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MHD FLOWS IN NONLINEAR VELOCITY DEPENDENT MAGNETIC FIELDS NEAR THE LEADING EDGE OF A FLAT PLATE OF NEGLIGIBLE THICKNESS

(Dedicated to the memory of Professor Reuben Olafenwa Ayeni)

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We prove the existence of unique classical similarity solutions to the MHD flows in nonlinear velocity dependent magnetic fields near the leading edge of a flat plate of negligible thickness.

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1. INTRODUCTION

Since the celebrated pioneering work of Prandtl [1] in 1904, several analyses of the boundary layer theory have been carried out under various conditions. Magnetohydrodynamic (MHD) flows are examples of such analyses. In the current analysis, we shall be concerned with two dimensional boundary layer problems. For an introduction to two dimensional boundary layer problems, the reader is referred to [2].

MHD flow of electrically conducting fluid in the presence of an applied magnetic field has received considerable attention because of the numerous applications to geophysics, astrophysics, engineering and other industrial areas. Magnetic fields are produced by charged particles in motion, and depend on the charge and velocity of these particles. In particular, moving point charges, such as electrons, produce complicated but well known magnetic fields that depend on the charge, velocity, and acceleration of the particles [5]. In the study of MHD flows, magnetic fields are either assumed to be uniform or time or/and space variables dependent. For the examples of a few of such studies, we refer the reader to [6]–[16] and the literature cited in them. To our knowledge, there is, to date,

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no analysis where the MHD flow has been considered with velocity dependent magnetic field.

Motivated by previous works, we study, first in our series of intended analyses, the MHD flow in nonlinear velocity dependent magnetic fields near the leading edge of a flat plate of negligible thickness.

The remaining part of this paper is organized as follows: The mathematical formulation is presented in Section 2. Some auxiliary results are stated and proved in Section 3. In Section 4, the existence of unique classical solution to the problem is proved in a convex subset of $C^2(0,1)$ for some small non-negative parameters; while the existence of non-unique classical solution is proved in a convex and compact subset of $C^2(0,1)$, without restriction on the parameters. Finally, we give illustrative examples for which our results apply in Section 5.

2. MATHEMATICAL FORMULATION

We consider a steady two-dimensional laminar flow of an incompressible and electrically conducting fluid of density ρ, near the leading edge of a flat plate of negligible thickness. We assume that (x, y) is the Cartesian coordinates of any point in the domain of flow, where $x - axis$ is along the plate and $y - axis$ is normal to it. We assume that u and v are the velocity components in the x and y directions respectively; and that a nonlinear magnetic field $H(x, u)$ is applied normally to the plate.

Within the boundary-layer approximation, the continuity and Navier-Stokes equations can be simplified to the following equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \frac{\sigma\mu^2 H^2(x,u)}{\rho}\left(u - u_\infty\right). \tag{2}$$

(1)–(2) are to be solved with the boundary conditions

$$v(x,0) = 0, \tag{3}$$

$$u(x,0) = 0,\ u(x,\infty) = u_\infty,\ u(0,y) = u_\infty, \tag{4}$$

where ν is the kinematic viscosity, σ is the electrical conductivity of the fluid and μ is the magnetic permeability. In the absence of the magnetic field (the case $H = 0$), an approximate solution of (1)–(3)

is obtained in [2]; while the exact numerical solution is obtained by Blasius [17].

Now, (1) can be solved for v with the boundary condition (3) and substituted into (2) to get the integro-differential equation

$$u\frac{\partial u}{\partial x}+\left(\int_0^y \frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}-\frac{\sigma\mu^2 H^2(x,u)}{\rho}\left(u-u_\infty\right). \tag{5}$$

Thus the nonlinear problems (1)–(4) are now reduced to the nonlinear problems (4)–(5). Consequently, we seek the solution to the problems (4)–(5).

