $\mathbf{T}_o$ is the deviatoric part of the Cauchy stress tensor.

5. ENERGY IMBALANCE, DISSIPATION AND CONSTITUTIVE RELATIONS OF THE CONVENTIONAL THEORY

Let ψ denote the free-energy per unit volume of an arbitrary portion Ω_1 of the body Ω. The free-energy imbalance has the form

$$\dot{\psi} - \mathbf{T} : \dot{\mathbf{E}} \leq 0. \tag{13}$$

If there is plastic flow, then using eqs. (2) and (12), the free-energy imbalance can be written as

$$\dot{\psi} - \mathbf{T} : \dot{\mathbf{E}}^e - \mathbf{T}^p : \dot{\mathbf{E}}^p \leq 0. \tag{14}$$

By convention and since elastic strain rate does not induce dissipation, the elastic stress is defined by

$$\mathbf{T} = \frac{\partial \psi}{\partial \mathbf{E}^e}. \tag{15}$$

In order to account for kinematic hardening, we assume that the free-energy admits the additive decomposition

$$\hat{\psi}(\mathbf{E}) = \hat{\psi}^e(\mathbf{E}^e) + \hat{\psi}^p(\mathbf{E}^p) \tag{16}$$

into elastic part ψ^e and plastic part ψ^p. Following eqs. (14) and (15), the free-energy reduces to the plastic free-energy imbalance

$$\dot{\psi}^p - \mathbf{T}^p : \dot{\mathbf{E}}^p \leq 0. \tag{17}$$

Assume also that the microscopic stress $\mathbf{T}^p$ admits additive decomposition into energetic and dissipative parts $\mathbf{T}^p_{en}$,$\mathbf{T}^p_{dis}$ respectively, in the sense: $\mathbf{T}^p = \mathbf{T}^p_{en} + \mathbf{T}^p_{dis}$, where the energetic microscopic stress is defined by

$$\hat{\mathbf{T}}^p_{en}(\mathbf{E}^p) = \frac{\partial \hat{\psi}^p(\mathbf{E}^p)}{\partial \mathbf{E}^p}. \tag{18}$$

A consequence of eqs. (17) and (18) is the reduced plastic dissipation $\delta(\dot{\mathbf{E}}^p, e^p)$ defined by

$$\delta(\dot{\mathbf{E}}^p, e^p) = \mathbf{T}^p_{dis} : \dot{\mathbf{E}}^p \geq 0. \tag{19}$$

Basic to what follows is the hypothesis of *strict dissipativity* which means that dissipation is positive whenever there is flow. By eq. (19), it implies that

$$\mathbf{T}^p_{dis} \neq \mathbf{0} \text{ whenever } \dot{\mathbf{E}}^p \neq \mathbf{0}. \tag{20}$$

The constitutive relation for the dissipative microscopic stress has the form

$$\mathbf{T}^p_{dis} = \hat{\mathbf{T}}^p_{dis}(\dot{\mathbf{E}}^p, e^p) \text{ for } \dot{\mathbf{E}}^p \neq \mathbf{0}, \tag{21}$$

where e^p has been included as constitutive independent variable to account for accumulation of the plastic strain. For *rate-independent processes*, the constitutive relation has the form

$$\mathbf{T}^p_{dis} = \hat{\mathbf{T}}^p_{dis}(\mathbf{N}^p, e^p) \text{ for } \dot{\mathbf{E}}^p \neq \mathbf{0}, \tag{22}$$

where $\mathbf{N}^p$ is the flow direction defined in eq. (9).

The *flow resistance* $Y(\mathbf{N}^p, e^p)$ is defined as dissipation per unit $|\dot{\mathbf{E}}^p|$

$$Y(\mathbf{N}^p, e^p) = \hat{\mathbf{T}}^p_{dis}(\mathbf{N}^p, e^p) : \mathbf{N}^p > 0. \tag{23}$$

The hypothesis of *strong isotropy* renders the flow resistance independent of the flow direction, and eq. (23) can be written as

$$Y(e^p) = \hat{\mathbf{T}}^p_{dis}(\mathbf{N}^p, e^p) : \mathbf{N}^p \text{ for } \dot{\mathbf{E}}^p \neq \mathbf{0}. \tag{24}$$

5.1. Maximum Plastic Dissipation Principle

From eqs. (12), (18) and (22) the constitutive equation for the deviatoric stress $\mathbf{T}_o$ has the form

$$\mathbf{T}_o = \hat{\mathbf{T}}^p_{en}(\mathbf{E}^p) + \hat{\mathbf{T}}^p_{dis}(\mathbf{N}^p, e^p), \text{ for } \dot{\mathbf{E}}^p \neq \mathbf{0}. \tag{25}$$

We say a stress $\mathbf{T}_o$ in the space $SymDev$ of symmetric deviatoric tensors is *associated with* the flow direction $\mathbf{N}^p$ if it satisfies eq. (25).

