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SOLUTIONS IN CLOSED-FORM FOR UNSTEADY UNIDIRECTIONAL FLOW OF A MAXWELL FLUID

(Dedicated to the memory of Professor R. O. Ayeni)

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ABSTRACT. An analytic approach based on the functional framework method is employed to solve a linear model for the flow of a Maxwell fluid past an infinite plate with or without the upper surface being free. Explicit solutions are given in the form of a theorem and the results are compared with those of Newtonian fluid when the relaxation time $\lambda = 0$ and the kinematical viscosity (ν) is unity. Also, we recover earlier known analytical solutions when $\lambda \to \infty$ and $\nu/\lambda = O(1)$.

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1. INTRODUCTION

Many fluids in manufacturing processes have been treated as viscoelastic fluids possessing both elastic and viscous properties. In [6], for instance, an exact solution corresponding to a Maxwell fluid was obtained. It was found that the solution for the special limiting case of Newtonian fluid cannot be obtained from the solution of the linear problem.

Recently, Fetecau and Fetecau [8] obtained a new exact solution for the Maxwell fluid using Fourier Sine Transform and deduced the well-known solution for Newtonian fluid.

However, other traditional analytic techniques for linear problems are Fourier Series, Laplace transform, and method of separation of variables [12]. For the fully nonlinear equations, the above techniques have no meanings at all. Perturbation techniques [13] can be applied to nonlinear case scenarios by regarding the variable, t, as a small variable. Consequently, such kind of solutions are often only valued for small time t (see for examples [4], [5], [11], [15]).

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It is worth noting that other techniques are Lyapunov's small parameter method [14], the δ-expansion method [11] and Adomian's decomposition method [2], [1] [16], [17].
Quite recently, a kind of analytic method, namely the homotopy analysis method [13] was developed to solve highly nonlinear problems (see for examples [3], [10], [19]). This technique does not depend upon any small or large parameters. Another method that is non-parameter dependent and useful for linear and nonlinear problem is the functional framework technique (see [20]-[22]) for nonlinear cases as well as [18], [23] and [24] for the linear cases)
In this paper, exact analytical solutions are obtained by the standard technique of separation of variables through the use of functional framework to reduce the governing partial differential equations to various ordinary differential equations. As a check on the functional scheme, it gives precisely the well known simple exact solutions for the Newtonian fluids when the relaxation time, $\lambda \to 0$ and $\nu = 1$ i.e Rubel, [24]. Also, simple solutions of Laplace and wave equations in two variables appear in limiting cases of our solutions. The general closed-form solutions derived in this study can be employed as a benchmark to validate numerical results.

2. PROBLEM STATEMENT

In this section, we discuss the formulation of the problem using the concept of the continuity equation, conservation of momentum and constitutive equations.
Let us consider a Maxwell fluid at rest lying over an infinitely extended plate which is situated in the $(x, z)-$ plane with the geometry to be of the form of Fig. 1. Here the y axis is perpendicular to the plate while the x-axis have negligible side effects. The fluid is set into motion through the action of the stress at the plate. Its velocity field can be written in the form

$$\mathbf{V}(t, W) = (0, 0, W(t, y)). \tag{1}$$

Hence, the continuity equation reduces to

$$\partial W / \partial z = 0, \tag{2}$$

and the momentum equation is given by

$$\rho \dot{\mathbf{V}} = \rho \mathbf{b} + \nabla . \mathbf{T}, \tag{3}$$

where $\mathbf{b}$ is the body force, ρ is the density of the fluid, $\mathbf{T}$ is the stress tensor, and the dot denotes the material time derivative.