We assume, as in [2], that the velocity profiles at various values of x have the same shape:

$$\frac{u}{u_\infty}=U(Y) \quad \text{where} \quad Y=\frac{y}{\delta(x)}, \tag{6}$$

where $\delta(x)$ is the boundary layer thickness at a distance x down the plate. Further, we assume that $u=u_\infty$ outside the boundary layer. For our current analysis we set

$$H(x,u):=\frac{A}{\delta(x)}f\left(U(Y)\right), \tag{7}$$

where A is the current (in amperes) and the nonlinear function $f:\mathbb{R}\to\mathbb{R}$ satisfies

$$|f(u)-f(v)|\le L|u-v| \quad \text{(Lipschitz continuity)}. \tag{8}$$

Using (6)–(7), (4)–(5) transform to the similarity problems

$$\begin{aligned} U'' &= R_m\left[-UU'Y+\left(\int_0^Y U'(s)s\,ds\right)U'\right]+Mf^2(U)(U-1)\\ &= -R_mU'\int_0^Y U(s)\,ds+Mf^2(U)(U-1),\ Y\in(0,1), \qquad (9)\\ U(0)&=0,\ U(1)=1, \qquad (10)\end{aligned}$$

where

$$R_m:=\frac{u_\infty\delta}{\nu}\frac{d\delta}{dx} \text{ is a modified Reynold's number,} \tag{11}$$

$$M:=\frac{\sigma\mu^2A^2}{\rho\nu} \text{ is the Hartmann number.} \tag{12}$$

Notice that (11) implies that the boundary layer thickness is

$$\delta(x)=\sqrt{\frac{2\nu R_m x}{u_\infty}}=\sqrt{2R_m}\sqrt{\frac{\nu x}{u_\infty}}, \tag{13}$$

where we have imposed the condition $\delta(0) = 0$. (13) implies that the boundary layer thickness depends on the modified Reynolds number R_m. We mention that the approximate solution presented in [2] is the special case where $R_m = \frac{140}{13}$.
The following Lemma can be easily proved:

Lemma 1. *The boundary value problems* (9)–(10) *are equivalent to the integral equation*

$$\begin{aligned} U = \Bigg[1 + R_m \int_0^1 \int_0^s \Big[U(s)U(t) - U^2(t)\Big] dtds - \\ M \int_0^1 \int_0^s f^2(U(t))(U(t) - 1)dtds\Bigg] Y - \\ R_m \int_0^Y \int_0^s \Big[U(s)U(t) - U^2(t)\Big] dtds + \\ M \int_0^Y \int_0^s f^2(U(t))(U(t) - 1)dtds. \end{aligned} \tag{14}$$

We apply the following theorems in this paper:

Theorem 1. *A set S of real numbers is compact if and only if it is closed and bounded. (see* [3]*).*

Theorem 2. *(Schauder's Fixed Point Theorem). Let X be a real Banach space. Suppose $K \subset X$ is compact and convex; and assume*

$$A : K \to K$$

is a continuous mapping. Then A has a fixed point in K. (see [4]*).*

3. AUXILIARY RESULTS

In this section, we state and prove two useful lemmas and a theorem.

Lemma 2. *Let $w \in C(0, 1)$. Suppose the condition* (8) *holds. Then $f(w) \in C(X)$, where*

$$X := \{w(Y) \in \mathbb{R} \mid 0 \le Y \le 1\}. \tag{15}$$

Proof. Notice that the condition (8) implies that f is continuous. Hence, we need only to prove that $\sup_X |f(w)| < \infty$. Using (8), we deduce:

$$|f(w)-f(0)|^2 = f^2(w)+f^2(0)-2f(w)f(0) \le L^2|w|^2 \text{ or}$$
$$f^2(w) \le L^2|w|^2 - f^2(0) + \frac{1}{2}f^2(w) + 2f^2(0)$$
$$\text{(by Young's inequality)}$$
$$\le L^2|w|^2 + f^2(0) + \frac{1}{2}f^2(w) \tag{16}$$

$$\sup_X |f(w)| \le \gamma(\|w\|_{C(0,1)}+1) < \infty, \tag{17}$$

here γ is defined as

$$\gamma := \sqrt{2\max\{L^2, f^2(0)\}} \tag{18}$$

□

Lemma 3. *Let $U, \tilde{U} \in C(0,1)$ and f satisfy* (8)*, then we have the estimates:*

$$|U^2 - \tilde{U}^2| \le \left(\|U\|_{C(0,1)} + \|\tilde{U}\|_{C(0,1)}\right)\|U-\tilde{U}\|_{C(0,1)} \tag{19}$$

$$|f^2(U)(U-1) - f^2(\tilde{U})(\tilde{U}-1)|$$
$$\le \gamma^2\left(\|U\|_{C(0,1)} + \|\tilde{U}\|_{C(0,1)} + 1\right)^2\|U-\tilde{U}\|_{C(0,1)}, \tag{20}$$