The principle of maximum plastic dissipation states:

Theorem 5.1.1[Maximum Plastic Dissipation] Given that $\mathbf{T}_o \in SymDev$ is associated with flow direction $\mathbf{N}^p$ then

$$\mathbf{T}^*_o : \mathbf{N}^p \leq \mathbf{T}_o : \mathbf{N}^p \text{ for all } \mathbf{T}^*_o \in SymDev. \tag{26}$$

By eqs. (25) and (26) the maximum plastic dissipation can be rewritten as

$$[\mathbf{T}^*_o - \hat{\mathbf{T}}^p_{en}(\mathbf{E}^p)] : \mathbf{N}^p \leq \hat{\mathbf{T}}^p_{dis}(\mathbf{N}^p, e^p) : \mathbf{N}^p \text{ for every } \mathbf{T}^*_o \in SymDev. \tag{27}$$

Eq. (27) can be written in terms of the flow resistance following eq. (24) as

$$[\mathbf{T}^*_o - \hat{\mathbf{T}}^p_{en}(\mathbf{E}^p)] : \mathbf{N}^p \leq Y(e^p) \text{ for every } \mathbf{T}^*_o \in SymDev. \tag{28}$$

If eq.(28) is satisfied, then $\mathbf{T}_o^*$ is said to admissible in the sense of maximum dissipation.

5.2 The Classical Mises Flow Rule

Theorem 5.2.1[Flow Rule] Given that $\mathbf{T}_o$ is associated with the flow direction $\mathbf{N}^p$ and admissible in the sense of maximum dissipation, then constitutive relation for $\mathbf{T}_o$ is defined by

$$\mathbf{T}_o - \hat{\mathbf{T}}_{en}^p(\mathbf{E}^p) = Y(e^p)\mathbf{N}^p, \tag{29}$$

where $\hat{\mathbf{T}}_{en}^p(\mathbf{E}^p)$ is the energetic plastic stress satisfying eq. (25)

Proof. Let $\mathfrak{N}$ be a unit sphere of symmetric deviatoric tensor so that $\mathfrak{N}$ defines a surface containing flow directions. Let $\mathbf{N}^p(\lambda)$ be a smooth curve defined on the surface $\mathfrak{N}$. It is assumed that the curve passes through a fixed but artbitrary point λ_o such that $\mathbf{N}^p(\lambda_o) = \mathbf{N}^p$ associated with $\mathbf{T}_o$. Then by maximum plastic dissipation principle we have

$$[\mathbf{T}_o - \hat{\mathbf{T}}_{en}^p(\mathbf{E}^p)] : \mathbf{N}^p(\lambda) \leq Y(e^p) \text{ for every } \mathbf{N}^p(\lambda) \in \mathfrak{N}. \tag{30}$$

Let $\mathbf{M}$ be an arbitrary symmetric deviatoric tensor tangent to the unit sphere $\mathfrak{N}$ such that

$$\mathbf{N}^p : \mathbf{M} = 0. \tag{31}$$

Define a function $\phi(\lambda)$ by

$$\phi(\lambda) = Y(e^p) - [\mathbf{T}_o - \hat{\mathbf{T}}_{en}^p(\mathbf{E}^p)] : \mathbf{N}^p(\lambda) \geq 0. \tag{32}$$

Then $\phi(\lambda)$ has a minimum at $\lambda = \lambda_o$ so that we have

$$[\mathbf{T}_o - \hat{\mathbf{T}}_{en}^p(\mathbf{E}^p)] : \left[\frac{d\mathbf{N}^p(\lambda)}{d\lambda}\right]_{\lambda=\lambda_o} = 0. \tag{33}$$

Since $\frac{d\mathbf{N}^p(\lambda)}{d\lambda}$ is arbitrary and a tangent to the unit sphere $\mathfrak{N}$, then there exist a scalar k such that

$$\mathbf{T}_o - \hat{\mathbf{T}}_{en}^p(\mathbf{E}^p) = k\mathbf{N}^p. \tag{34}$$

Furthermore, by eqs. (24) and (25) we have

$$\mathbf{T}_o - \hat{\mathbf{T}}_{en}^p(\mathbf{E}^p) = Y(e^p)\mathbf{N}^p. \tag{35}$$

□

5.3. Yield Condition, Elastic Region and Boundedness

Eq. (29) is called the classical Mises Flow rule. It is clear that whenever $\mathbf{T}_o$ is associated with $\dot{\mathbf{E}}^p \neq \mathbf{0}$ then we have have the ***yield condition***

$$|\mathbf{T}_o - \hat{\mathbf{T}}_{en}^p(\mathbf{E}^p)| = Y(e^p). \tag{36}$$

Elastic Region is defined as the set of $\mathbf{T}_o \in SymDev$ that are admissible in the sense of maximum dissipation. This implies that

$$|\mathbf{T}_o - \hat{\mathbf{T}}^p_{en}(\mathbf{E}^p)| \leq Y(e^p) \text{ for all } \dot{\mathbf{E}}^p \in SymDev. \tag{37}$$

This inequality is known as the boundedness property. The converse of this statement is true i.e. Boundedness also implies elastic region.