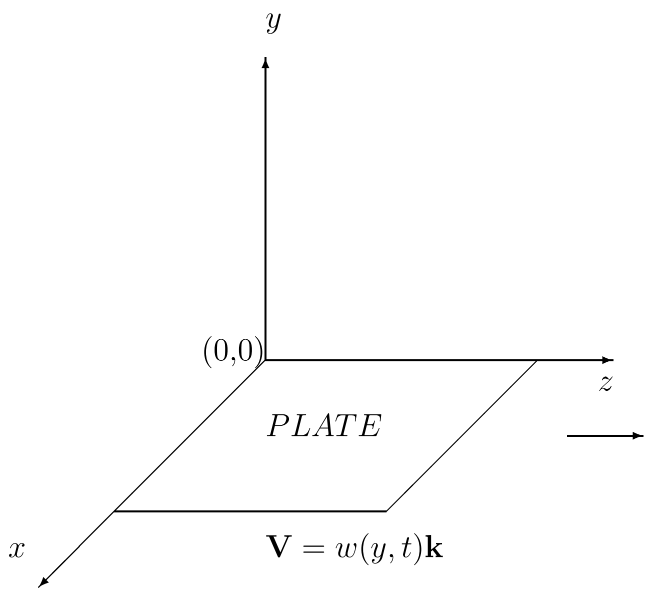

Fig. 1 : Direction of flow over an extended plate.

The incompressible Maxwell fluid is characterized by a stress tensor $\mathbf{T}$ which is related to kinematic variables by

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S} \tag{4}$$

$$\mathbf{S} + \lambda(\dot{\mathbf{S}} - \mathbf{LS} - \mathbf{SL}^T) = \mu\mathbf{A}, \tag{5}$$

where p is the pressure, $\mathbf{I}$ is a unit tensor, $\mathbf{S}$ is the extra-stress tensor, $\mathbf{L}$ is the velocity gradient, $\mathbf{A} = \mathbf{L} + \mathbf{L}^T$ which is the Rivlin-Ericksen tensor, μ is the dynamic viscosity, λ is the relaxation time and T is the transpose.

The classical formulation of this type of problem assume that the extra-stress $\mathbf{S}$ is a function of t and y only. Substituting equation (2) into equation (5) and taking into account the initial condition (the fluid being at rest up to the moment $t = 0$).

$$S(t; 0, y, 0) = 0, \tag{6}$$

we obtain $S_{xx} = S_{xy} = S_{xz} = S_{yy} = 0$, and $S_{yz} = S_{zy}$ (symmetry principle) is the shear stress and S_{zz} is the normal stress (see [8] for example).

Then (5) reduces to

$$\tau + \lambda\frac{\partial\tau}{\partial t} = \mu\frac{\partial W}{\partial y}, \tag{7}$$

where $\tau = S_{zy}$.

Neglecting body forces and in the absence of pressure gradient in the z-direction, the linear momentum balance equation take the form [7]

$$0 = -\frac{\partial P}{\partial x}, \tag{8}$$

$$0 = -\frac{\partial P}{\partial y}, \tag{9}$$

$$\rho\frac{\partial W}{\partial t} = \frac{\partial \tau}{\partial y}. \tag{10}$$

On differentiating equation (7) with respect to y, we obtain

$$\frac{\partial \tau}{\partial y} + \lambda\frac{\partial^2 \tau}{\partial y \partial t} = \mu\frac{\partial^2 W}{\partial y^2}. \tag{11}$$

Substituting equation (10) into (11), we finally obtain

$$\frac{\partial W(t,y)}{\partial t} + \lambda\frac{\partial^2 W(t,y)}{\partial t^2} = \nu\frac{\partial^2 W(t,y)}{\partial y^2},\ y > t > 0 \tag{12}$$

where $\nu = \mu/\rho$ is the kinematic viscosity.

3. MAIN RESULTS

We now state and prove our main theorem.

Theorem: Suppose $W(t,y) = \phi(A(y) + B(t))$ is a non-constant real solution of the Maxwell fluid equation $\lambda W_{tt} + W_t = \nu W_{yy}$ for non-negative constants λ, ν, m_0, α_1, α_4, α_{44}, m_1, a_0, e, R, C, D, D_*, F, G_0, G_1, H_0, H_1, K_0, K_1, L_0 and L_1 such that $A(y)$ and $B(t)$ are real-analytic functions. Then w must have at least one of the following:

$$W(t,y) = m_0(a_0 y - C\lambda \exp\left(-\frac{t}{\lambda}\right) + R) + m_1, \tag{13}$$