Proof. The proof of (19) is trivial, and therefore omitted. We render the proof of (20):

$$|f^2(U)(U-1) - f^2(\tilde{U})(\tilde{U}-1)|$$
$$= |f^2(U)(U-1) - f^2(U)(\tilde{U}-1) + f^2(U)(\tilde{U}-1) - f^2(\tilde{U})(\tilde{U}-1)|$$
$$\le |f^2(U)(U-\tilde{U}) + (\tilde{U}-1)(f^2(U) - f^2(\tilde{U}))| \tag{21}$$
$$\le L\left(\|\tilde{U}\|_{C(0,1)}+1\right)\left(\|f(U)\|_{C(X)} + \|f(\tilde{U})\|_{C(X)}\right)\|U-\tilde{U}\|_{C(0,1)}$$
$$+\|f(U)\|^2_{C(X)}\|U-\tilde{U}\|_{C(0,1)}, \tag{22}$$

where we have used (8) and (19) to estimate the second term on the right side of (21). Using (17) in (22) and simplifying, we deduce (20). □

Consider the linear boundary value problem:

$$U'' = -R_m r'\int_0^Y r(s)ds + Mf^2(r)(r-1),\ Y \in (0,1), \tag{23}$$
$$U(0) = 0,\ U(1) = 1, \tag{24}$$

where $r \in C^2(0,1)$ is a known function of Y; and R_m, M are fixed positive numbers.

Theorem 3. *(A priori estimates). Let ψ be a solution of* (23)–(24). *Then $\psi \in C^2(0,1)$, and we have the estimate:*

$$\|\psi\|_{C^2(0,1)} \leq 2 + 9R_m\|r\|^2_{C^2(0,1)} + 5M\|f\|^2_{C(X)}(\|r\|_{C^2(0,1)} + 1) =: \Lambda \tag{25}$$

Proof. Using Lemma 1, the problem (23)–(24) admits the unique solution:

$$\begin{aligned}\psi = \Bigg[1 + R_m \int_0^1 \int_0^s \Big[r(s)r(t) - r^2(t)\Big] dtds - \\ M \int_0^1 \int_0^s f^2(r(t))(r(t)-1)dtds\Bigg] Y - \\ R_m \int_0^Y \int_0^s \Big[r(s)r(t) - r^2(t)\Big] dtds + \\ M \int_0^Y \int_0^s f^2(r(t))(r(t)-1)dtds.\end{aligned} \tag{26}$$

It is clear that ψ is twice continuously differentiable. We readily estimate:

$$\begin{aligned}\sup_{C(0,1)} |\psi| \leq 1 + 4R_m\|r\|^2_{C^2(0,1)} + \\ 2M\|f\|^2_{C(X)}(\|r\|_{C(0,1)} + 1) < \infty,\end{aligned} \tag{27}$$

$$\begin{aligned}\sup_{C(0,1)} |\psi'| \leq 1 + 4R_m\|r\|^2_{C^2(0,1)} + \\ 2M\|f\|^2_{C(X)}(\|r\|_{C(0,1)} + 1) < \infty,\end{aligned} \tag{28}$$

$$\begin{aligned}\sup_{C(0,1)} |\psi''| \leq R_m\|r\|^2_{C^2(0,1)} + \\ M\|f\|^2_{C(X)}(\|r\|_{C^2(0,1)} + 1) < \infty,\end{aligned} \tag{29}$$

where we have used Lemma 2 to estimate f. Combining (27)–(29) yields (25). □

4. MAIN RESULT

Theorem 4. *(i) Let* $\max\{9R_m, 5\gamma^2 M\} > 0$ *be sufficiently small. Then there exist unique classical solutions to the boundary value problems* (9)–(10).