Whenever the inequality (37) is strict, then it implies there is no flow i.e. $\dot{\mathbf{E}}^p = \mathbf{0}$.

6. MISES FLOW EQUATIONS FOR STRAIN GRADIENT PLASTICITY

6.1. Burgers Tensor and its Rate

In the absence of plastic rotation, the Burgers tensor and its rate are defined by

$$\mathbf{G} = \nabla \times \mathbf{E}^p, \tag{38}$$

$$\dot{\mathbf{G}} = (\nabla \dot{e}^p \times)\mathbf{N}^p + (\nabla \times \mathbf{N}^p)\dot{e}^p. \tag{39}$$

The microscopic force balance of the Gurtin-Anand model is given by (Gurtin and Anand, 2005)

$$\mathbf{T}_o = \mathbf{T}^p - \text{div}\mathbb{K}^p \text{ with } \mathbb{K}^p\mathbf{n} = \mathbf{K}(\mathbf{n}), \tag{40}$$

where $\mathbb{K}^p$ is the polar microscopic stress, power-conjugate to the gradient of plastic strain rate $\nabla\dot{\mathbf{E}}^p$, and $\mathbf{K}(\mathbf{n})$ is the microtraction, power-conjugate to the plastic strain rate $\dot{\mathbf{E}}^p$.

6.2. Free Energy Imbalance, Codirectionality and Equivalent Microscopic force balance

The free energy ψ satisfies the inequality

$$\dot{\psi} - \mathbf{T} : \dot{\mathbf{E}}^e - \mathbf{T}^p : \dot{\mathbf{E}}^p - \mathbb{K}^p \vdots \nabla\dot{\mathbf{E}}^p \leq 0. \tag{41}$$

The plastic free-energy ψ^p satisfies

$$\dot{\psi}^p - \mathbf{T}^p : \dot{\mathbf{E}}^p - \mathbb{K}^p \vdots \nabla\dot{\mathbf{E}}^p \leq 0. \tag{42}$$

Following Borokinni (2018), the choice $\psi^p = \hat{\psi}^p(\mathbf{G})$ renders the plastic microscopic stress $\mathbf{T}^p$ dissipative.

The inner product $\mathbb{K}^p \vdots \nabla\dot{\mathbf{E}}^p$ can be written as

$$\mathbb{K}^p \vdots \nabla\dot{\mathbf{E}}^p = \vec{\xi}^p \cdot \nabla\dot{e}^p + \phi\dot{e}^p, \tag{43}$$

where

$$\xi^p_k = (\mathbb{K}^p)_{ijk} N^p_{ij} \text{ and } \phi^p = \mathbb{K}^p \vdots \nabla\mathbf{N}^p. \tag{44}$$

The defect free-energy imbalance in terms of $\vec{\xi}^p$ and ϕ^p is

$$\dot{\psi}^p - (\mathbf{T}^p : \mathbf{N}^p)\dot{e}^p - \vec{\xi}^p \cdot \nabla \dot{e}^p - \phi \dot{e}^p \leq 0. \tag{45}$$

Assume the codirectionality hypothesis holds i.e.

$$\mathbf{N}^p = \frac{\mathbf{T}_o}{|\mathbf{T}_o|} \text{ for } \dot{\mathbf{E}}^p \neq \mathbf{0}, \tag{46}$$

then the microscopic force balance eq. (40) can be written as

$$\int_{\Omega_1} \left[\mathbf{T}_o : \dot{\mathbf{E}}^p - \mathbf{T}^p : \dot{\mathbf{E}}^p - \mathbb{K}^p \vdots \nabla \dot{\mathbf{E}}^p \right] dV = - \int_{\partial \Omega_1} \mathbf{K}^p(\mathbf{n}) : \dot{\mathbf{E}}^p dA. \tag{47}$$

In local form, following eq. (44), we have the microscopic force balance

$$\mathbf{T}_o : \mathbf{N}^p = \mathbf{T}^p : \mathbf{N}^p - (\text{div}\vec{\xi}^p - \phi^p) \text{ with } \vec{\xi}^p \cdot \mathbf{n} = \mathbf{K}^p(\mathbf{n}) : \mathbf{N}^p. \tag{48}$$

By comparing eq. (40) with eq. (48), it is clear that

$$\mathbf{N}^p : \text{div}\mathbb{K}^p = \text{div}\vec{\xi}^p - \phi^p. \tag{49}$$

Given that the plastic free energy $\psi^p = \hat{\psi}^p(\mathbf{G})$ is a function of the Burgers tensor, then increase in the plastic free energy is deduced as

$$\dot{\psi}^p = \vec{\xi}^p_{en} \cdot \nabla \dot{e}^p + \phi^p_{en} \dot{e}^p = \mathbb{K}^p_{en} \vdots \nabla \dot{\mathbf{E}}^p, \tag{50}$$

where

$$(\vec{\xi}^p_{en})_k = \frac{\partial \psi^p}{\partial G_{ij}} \epsilon_{iks} N^p_{sj}, \; \phi^p_{en} = \frac{\partial \psi^p}{\partial G_{ij}} \epsilon_{iks} N^p_{js,k} \text{ and } (\mathbb{K}^p_{en})_{jsk} = \frac{\partial \psi^p}{\partial G_{ij}} \epsilon_{iks}. \tag{51}$$