$$W(t,y) = m_0(\frac{\alpha}{2\nu}y^2 + e_0 y + \alpha t - \lambda D \exp\left(-\frac{t}{\lambda}\right) + D_*) + m_1, \tag{14}$$

$$W(t,y) = F\exp\left(\left(\sqrt{\frac{1}{\nu}(\alpha_1 + \lambda\alpha_1^2)}\right) y + \alpha_1 t\right), \tag{15}$$

$$W(t,y) = F\exp\left(\left(-\sqrt{\frac{1}{\nu}(\alpha_1 + \lambda\alpha_1^2)}\right) y + \alpha_1 t\right), \tag{16}$$

$$W(t,y) = G_1\exp(\alpha_1 t)\sinh\left(\sqrt{\frac{1}{\nu}(\alpha_1 + \lambda\alpha_1^2)}\right)(y - G_0), \tag{17}$$

$$W(t,y) = H_1 \exp(-\alpha_1 t) \sin\left(\sqrt{\frac{1}{\nu}(\alpha_1 - \lambda\alpha_1^2)}\right)(y - H_0), \tag{18}$$

$$W(t,y) = K_1 \exp\left(\alpha_4 y - \frac{t}{2\lambda}\right) \sinh\sqrt{\frac{\nu\alpha_4^2}{\lambda} + \frac{1}{4\lambda^2}}(t - K_0), \tag{19}$$

$$W(t,y) = L_1 \exp\left(\pm i\alpha_{44} y - \frac{t}{2\lambda}\right) \cos\sqrt{\frac{\nu\alpha_{44}^2}{\lambda} - \frac{1}{4\lambda^2}}(t + L_0). \tag{20}$$

In addition, each of (13)-(20) is a solution of the form $\phi(A(y) + B(t))$.

Proof: We start by supposing

$$W(t,y) = \phi(A(y) + B(t)). \tag{21}$$

In what follows, the argument of ϕ, $\phi^{''}$, $\phi^{'''}$ is always $A + B$, the argument of A, $A^{'}$, $A^{''}$, $A^{'''}$ is always y and the argument of B, $B^{'}$, $B^{''}$, $B^{'''}$ is always t, So that

$$\lambda W_{tt} + W_t = \nu W_{yy}, \tag{22}$$

becomes

$$\phi^{''}(\nu(A^{'})^2 - \lambda(B^{'})^2) = \phi^{'}(B^{'} + \lambda B^{''} - \nu A^{''}), \tag{23}$$

which can also be written as

$$\frac{\phi^{''}}{\phi^{'}} = \frac{B^{'} + \lambda B^{''} - \nu A^{''}}{\nu(A^{'})^2 - \lambda(B^{'})^2}. \tag{24}$$

We suppose that,

$$B^{'} + \lambda B'' - \nu A^{''} = 0, \tag{25}$$

then

$$B^{'} + \lambda B^{''} = \nu A^{''} = \alpha, \tag{26}$$

for some constant α.

Case 1: Let $\alpha = 0$, then it is easy to show that

$$B = -C\lambda \exp(-t/\lambda) \text{ and } A = a_0 y + a_1, \tag{27}$$

which leads to equation (13). When $\lambda = R = m_1 = 0$ and $\nu = m_0 = a_0 = 1$ then we obtain $W(t,y) = y$, which is in accordance with Theorem 3 and equation (3.1) of Rubel [24].

Case 2: Let us do the case for any positive or negative number α. In this case, it is evident that $A = \alpha y^2/2\nu + e_0 y + e_1$ and $B = \alpha t - \lambda D \exp(-t/\lambda) + D_1)$ satisfied equation (26). Then

$$W(t,y) = m_0(\frac{1}{2\nu}y^2 + e_0 y + t - \lambda D \exp(-t/\lambda) + D_1) + m_1, \tag{28}$$

so that we have the solution given in eqn (14).
In the limit case when $\lambda = 0$ and $\nu = 1$,$m_0 = 1$ and $e_0 = D_* = m_1 = 0$ our solution goes to $W(t, y) = (y^2/2 + t)$ which is Theorem 3 and equation (3.2) of (Rubel [24]).