(ii) For any $\max\{9R_m, 5\gamma^2 M\} > 0$, there exists classical solutions to the boundary value problems (9)–(10).

Proof. We split the proof in 8 steps.
Step 1. The fixed point argument to (9)–(10) is

$$w'' = -R_m U' \int_0^Y U(s)ds + Mf^2(U)(U-1), \ Y \in (0,1), \tag{30}$$

$$w(0) = 0, \ w(1) = 1 \tag{31}$$

Define a mapping

$$A : C^2(0,1) \to C^2(0,1)$$

by setting $A[U] = w$ whenever w is derived from U via (30)–(31).
Step 2. Choose $U, \tilde{U} \in C^2(0,1)$, and define $A[U] = w, \ A[\tilde{U}] = \tilde{w}$. For two solutions $w, \tilde{w}$ of (30)–(31), we employ (26) to deduce:

$$\begin{aligned} &w - \tilde{w} = \\ &\Bigg[R_m \int_0^1 \int_0^s \Big[U(s)U(t) - \tilde{U}(s)\tilde{U}(t) - (U^2(t) - \tilde{U}^2(t))\Big]\, dtds \\ &-M \int_0^1 \int_0^s [f^2(U(t))(U(t)-1) - f^2(\tilde{U}(t))(\tilde{U}(t)-1)]dtds\Bigg] Y \\ &-R_m \int_0^Y \int_0^s \Big[U(s)U(t) - \tilde{U}(s)\tilde{U}(t) - (U^2(t) - \tilde{U}^2(t))\Big]\, dtds \\ &+M \int_0^Y \int_0^s [f^2(U(t))(U(t)-1) - f^2(\tilde{U}(t))(\tilde{U}(t)-1)]dtds. \end{aligned} \tag{32}$$

Using Lemma (3) and (32) we deduce:

$$\begin{aligned} &\|w - \tilde{w}\|_{C^2(0,1)} \\ &\qquad \le C \left(\|U\|_{C^2(0,1)} + \|\tilde{U}\|_{C^2(0,1)} + 1 \right)^2 \|U - \tilde{U}\|_{C^2(0,1)} \end{aligned} \tag{33}$$

where $C := \max\{9R_m, 5\gamma^2 M\}$.

Notice that the bound in (33) is not uniform. Consequently, we need to prove the existence of solutions in a subset of $C^2(0,1)$. Define a convex set

$$K := \{U \in C^2(0,1) | \ \|U\|_{C^2(0,1)} \le 2\Lambda\}, \tag{34}$$

where $\Lambda =$ constant is the bound in (25) of Theorem 3.
Step 3. We now prove Theorem 4(i). We claim that, if $\max\{9R_m, 5\gamma^2 M\} > 0$ is sufficiently small, then A is a contraction mapping. We will show that if $\max\{9R_m, 5\gamma^2 M\}$ is sufficiently small, then

$$A[K] \subseteq K, \ \|A[U] - A[\tilde{U}]\|_{C^2(0,1)} \le \frac{1}{4}\|U - \tilde{U}\|_{C^2(0,1)}, \tag{35}$$

for all $U, \tilde{U} \in K$. Using (25), we have

$$\|A[r]\|_{C^2(0,1)} = \|\psi\|_{C^2(0,1)} \leq \Lambda < 2\Lambda. \tag{36}$$

Thus, ψ defined by (25) belongs to K; so that the set K is not empty. Using (33) and (36), we have for any $U \in K$,

$$\begin{aligned}
&\|A[U]\|_{C^2(0,1)} \leq \|A[r]\|_{C^2(0,1)} + \|A[U] - A[r]\|_{C^2(0,1)} \\
&\leq \Lambda + C\left(\|U\|_{C^2(0,1)} + \|r\|_{C^2(0,1)} + 1\right)^2 \|U - r\|_{C^2(0,1)} \\
&\leq \Lambda + C\,(4\Lambda + 1)^2\,(4\Lambda) \leq 2\Lambda,
\end{aligned}$$