The dissipative microscopic stresses $\vec{\xi}^p_{dis}, \phi^p_{dis}$ and $\mathbb{K}^p_{dis}$ are defined by

$$\vec{\xi}^p_{dis} = \vec{\xi}^p - \vec{\xi}^p_{en}, \; \phi^p_{dis} = \phi^p - \phi^p_{en} \text{ and } \mathbb{K}^p_{dis} = \mathbb{K}^p - \mathbb{K}^p_{en}. \tag{52}$$

Hence following eqs. (45), (49)-(52), we have the force balance

$$\mathbf{N}^p : \text{div}\mathbb{K}^p_{en} = \text{div}\vec{\xi}^p_{en} - \phi^p_{en} \text{ and } \mathbf{N}^p : \text{div}\mathbb{K}^p_{dis} = \text{div}\vec{\xi}^p_{dis} - \phi^p_{dis} \tag{53}$$

and the dissipation δ given by

$$\delta = \mathbf{T}^p : \dot{\mathbf{E}}^p + \vec{\xi}^p_{dis} \cdot \nabla \dot{e}^p + \phi^p_{dis} \dot{e}^p = \mathbf{T}^p : \dot{\mathbf{E}}^p + \mathbb{K}^p_{dis} \vdots \nabla \dot{\mathbf{E}}^p \geq 0. \tag{54}$$

6.3. Mises-Aifantis Flow Rule Accounting for Kinematic and Isotropic Hardening

Here, we use the Gurtin-Anand model to establish the Aifantis flow equation. The microscopic force balance eq. (40) can be written as

$$\mathbf{T}_o + \text{div}\mathbb{K}^p_{en} = \mathbf{T}^p - \text{div}\mathbb{K}^p_{dis}. \tag{55}$$

This would imply that we have

$$(\mathbf{T}_o + \text{div}\mathbb{K}^p_{en}) : \mathbf{N}^p = \mathbf{T}^p : \mathbf{N}^p - \text{div}\vec{\xi}^p_{dis} + \phi^p_{dis}. \tag{56}$$

To obtain the Aifantis flow equation, it would be assumed that

$$\vec{\xi}^p_{dis} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega_1 \iff \text{div}\vec{\xi}^p_{dis} = 0 \text{ in } \Omega_1. \tag{57}$$

Hence, the dissipation per unit plastic flow is equivalent to the right hand side of eq. (56). This is clear from

$$\int_{\Omega_1} \delta dV = \int_{\Omega_1} \Big[(\mathbf{T}^p : \mathbf{N}^p)\dot{e}^p + \vec{\xi}^p_{dis} \cdot \nabla\dot{e}^p + \phi^p_{dis}\dot{e}^p\Big] dV. \tag{58}$$

Following eq. (57), the dissipation in local form is

$$\delta = (\mathbf{T}^p : \mathbf{N}^p)\dot{e}^p + \phi^p_{dis}\dot{e}^p. \tag{59}$$

Given that

$$\mathbf{T}^p = \hat{\mathbf{T}}^p(\mathbf{N}^p, e^p) \text{ and } \phi^p_{dis} = \hat{\phi}^p_{dis}(\mathbf{N}^p, e^p), \tag{60}$$

the flow resistance $Y(e^p)$ (assume strong isotropy) which is defined as dissipation per unit unit magnitude of the plastic flow is given by

$$Y(e^p) = \hat{\mathbf{T}}^p(\mathbf{N}^p, e^p) : \mathbf{N}^p + \phi^p_{dis}(\mathbf{N}^p, e^p). \tag{61}$$

Hence, if the flow stress $\mathbf{T}_o$ is associated with the plastic flow $\dot{\mathbf{E}}^p \neq \mathbf{0}$, then the Aifantis flow rule is of the form

$$(\mathbf{T}_o + \text{div}\mathbb{K}^p_{en}) : \mathbf{N}^p = Y(e^p). \tag{62}$$

Using the principle of Maximum plastic dissipation, it is easy to show that the flow rule also has the form

$$\mathbf{T}_o + \text{div}\mathbb{K}^p_{en} = Y(e^p)\mathbf{N}^p \text{ Compare eq. (35)} \tag{63}$$

To derive the specific form of the Aifantis flow rule, we assume the Burgers tensor has the constitutive form as

$$\mathbf{G} = \hat{\mathbf{G}}(e^p, \nabla e^p) \tag{64}$$

Following eq. (30), we deduce that

$$\frac{\partial \mathbf{G}}{\partial e^p} = \text{curl}\mathbf{N}^p \text{ and } \left(\frac{\partial \mathbf{G}}{\partial \nabla e^p}\right)_{jik} = N^p_{jr}\epsilon_{ikr}. \tag{65}$$