Case 3 & 4: We will now provide arguments for the case

$$B^{'} + \lambda B^{''} - \nu A^{''} \neq 0. \tag{29}$$

First, we suppose that $B^{'''} = 0$ then we have $B = \alpha_0 t^2 + \alpha_1 t + \alpha_2$ with α_i ($i = 1, 2$ and 3) being constants. Then, from equation (24), we have

$$\frac{\phi^{''}}{\phi^{'}} = \frac{\alpha_1 - \nu A^{''}}{\nu(A^{'})^2 - \lambda\alpha_1^2} = \Phi(A + B), \tag{30}$$

provided $\alpha_0 = 0$. and taking $\partial/\partial t$, we get

$$\Phi^{'} B^{'} = 0, \tag{31}$$

so that,

$$\Phi^{'}(s) = 0, \tag{32}$$

where $s = A + B$, then

$$\Phi = \sigma = \text{constant}. \tag{33}$$

From (30), we have

$$\frac{\phi^{''}}{\phi^{'}} = \sigma. \tag{34}$$

Therefore,

$$\phi(s) = \frac{D_3}{\sigma}\exp(\sigma s) + D_4, \tag{35}$$

where D_3 and D_4 are constants. Without loss of generality, we assume that $D_3 = 1$ and $D_4 = 0$. Take $\sigma = 1$ and since the solution is of the form $W(t, y) = \phi(A(y) + B(t))$, then

$$W(t, y) = \exp(A(y) + \alpha_1 t + \alpha_2). \tag{36}$$

Substituting (36) into (12), we get the nonlinear ordinary differential equation

$$A^{''} + (A^{'})^2 = \frac{1}{\nu}(\alpha_1 + \lambda\alpha_1^2). \tag{37}$$

The simple solution of the differential equation (37) is found to be

$$A = \pm\sqrt{\frac{1}{\nu}(\alpha_1 + \lambda\alpha_1^2)}y + F_1, \tag{38}$$

where F_1 is a constant. Then we have (15) and (16) after substituting (38) into (36).
Investigating the limiting cases with the value of $\lambda = F_1 = 0$

and $\nu = \alpha_1 = F = 1$, we have the $W(t,y) = \exp(y+t)$ and $W(t,y) = \exp(-y+t)$ which yields Theorem 3, equations (3.3) and (3.4) in Rubel [24].

Case 5: To obtain another solution of equation (37), we employed the method of reduction of order.
Let $A^{'} = P$, so we have

$$\frac{dP}{dy} + P^2 = \frac{1}{\nu}(\alpha_1 + \lambda\alpha_1^2). \tag{39}$$

Using method of separation of variables in (39), we have

$$\frac{dP}{\left(\sqrt{\frac{1}{\nu}(\alpha_1 + \lambda\alpha_1^2)}\right)^2 - P^2} = dy. \tag{40}$$

Integrating (40), we obtain

$$y = \frac{1}{\left(\sqrt{\frac{1}{\nu}(\alpha_1 + \lambda\alpha_1^2)}\right)} \coth^{-1}\left(\frac{P}{\sqrt{\frac{1}{\nu}(\alpha_1 + \lambda\alpha_1^2)}}\right) + G_0, \tag{41}$$

where G_0 is a constant. It follows that

$$P = \sqrt{\frac{1}{\nu}(\alpha_1 + \lambda\alpha_1^2)} \coth\left(\sqrt{\frac{1}{\nu}(\alpha_1 + \lambda\alpha_1^2)}\right)(y - G_0) = A^{'}. \tag{42}$$

Hence

$$A = \ln\sinh\left(\sqrt{\frac{1}{\nu}(\alpha_1 + \lambda\alpha_1^2)}\right)(y - G_1) \tag{43}$$

Substituting (43) into (36), we have (17).
The solution gives the fifth solution of an existing one () with the value of Let $\lambda = G_0 = 0$ and $\nu = \alpha_1 = G_1 = 1$ then we have precisely the form $W(t,y) = \exp(t)\sinh y$. It follows that we have Theorem 3 and equation (3. 5) of Rubel [24].