for $C := \max\{9R_m, 5\gamma^2 M\} > 0$ sufficiently small that

$$\max\{9R_m, 5\gamma^2 M\}\,(4\Lambda + 1)^2 < \frac{1}{4}. \tag{37}$$

Thus $A[K] \subseteq K$. Furthermore, using (37), (33) implies that

$$\|A[U] - A[\tilde{U}]\|_{C^2(0,1)} = \|w - \tilde{w}\|_{C^2(0,1)} < \frac{1}{4}\|U - \tilde{U}\|_{C^2(0,1)},$$

for all $U, \tilde{U} \in K$. Consequently, the mapping A is a strict contraction for sufficiently small parameter $\max\{9R_m, 5\gamma^2 M\} > 0$.

Step 4. Write $U_0 = Y$. For $k = 0, 1, 2, ...$, inductively define $U_{k+1} \in K$ to be the unique weak solution of the linear boundary value problem

$$U''_{k+1} = -R_m U'_k \int_0^Y U_k(s)ds + Mf^2(U_k)(U_k - 1),\ Y \in (0,1), \tag{38}$$

$$U_{k+1}(0) = 0,\ U_{k+1}(1) = 1. \tag{39}$$

By the definition of the mapping A, we have for $k = 0, 1, 2, ...$:

$$U_{k+1} = A[U_k].$$

Since A is a contraction mapping, there exists, by the Banach fixed point theorem (see for example [18]), a unique $U \in K$ such that

$$\lim_{k\to\infty} U_{k+1} = \lim_{k\to\infty} A[U_k] = A[U] = U \tag{40}$$

Step 5. Using (17), we deduce

$$\sup_X |f(U_k)| \leq \gamma(\|U_k\|_{C(0,1)} + 1) < \infty, \tag{41}$$

where γ and X are defined by (15) and (18) respectively. We now use (40) to take the limit on the right side of (41) to conclude

$$\sup_k \|f(U_k)\|_{C(X)} < \infty. \tag{42}$$

(42) implies the existence of a subsequence of $\{f(U_k)\}_{k=1}^{\infty}$, still denoted by $\{f(U_k)\}_{k=1}^{\infty}$, converging strongly in X to $f(U)$ in X.

Step 6. We now verify that U is the classical solution of (9)–(10). Using (40) and the deduction of the last step, we let $k \to \infty$ in (38)–(39) to obtain (9)–(10) as desired.
Step 7. We next prove 4(ii). For any $U \in K$, define

$$U^* := \frac{\alpha\Lambda}{1+2\Lambda^2} U \text{ for some } \alpha \in [1, \|r\|_{C^2(0,1)}],$$

where $r = r(Y)$ is the assumed known function in (23). It is not difficult to check that

$$U^* \in K^* := \{V^* \in C^2(0,1) | \ \|V^*\|_{C^2(0,1)} \le \alpha\Lambda\}.$$

Notice that the set K^* is bounded and closed; and hence, by Theorem 1, compact. Notice also that K is convex. Without restriction on the parameters, we claim that A is a continuous mapping. We will prove our claim in the next step. We will show that

$$A[K^*] \subseteq K^*, \ \|A[U] - A[\tilde{U}]\|_{C^2(0,1)} \le C\|U - \tilde{U}\|_{C^2(0,1)},$$

for all $U, \tilde{U} \in K^*$, where $C := \max\{9R_m, 5\gamma^2 M\} \left(2\alpha\Lambda + 1\right)^2$.
Step 8. Using (25), we deduce that:

$$\|A[r]\|_{C^2(0,1)} = \|\psi\|_{C^2(0,1)} \le \Lambda \le \alpha\Lambda. \tag{43}$$