Consequences of eqs. (51) and (65) are

$$\vec{\xi}^p_{en} = \frac{\partial \psi^p}{\partial \nabla e^p} \text{ and } \phi^p_{en} = \frac{\partial \psi^p}{\partial e^p}. \tag{66}$$

Basic to Aifantis model is the assumption that $\phi^p_{en} = 0$ so that (Gurtin and Anand, 2009)

$$\hat{\psi}^p(\mathbf{G}) = \hat{\psi}^p(\nabla e^p). \tag{67}$$

The defect energy takes the quadratic form

$$\hat{\psi}^p(\nabla e^p) = \frac{\beta}{2}|\nabla e^p|^2, \tag{68}$$

where β is a constant. Eq. (62) can be written as

$$\tau + \text{div}\vec{\xi}^p_{en} - \phi^p_{en} = Y(e^p), \tag{69}$$

where $\tau = \mathbf{T}_o : \mathbf{N}^p$. Substitute eq. (68) in eq. (66) and substitute the result in eq. (69) (where $\phi^p_{en} = 0$) we have the Aifantis flow rule specifically written as

$$\tau + \beta\Delta e^p = Y(e^p). \tag{70}$$

Hence, given that $\mathbf{T}_o$ is the flow stress associated with $\mathbf{N}^p$ or $\dot{\mathbf{E}}^p$ the *Mises-Aifantis Flow Equation* is given by

$$\mathbf{T}_o + \mathrm{div}\mathbb{K}^p = Y(e^p)\mathbf{N}^p \implies \tau + \beta\Delta e^p = Y(e^p). \tag{71}$$

The yield criterion is given by

$$|\mathbf{T}_o + \mathrm{div}\mathbb{K}^p_{en}| = Y(e^p), \tag{72}$$

where the term $-\mathrm{div}\mathbb{K}^p_{en}$ is the backstress which accounts for kinematic hardening.

6.4. Generalized Plastic Strain Rate and Dissipative Stresses

For a gradient theory that mimicks the conventional theory, we introduce generalized plastic strain and dissipative plastic stress

$$\mathsf{E}^p = (\mathbf{E}^p, l\nabla\mathbf{E}^p) \text{ and } \mathsf{T}^p_{dis} = (\mathbf{T}^p, l^{-1}\mathbb{K}^p_{dis}) \tag{73}$$

respectively. The rate of the generalized plastic strain is denoted as $\dot{\mathsf{E}}^p$ and magnitude $|\dot{\mathsf{E}}^p|$ are defined as

$$\dot{\mathsf{E}}^p = (\dot{\mathbf{E}}^p, l\nabla\dot{\mathbf{E}}^p) \text{ and } |\dot{\mathsf{E}}^p| = \sqrt{|\dot{\mathbf{E}}^p|^2 + l^2|\nabla\dot{\mathbf{E}}^p|^2} \tag{74}$$

respectively.

Define the operation $\odot$ on the pair $(\mathsf{T}^p_{dis}, \dot{\mathsf{E}}^p)$ by

$$\mathsf{T}^p_{dis} \odot \dot{\mathsf{E}}^p = \mathbf{T}^p : \dot{\mathbf{E}}^p + \mathbb{K}^p_{dis} \vdots \nabla\dot{\mathbf{E}}^p. \tag{75}$$

Eq. (75) defines dissipation i.e.

$$\delta = \mathsf{T}^p_{dis} \odot \dot{\mathsf{E}}^p. \tag{76}$$

We assume that whenever $\dot{\mathsf{E}}^p \neq \mathbf{0}$ that the generalized plastic stress is defined via the constitutive relation

$$\mathsf{T}^p_{dis} = \hat{\mathsf{T}}^p_{dis}(\dot{\mathsf{E}}^p, \mathfrak{e}^p), \tag{77}$$

where $\mathfrak{e}^p$ is an hardening variable which we will call *accumulation of the generalized plastic strain* satisfying the initial value problem:

$$\dot{\mathfrak{e}}^p = |\dot{\mathsf{E}}^p| \text{ with } \mathfrak{e}^p(\mathbf{X}, 0) = 0. \tag{78}$$

For rate-independent plastic materials eq. (77) reduces to

$$\mathsf{T}^p_{dis} = \hat{\mathsf{T}}^p_{dis}(\mathsf{N}^p, \mathfrak{e}^p) \text{ for } \dot{\mathsf{E}}^p \neq \mathbf{0}, \tag{79}$$

where N^p is called generalized flow direction defined by eq. (10).