Case 6: Next we continue the solution by considering the case when $B(t) = -\alpha_1 t + \alpha_2$, so that $W(t,y) = \phi(A(y) + B(t))$ becomes

$$W(t,y) = \exp(A(y) - \alpha_1 t + \alpha_2). \tag{44}$$

Substituting (44) into (12), we get

$$A^{''} + (A^{'})^2 = -\frac{1}{\nu}(\alpha_1 - \lambda\alpha_1^2) \tag{45}$$

The computation for (45) is similar to the previous case and therefore

$$A = \ln \sin \left(\sqrt{\frac{1}{\nu}(\alpha_1 - \lambda \alpha_1^2)} \right) (y - H_0) + H_1. \tag{46}$$

Substituting A into (44), we get

$$W(t, y) = H_1 \exp(-\alpha_1 t) \sin \sqrt{\frac{1}{\nu}(\alpha_1 - \lambda \alpha_1^2)}(y - H_0), \tag{47}$$

which is equivalent to (18). Suppose $\lambda = H_0 = 0$, $\nu = \alpha_1 = H_1$ we obtain the solution $W(y, t) = \exp(-t) \sin y$ verifying Theorem 3 and equation (3.6) in Rubel [24].

Case 7: Secondly, we suppose that $A''' = 0$ then we have $A = \alpha_3 y^2 + \alpha_4 y + \alpha_5$ with α_i ($i = 3, 4$ and 5) being constants which means that (24) reduces to

$$\lambda(B'' + (B')^2) + B' = \nu \alpha_4^2, \tag{48}$$

provided that $\alpha_3 = 0$. Using the method of reduction of order, separation of variables and integrating twice, it can be verified that

$$B = \ln \sinh \left(\sqrt{\frac{\nu \alpha_4^2}{\lambda} + \frac{1}{4\lambda^2}}(t - K_0) \right) - \frac{t}{2\lambda} + K_2,\ \lambda \neq 0, \tag{49}$$

where k_0 and K_2 are constants.
Therefore, the general solution is

$$W(t, y) = K_1 \exp(\alpha_4 y - \frac{t}{2\lambda}) \sinh \sqrt{\frac{\nu \alpha_4^2}{\lambda} + \frac{1}{4\lambda^2}}(t - K_0),\ \lambda \neq 0. \tag{50}$$

Case 8: Finally, we suppose $A = i\alpha_{44} y + \alpha_5$ where α_{44} is a constant and $i^2 = -1$, it follows that the equation for B is

$$\lambda(B'' + (B')^2) + B' = -\nu \alpha_{44}^2. \tag{51}$$

Problem (51) has the solution

$$B = \ln \cos \left(\sqrt{\frac{\nu \alpha_{44}^2}{\lambda} - \frac{1}{4\lambda^2}}(t + L_0) \right) - \frac{t}{2\lambda} + L_2,\ \lambda \neq 0, \tag{52}$$

The general solution can be written as

$$W(t, y) = L_1 \exp(\pm i\alpha_{44} y - \frac{t}{2\lambda}) \cos \sqrt{\frac{\nu \alpha_{44}^2}{\lambda} - \frac{1}{4\lambda^2}}(t + L_0),\ \lambda \neq 0. \tag{53}$$

Observations 1: It is not difficult to show that when $1/\lambda \to 0$ and $\nu/\lambda = O(1)$, then equation (12) reduces to the wave equation $w_{tt} = w_{yy}$ with the solution $w(t,y) = \cos t \cos y$ when $L_0 = 0$ and $L_1 = 1$. Hence Theorem 2 and equation (2.1) holds in Rubel [24].

Observations 2: Since W(t, y) is the solution of Laplace equation in two variables, if W(t, iy) is a solution of the wave equation in two variables, then we have that $W(t,y) = \cos t \cosh y$ as contained in Theorem 1 and equation (1.1) of Rubel [24]..