Thus ψ defined by (25) belongs to K^*; so that K^* is not empty. We have for any $U \in K^*$ that $U = \frac{\alpha\Lambda}{1+2\Lambda^2} V$ for some $V \in K$. Employing (25) once more, we compute:

$$\begin{aligned}
&\|A[U]\|_{C^2(0,1)} = \left\| A\left[\frac{\alpha\Lambda}{1+2\Lambda^2} V\right]\right\|_{C^2(0,1)} \\
&\le 2 + 9R_m \frac{\alpha^2\Lambda^2}{(1+2\Lambda^2)^2}\|V\|^2_{C^2(0,1)} \\
&\qquad\qquad +5M\|f\|^2_{C(X)}\left(\frac{\alpha\Lambda}{1+2\Lambda^2}\|V\|_{C^2(0,1)} + 1\right) \\
&\le 2 + 9R_m \frac{\alpha^2\Lambda^2(2\Lambda)^2}{(1+2\Lambda^2)^2} \\
&\qquad\qquad +5M\|f\|^2_{C(X)}\left(\frac{\alpha\Lambda(2\Lambda)}{1+2\Lambda^2} + 1\right) \text{ (since } V \in K) \\
&\le 2 + 9R_m\alpha^2 + 5M\|f\|^2_{C(X)}(\alpha+1) \\
&\le 2 + 9R_m\alpha^2 + 5M\|f\|^2_{C(X)}(\alpha+1) \qquad (44) \\
&\le 2 + 9R_m\|r\|^2_{C^2(0,1)} + 5M\|f\|^2_{C(X)}(\|r\|_{C^2(0,1)} + 1) = \Lambda \le \alpha\Lambda,
\end{aligned}$$

using $1 \le \alpha \le \|r\|_{C^2(0,1)}$. Thus $A[K^*] \subseteq K^*$, since U was arbitrarily chosen. Hence for all $U, \tilde{U} \in K^*$, (33) implies that

$$\|A[U]-A[\tilde{U}]\|_{C^2(0,1)} \le \max\{9R_m, 5\gamma^2 M\}\,(2\alpha\Lambda+1)^2\,\|U-\tilde{U}\|_{C^2(0,1)}. \tag{45}$$

Thus, the mapping A is Lipschitz continuous on K^*, and hence continuous on K^*. Since the set K^* is convex and compact; and the mapping A is continuous on K^*, by Schauder's fixed point Theorem 2, A has a fixed point in K^*. Consequently, employing once more the linear boundary value problem (38)–(39), there exists $U \in K^*$ such that

$$\lim_{k\to\infty} U_{k+1} = \lim_{k\to\infty} A[U_k] = A[U] = U. \tag{46}$$

Steps 5 and 6 can now be repeated to conclude the proof of Theorem (4)(ii). □

Remark Notice that for sufficiently small R_m and M, we can infer from (14) that U is approximately linear. Since we have proved uniqueness of solutions for small parameters, we infer that the uniqueness of solution is provable when U is sufficiently approximately linear.

5. ILLUSTRATIVE EXAMPLE

Example 1: Consider the boundary value problem

$$\begin{aligned} U'' &= -0.0002U'\int_0^Y U(s)ds + \\ &\qquad 2\left(0.01\sqrt{U^2+1}\right)^2(U-1),\ Y\in(0,1), \end{aligned} \tag{47}$$

$$U(0)=0,\ U(1)=1. \tag{48}$$

Here $R_m = 0.0002,\ M = 2,\ f(U) = 0.01\sqrt{U^2+1},\ \gamma = \sqrt{0.0002}$. Consider the auxiliary linear problem

$$\begin{aligned} U'' &= -0.0002Y'\int_0^Y sds + \\ &\qquad 2\left(0.01\sqrt{Y^2+1}\right)^2(Y-1),\ Y\in(0,1), \end{aligned} \tag{49}$$

$$U(0)=0,\ U(1)=1. \tag{50}$$

It is clear that $\|Y\|_{C(0,1)} = 1$ and $\|Y\|_{C^2(0,1)} = 2$. By analogy with (23)–(24), Λ for the current auxiliary problem can be obtained, using (25), as

$$\Lambda = 2 + 9(0.0002) \times 2 + 5 \times 2(.01\sqrt{2})^2 \times 2 = 2.0076.$$

Hence, using (34), we thus have the convex set

$$K := \{U \in C^2(0,1) | \ \|U\|_{C^2(0,1)} \leq 4.0152\},$$

in which we seek the solution of (47)–(48). Now

$$\begin{aligned} \max\{9R_m, 5\gamma^2 M\}\,(4\Lambda+1)^2 \\ &= \max\{9(.0002), 5(.0002) \times 2\}(4(2.0076)+1)^2 = \\ &= (0.002)(9.0304)^2 < \frac{1}{4}, \end{aligned} \tag{51}$$

so that the condition (37) is satisfied. By Theorem 4(i), there exists a unique solution $U \in K$ of (47)–(48). We display in Figure 1, the numerical solution of (47)–(48), which is approximately linear; and shows monotonic increase of the dimensionless velocity U from the plate across the dimensionless normal space variable Y within the boundary layer. The observed approximate linearity jells with the remark at the end Section 4.