We assume strong isotropy and define the generalized flow resistance $Y(\mathfrak{e}^p)$ as the dissipation per unit magnitude of the generalized plastic strain rate viz:

$$Y(\mathfrak{e}^p) = \mathsf{T}^p_{dis}(\mathsf{N}^p, \mathfrak{e}^p) \odot \mathsf{N}^p. \tag{80}$$

6.5. Mises-Gurtin-Anand Flow Equation

The Principle of Maximum Plastic dissipation requires that the generalized dissipative stress T^p_{dis} is admissible in the sense of maximum plastic dissipation written as

$$\mathsf{T}^p_{dis} : \mathsf{N}^p \leq \hat{\mathsf{T}}^p_{dis}(\mathsf{N}^p, \mathfrak{e}^p) : \mathsf{N}^p \text{ for every } \mathsf{N}^p. \tag{81}$$

Now, following the arguments that established eq. (35), it is clear that the generalized Mises flow rule for strain gradient plasticity is

$$\mathsf{T}^p_{dis} = Y(\mathfrak{e}^p)\mathsf{N}^p \text{ for } \dot{\mathsf{E}}^p \neq \mathbf{0}. \tag{82}$$

In Component form, following eqs. (10), (73) and (74) we have

$$\mathbf{T}^p = Y(\mathfrak{e}^p)\frac{\dot{\mathbf{E}}^p}{\dot{\mathfrak{e}}^p} \text{ and } \mathbb{K}^p_{dis} = l^2 Y(\mathfrak{e}^p)\frac{\nabla\dot{\mathbf{E}}^p}{\dot{\mathfrak{e}}^p}. \tag{83}$$

The yield condition for Mises flow rule eq. (82) is given by

$$|\mathsf{T}^p_{dis}| = Y(\mathfrak{e}^p) \iff \sqrt{|\mathbf{T}^p|^2 + l^{-2}|\mathbb{K}^p_{dis}|^2} = Y(\mathfrak{e}^p). \tag{84}$$

Eq. (84) can be written in the standard form that shows the presence of kinematic and isotropic hardening terms as

$$|\mathsf{T}_o - \mathsf{T}_{back}| = Y(\mathfrak{e}^p), \tag{85}$$

where T_o is the generalized flow stress, and T_{back} is the generalized back stress accounting for kinematic hardening defined as

$$\mathsf{T}_o = (\mathbf{T}_o + \mathrm{div}\mathbb{K}^p, l^{-1}\mathbb{K}^p) \text{ and } \mathsf{T}_{back} = (0, l^{-1}\mathbb{K}^p_{en}). \tag{86}$$

Clearly, we have

$$\mathsf{T}_o - \mathsf{T}_{back} = \mathsf{T}^p_{dis}. \tag{87}$$

Hence the Generalized Mises flow rule is given by

$$\mathsf{T}_o - \mathsf{T}_{back} = Y(\mathfrak{e}^p)\mathsf{N}^p \text{ for } \dot{\mathsf{E}}^p \neq \mathbf{0}. \tag{88}$$

The generalized flow equations are given by eq. (88) with the boundedness property

$$|\mathsf{T}_o - \mathsf{T}_{back}| \leq Y(\mathfrak{e}^p) \text{ for every } \dot{\mathsf{E}}^p, \tag{89}$$

which defines the elastic region, and

$$\dot{\mathsf{E}}^p = \mathbf{0} \text{ for } |\mathsf{T}_o - \mathsf{T}_{back}| < Y(\mathfrak{e}^p), \tag{90}$$

which is the no-flow condition.

From eqs. (83) and (88), it is clear that

$$\mathbf{T}_o + \mathrm{div}\mathbb{K}^p = Y(\mathfrak{e}^p)\frac{\dot{\mathbf{E}}^p}{\dot{\mathfrak{e}}^p} \text{ and } \mathbb{K}^p - \mathbb{K}^p_{en} = l^2 Y(\mathfrak{e}^p)\frac{\nabla\dot{\mathbf{E}}^p}{\dot{\mathfrak{e}}^p}. \tag{91}$$

Taking the divergence of eq. $(91)_2$ and substituting in eq. $(91)_1$ we have the Gurtin-Anand rate-independent plastic flow rule given as

$$\mathbf{T}_o + \mathrm{div}\hat{\mathbb{K}}^p_{en}(\mathbf{G}) = Y(\mathfrak{e}^p)\frac{\dot{\mathbf{E}}^p}{\dot{\mathfrak{e}}^p} - \mathrm{div}\left(l^2 Y(\mathfrak{e}^p)\frac{\nabla\dot{\mathbf{E}}^p}{\dot{\mathfrak{e}}^p}\right). \tag{92}$$

7. CONSTRAINED SHEAR STRIP PROBLEM

Consider an infinite slab of thickness h under a pure shear T along the thickness of the slab. In this problem, the plastic strain is a function of the single space variable y -measured along thickness of the slab- and time t. If the body force is neglected then from the macroscopic equation of motion, T is independent of y and we assume that $T = T(t)$ is a function of time only. Here we will denote the plastic shear strain by e so that gradient of plastic strain is $\frac{\partial e}{\partial y}$. Also, assume the absence of energetic contribution from the plastic microstress $\mathbf{T}^p$ and that $h = 1$ so that $y \in [0, 1]$.