Observations 3: When both of the plates (at a distance h apart) execute simple harmonic motion with the same amplitude and frequency in the form $W = \alpha \exp(-i\lambda_0 t)$ as contained in [7], we have for this special case that

$$W(t,y) = \alpha \exp(-i\lambda_0 t) \frac{\cos \alpha_{44}(h/2 - y)}{\cos \alpha_{44} h/2}, \tag{54}$$

where $\lambda_0 = \sqrt{\nu\alpha_{44}^2/\lambda - 1/(4\lambda^2)}$ and $\lambda \neq 0$.
A quantity of interest is the frictional force at the upper plate $z = h$ which is given by the following equation

$$P_x = \mu \frac{\partial W}{\partial z}\bigg|_{z=h} = -\alpha\alpha_{44}\mu \exp(-i\lambda_0 t) \tan \frac{\alpha_{44} h}{2}. \tag{55}$$

Observations 4: When the lower plate at time $t = 0^+$ is impulsively brought to the constant velocity and the upper surface being free as in [6] and [8], our solution satisfied only three out of the five conditions.

For the second ("In addition") part, we can easily check by hand computation.
The proof of the theorem is now complete.

Remarks: (1) At this point, we want to scrutinize the only case remaining. Here,

$$B^{''}(t) \neq 0, \; B^{'} + \lambda B^{''} - \nu A^{''} \neq 0. \tag{56}$$

Then equation (22) reduces to

$$-\frac{\phi^{'}}{\phi^{''}} = \frac{\nu(A^{'})^2 - \lambda(B^{'})^2}{B^{'} + \lambda B^{''} - \nu A^{''}} = \Phi(A+B). \tag{57}$$

Making use of the Jacobian determinant J, we can also write

$$J\left(\frac{\nu(A^{'})^2 - \lambda(B^{'})^2}{B^{'} + \lambda B^{''} - \nu A^{''}}, A+B\right) = 0. \tag{58}$$

Expanding (58) and simplifying, we get

$$\nu \frac{B'' + \lambda B'''}{B'} = \frac{F(A', A'', A''', B', B'', B''')}{(A')^3}, \tag{59}$$

where $F(\cdot) = -\lambda A' B' B'' - 2\nu A' A'' B' - 2\lambda^2 A' (B'')^2 + \nu A' (A'')^2 - \nu (A')^2 A''' + \lambda^2 A' B' B''' + \nu \lambda A''' (B')^2$. Differentiating equation (59) with respect to t, this yields the equation

$$\left(\nu \frac{B'' + \lambda B'''}{B'} \right)' = \frac{F_t(\cdot)}{(A')^3}, \tag{60}$$

where $F_t(\cdot) = -\lambda A' (B' B'')' - 2\nu A' A'' B'' - 2\lambda^2 A' \{(B'')^2\}'$ $+\lambda^2 A' (B' B''')' + \nu \lambda A''' \{(B')^2\}'$. Continuing with the differentiation reveals that the variables are not separable. Letting $\lambda = 0$ and $\nu = 1$ in equation (60) and supposing also that $B'' \neq 0$, we get

$$\frac{2A''}{(A')^2} = \left(\frac{B''}{B'} \right)' / B''. \tag{61}$$

The solution to equation (61) which satisfies the functional dependence (24) was obtained by Rubel [24], who also showed that

$$w = \mathrm{erf} \left(\frac{y}{\sqrt{t}} \right). \tag{62}$$

(2) We present new possible set of solutions for the Maxwell fluid flow and our results generalizes earlier known solutions.

4. CONCLUDING REMARKS

Emerging equations from an unsteady flow of a non-Newtonian fluid past an infinite plate is investigated analytically using functional framework. Maxwell rheological model is used as the constitutive relation. The results show that at least eight solutions can be obtained using separation of variables method.
The investigation carried out in this study gives possible general solutions of a Newtonian fluid (heat equation) which can be obtained when the relaxation time $\lambda = 0$ and the kinematic viscosity $\nu = 1$. In addition, when $\lambda \to \infty$ and $\nu/\lambda = O(1)$, we recover earlier published exact solutions.

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