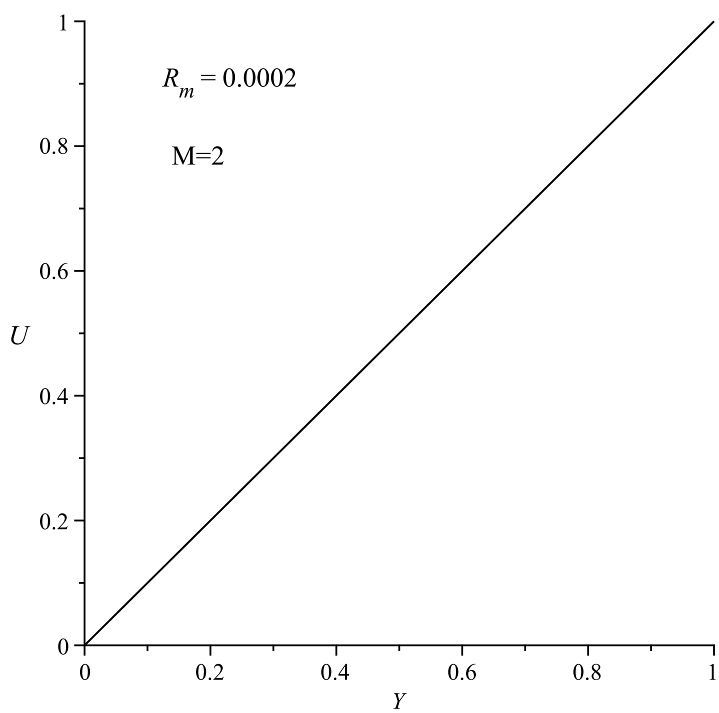

FIG. 1: Profile of the unique U against Y.

Example 2: Consider next, the boundary value problem

$$U'' = -0.2U' \int_0^Y U(s)ds + M\left(0.1\sqrt{U^2+1}\right)^2 (U-1),\ Y \in (0,1), \tag{52}$$

$$U(0) = 0,\ U(1) = 1. \tag{53}$$

Theorem 1(ii) is applicable to this example, in which the operator A defined for (52)–(53) is not contractive; so that uniqueness of solutions are not guaranteed. Numerical solutions of (52)–(53) are displayed in Figure 2 for various values of the Hartmann number M. The figure shows that the effects of the magnetic field on the dimensionless velocity become more pronounced with increase in the field. This result is in agreement with the deduction of DePuy [19], in investigating the steady state channel flow of an electrically conductive liquid exposed to transverse magnetic and electric fields.

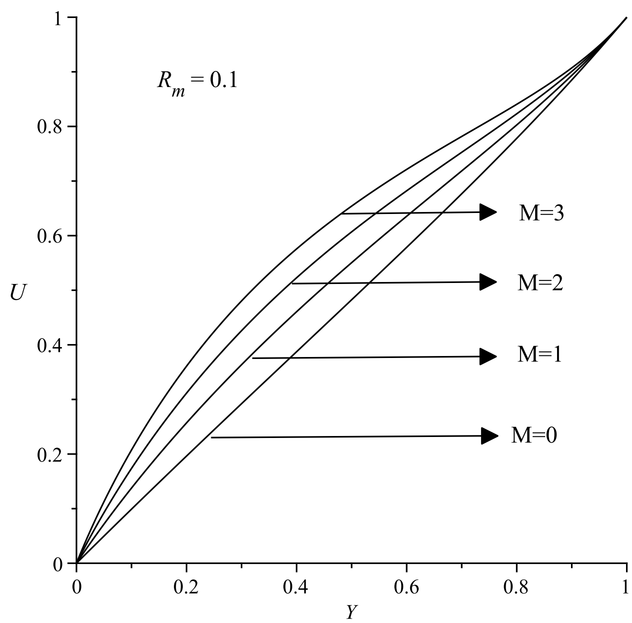

FIG 2: Profiles of U against Y for various values of M.