7.1. Classical Theory Versus Aifantis Flow

We assume that the flow resistance $Y(e^p)$ in eqs. (35) and (70) takes

$$Y(e) = k \cdot e, \tag{93}$$

where k is a constant.

The classical flow rule Eq. (35) reduces to the form

$$e = \frac{T(t)}{k} \tag{94}$$

and the Aifantis flow rule reduces to the non local form

$$T(t) + \mu L^2 \frac{\partial^2 e}{\partial y^2} = k \cdot e, \tag{95}$$

where $\beta = \mu L^2$ (see eq. (35)), μ is the shear modulus and L is an energetic length scale. Assume the boundary conditions

$$e(0, t) = 0, \text{ and } e_y(0, t) = 0, \tag{96}$$

where e_y is the first partial derivative of e with respect to y. Given the boundary condition eq. (96), the solution of eq. (95) is given by

$$e = \frac{T(t)}{k}\left[1 - \frac{1}{1 + e^{\left(\frac{-2}{L}\sqrt{\frac{k}{\mu}}h\right)}}\left(e^{\left(\frac{-1}{L}\sqrt{\frac{k}{\mu}}y\right)} + e^{\left(\frac{1}{L}\sqrt{\frac{k}{\mu}}(y-2h)\right)}\right)\right],$$

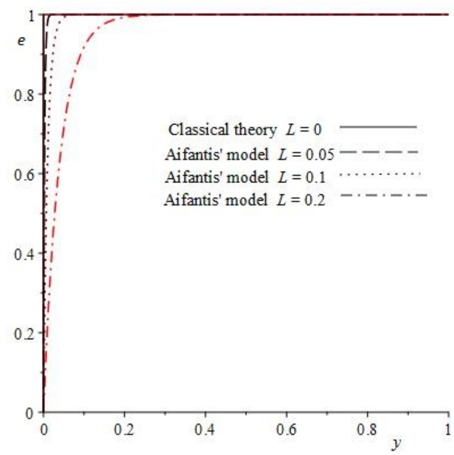

FIGURE 1. Classical theory versus Aifantis model

which can be written as

$$e = \frac{T(t)}{k}\left[1 - \operatorname{sech}\left(\frac{h}{L}\sqrt{\frac{k}{\mu}}\right)\cosh\left(\frac{y-h}{L}\sqrt{\frac{k}{\mu}}\right)\right]. \tag{97}$$

A plastic strain profile is shown in figure 1 for $k = \mu$ and varying values of the energetic length scale L. It is shown from the graph that the Aifantis' model approaches the classical theory as $L \to 0$.

7.2. Gurtin-Anand Flow

Following Gurtin and Anand (2005), we assume that

$$\operatorname{div}\hat{\mathbb{K}}^p_{en}(\mathbf{G}) = \mu L^2 \Delta \mathbf{E}^p.$$

For the present constrained shear strip problem, eq. (92) reduces to

$$T(t) + \mu L^2 \frac{\partial^2 e}{\partial y^2} = Y(\mathfrak{e}^p)\frac{\dot{e}}{\dot{\mathfrak{e}}^p} - \frac{\partial}{\partial y}\left(l^2 \frac{Y(\mathfrak{e}^p)}{\dot{\mathfrak{e}}^p}\frac{\partial \dot{e}}{\partial y}\right) \tag{98}$$

with the initial-boundary conditions

$$e(y,0) = 0,\ e(0,t) = 0 \text{ and } \mu L^2 e_y(h,0) + l^2\frac{Y(\mathfrak{e}^p)}{\dot{\mathfrak{e}}^p}\dot{e}(h,t) = 0. \tag{99}$$

Consistent with the classical theory we assume that the flow resistance takes the form

$$Y(\mathfrak{e}^p) = kt\dot{\mathfrak{e}}^p \text{ such that } e = t\dot{e}, \tag{100}$$

where k is a constant. Then the solution eq. (98) with the given boundary condition is

$$e = \frac{T(t)}{k}\left[1 - \operatorname{sech}\left(\frac{h}{L^*}\sqrt{\frac{k}{\mu}}\right)\cosh\left(\frac{y-h}{L^*}\sqrt{\frac{k}{\mu}}\right)\right]. \tag{101}$$

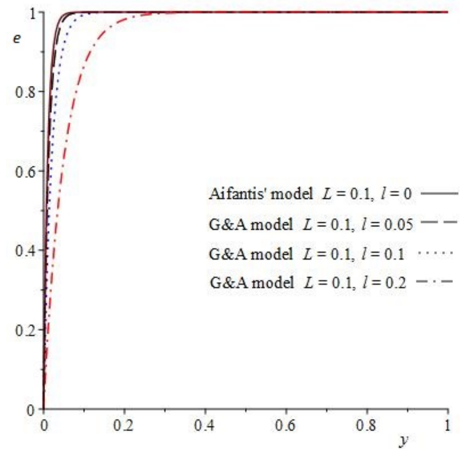

FIGURE 2. Aifantis versus Gurtin-Anand models

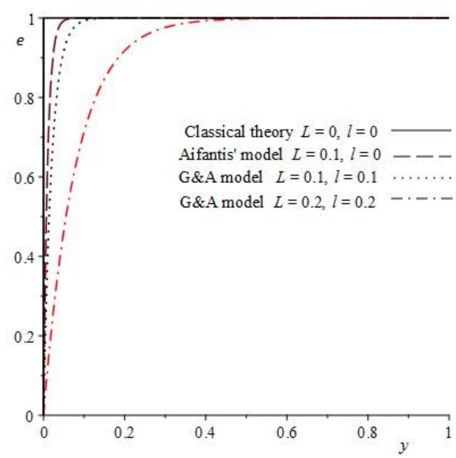

FIGURE 3. Classical, Aifantis and Gurtin-Anand theories

where $L^{*2} = L^2 + \frac{kl^2}{\mu}$.

Assuming the thickness of constrained strip is of unit length and $k = \mu$, the plastic strain profile is shown in figures 2 and 3. Figure 2 compares the Aifantis model with the Gurtin-Anand (G&A) model where it is observed that G&A model approaches the Aifantis' model as the dissipative length scale l approaches zero. Figure 3 shows the profiles of the plastic strain for G&A model, Aifantis' model and the classical theory. It is observed that G&A model generalizes both classical theory and Aifantis' model because of the presence of dissipative length scales.

8. CONCLUSION

This work obtains generalized Mises flow equations for rate independent strain gradient plasticity with the aim of providing a framework for which other gradient theories can be studied through thermodynamic consistency and principle of maximum plastic dissipation. It was deduced that gradient plasticity theories are simply extensions of the conventional theory only by the inclusion of the plastic strain gradient and its rate. Even with that, it was obtained that the mathematical structure leading to the classical theory and that leading to gradient theories are similar as observed in the flow equations. It is demonstrated by a simple constrained problem that the Gurtin-Anand model differs from the Aifantis' theory only in the dissipative length scale, so that as this length scale vanishes the Gurtin-Anand flow rule and Aifantis' flow rule are the same.

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REFERENCES

[1] E. C. Aifantis, *On the microstructural origin of certain inelastic models*, Trans. ASME J. Eng. Mater. Tech. **106**, 326-330, 1984.

[2] E. C. Aifantis, *The physics of plastic deformation*, Int. J. Plast. **3**, 211-247, 1987.

[3] M. F. Ashby, *The deformation of plastically non-homogeneous materials*, Phil. Mag. **21**, 399-424, 1970.

[4] A. S. Borokinni, *A gradient theory based on Aifantis theory using the Gurtin-Anand strain gradient plasticity approach*, J. Mech. Behav. Mater. DOI: 10.1515/jmbm-2018-0012, 2018.

[5] G. Del Piero, *The variational structure of classical plasticity*, Math. Mech. Complex Syst. **6**, 37-180, 2018.

[6] N. A. Fleck and J. W. Hutchinson, *A phenomenological theory for strain gradient effects in plasticity*, J. Mech. Phys. Solids. **41**, 1825-1857, 1994.

[7] N. A. Fleck and J. W. Hutchinson, *Strain gradient plasticity*, Adv. Appl. Mech. **33**, 295-361, 1997.

[8] P. Gudmunson and C. F. O. Dahlberg, *Isotropic strain gradient plasticity model based on self-energies of dislocations and the Taylor model for plastic dissipation*, Int. J. Plast. **121**, 1-20, 2019.

[9] M. E. Gurtin, and L. Anand, *A theory of strain-gradient plasticity for isotropic plastically irrotational materials. Part I: Small deformations*, J. Mech. Phys. Solids. **53**, 1624-1649, 2005.

[10] M. E. Gurtin and L. Anand, *Thermodynamics applied to gradient theories involving the accumulated plastic strain: The theories of Aifantis and Fleck and Hutchinson and their generalization*, J. Mech. Phys. Solids. **57**, 405-421, 2009.

[11] M. E. Gurtin, E. Fried and L. Anand, *The mechanics and thermodynamics of continua*, Cambridge University Press, Cambridge, 2010.

[12] W. Han and B. D. Reddy, *Plasticity: Mathematical theory and numerical analysis*, New York: Springer-Velag, 2013.

[13] R. Hill, *A theory of yielding and plastic flow of anisotropic metals*, Proc. Roy. Soc. A. **193**, 281-297, 1948.

[14] R. Hill, *Constitutive dual potentials in classical plasticity*, J. Mech. Phys. Solids. **35**, 39-63, 1987.

[15] J. W. Hutchinson, *Plasticity at the micron scale*, Int. J. Solids Struct. **37**, 225-238, 2000.

[16] B. D. Reddy, F. Ebobisse and A. McBride, *Well-posedness of a model of strain gradient plasticity for plastically irrotational materials*, Int. J. Plast. **24**, 55-73, 2008.

[17] J. R. Willis, *Some forms and properties of model of strain-gradient plasticity*, J. Mech. Phys. Solids. **123**, 348-356, 2019.

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