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REFERENCES

[1] L. Prandtl, *"Uber Flüssigkeitsbewegung bei sehr kleiner Reibung*, Verh. III. Intern. Math. Kongr., Heidelberg, 1904, Teubner, Leipzig, 1905.

[2] R. B. Bird, W. E. Stewart and E. N. Lightfoot, *Transport Phenomena*, John Wiley & Sons, Inc., 140–146, 1960.

[3] A. Olubumma, *Introduction to Real analysis*, Heinemann Educational Books, Nigeria, 1979.

[4] L. C. Evans, *Partial Differential Equations*, American Mathematical Society, Providence, Rhode Island, 1998.

[5] D. J. Griffiths, *Introduction to Electrodynamics* (3rd ed.), Prentice Hall, 1999.

[6] H. S. Takhar, R.S.R. Gorla and V.M. Soundelgekar, *Non-linear one-step method for initial value problems*, Int. Num. Meth Heat Fluid Flow, **6** 22–83, 1996.

[7] B. R. Kumar, D.R.S. Raghuraman and R. Muthucumaraswamy, *Hydromagnetic flow and heat transfer on a continuously moving vertical surface,* Acta Mech., **153** 249–253, 2002.

[8] M. A. A. Mahmoud, *Variable viscosity effects of hydromagnetic boundary layer flow along a continuously moving vertical plate in the presence of radiation*, Appl. Math. Sci., **1** (17) 799–814, 2007.

[9] S. Shateyi, P. Sibanda and S.S. Motsa,*Magnetohydrodynamic flow past a vertical plate with radiative heat transfer,* J. Heat Trans., **129** (12) 1708–1713, 2007

[10] R. K. Deka, *Hall effects on MHD flow past an accelerated plate,* Theoret. Appl. Mech., **35** (4) 333–346, 2008.

[11] Z. Hammouch, *Multiple solutions of steady MHD flow of dilatant fluids,* Eur. J. Pure Appl. Math., **1** (2) 11–20, 2008.

[12] M. A. Samad and M. Mohebujjaman, *MHD heat and mass transfer free convection flow along a vertical stretching sheet in presence of magnetic field with heat generation,* Res. J. Appl. Sci. Eng. Technol., **1** (3) 98–106, 2009

[13] T. S. Khaleque and M. A. Samad, *Effects of radiation, heat generation and viscous dissipation on MHD free convection flow along a stretching sheet,* Res. J. Appl. Sci. Eng. Technol., **2** (4) 368–377, 2010.

[14] T. Hayat, M. Nawaz and S. Obaidat, *Axisymmetric magnetohydrodynamic flow of micropolar fluid between unsteady stretching surfaces,* Appl. Math. Mech. (Engl. Ed.), **32** (3) 361–374, 2011.

[15] O. D. Makinde, *Computational modelling of MHD unsteady Flow and heat transfer over a flat plate with Navier slip and Newtonian heating,* Brazilian Journal of Chemical Engineering, **29** (1) 159–166, 2012.

[16] O. D. Makinde, W. A. Khan and Z. H. Khan, *Buoyancy effects on MHD stagnation point flow and heat transfer of a nanofluid past a convectively heated stretching/shrinking sheet,* International Journal of Heat and Mass Transfer **62** 526–533, 2013.

[17] H. Blasius, *Grenzschichten in Flüssigkeiten mit kleiner Reibung,* Z. Math. Phys., **56** 1–37, 1908.

[18] L. C. Evans, *Partial Differential Equations, American Mathematical Society,* Providence, Rhode Island, 1998.

[19] T. R. DePuy, *Fluid Dynamics and Heat Transfer in a Hartmann Flow,* MSc Thesis: Graduate Faculty of Rensselaer Polytechnic Institute, 2010.

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