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ON THE DYNAMIC ANALYSIS OF A TAPERED TIMOSHENKO BEAM UNDER A UNIFORM PARTIALLY DISTRIBUTED MOVING LOAD

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ABSTRACT. In this paper, the analysis of a variable cross-section Timoshenko beam subjected to a moving partially distributed load was carried out. Finite element method with Langrangian interpolation functions and reduced integration element was used to model the structure by discretizing the structure's domain into finite beam elements and deriving the semi-discrete differential equations which are the elemental and assembled stiffness, mass, and centripetal matrices and load vector. The Newmark direct integration method was used to solve the resulting semi-discrete time dependent equations to obtain the desired responses. Important features of the analysis were investigated and discussed.

Keywords and phrases: Dynamic Analysis, Tapered Timoshenko Beam, Finite Element method, Moving load, Reduced integration.
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1. INTRODUCTION

It is well known that static loads (forces) are functions of the spatial variables only, while dynamic loads are functions of time as well as the spatial variables. However, dynamic loads which are, in addition, continuously changing their positions are known as moving loads. Examples include trains, cars, trucks, cranes, and pedestrians walking or running across bridges. Moving loads usually cause elastic structures, such as beams, on which they act to vibrate intensively particularly when high velocities are involved. The problem of assessing the response of elastic structures to moving loads known as moving load problem is of technological importance. For instance, elastic structures are commonly used in the design of aircrafts which are under the influence of various types of moving pressure loads during flight [5]. Hence, problems of analysing the

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dynamic response of elastic structures under the action of moving loads continue to motivate a variety of investigations [1-13, 15-21, 23-25, 28-30]. Such investigations are found in civil, mechanical, transport, astronautical and marine engineering as well as applied mathematics, since moving loads are present in all these fields [9]. Significant early contribution towards solving various types of moving load problems were made by Wills[28], Stokes[25], Zimmermann[30], Krylov[17], Timoshenko[27], Lowan[19], Bolotin[4], Inglis[15], Hillerborg[13], and Kolousek[16]. Later, an extended review on this subject was carried out by Fryba[9] in his excellent monograph. The dynamic response of a simply supported beam traversed by a concentrated moving load was determined by Stanistic and Hardin [24]. They developed an interesting technique which, however, cannot easily be applied to various boundary conditions which are of practical interest. Akin and Mofid [3] presented an analytic numerical method that can be used to determine the dynamic behaviours of beams carrying a concentrated moving mass. The problem of dynamic behaviour of an elastic beam subject to a moving concentrated mass was also studied by Sadiku and Leipholz [23]. Gbadeyan and Oni [12] presented a more versatile technique which can be used to determine the dynamic behaviour of beams having arbitrary end supports. Michaltos, Sophianopoulos and Kaounadis [21] studied the effect of the mass of a moving load on the dynamic response of a simply supported beam. A detailed analysis of the effect of centripetal and coriolis forces on the dynamic response of light (steel) bridges under moving loads was also carried out by Michaltos and Kounadis [20]. It is remarked at this juncture that the elastic parameter of the beams in all the work discussed hitherto, are assumed constant. In other words, uniform beams were considered. The reason for this is not farfetched since by making such an assumption, the various researchers ended up with the governing partial differential equations having constant coefficients only and thereby based the aforementioned investigations, in general, on analytical approaches. Otherwise, the researcher could have found it very difficult, if not impossible, to obtain analytical closed form solutions to the problems. However, practical structures for which the elastic properties are functions of spatial coordinates abound. Hence in this paper, beams that are not uniform are considered.

Some of the previous works involving non-uniform beams include that of Wu and Dai [29]. They studied the dynamic response of

multi-span non-uniform beams under moving loads using the transfer matrix method. Dugush and Eisenberger [6] also investigated the dynamic behaviour of multi-span non-uniform beams traversed by a moving load at constant and variable velocities. They used both modal analysis and direct methods. The analysis of a variable cross-section beam subject to a moving concentrated mass is investigated in [2] using finite element method.

Although, the above completed works on both uniform and non-uniform beams are impressive, only concentrated moving loads were considered. However, such loads do not represent the physical reality of the problem formulation. As a matter of fact, concentrated loads do not exist physically. For practical applications, it is realistic to consider the moving loads as distributed moving loads as opposed to concentrated moving loads. Hence, the present paper deals with the more realistic moving load, namely, uniform partially distributed moving load. The first work on moving loads, to the best knowledge of the authors, to consider distributed moving loads was that of Esmailzade and Ghorashi [7]. The work in [7] was extended by the same authors [8] to include the vibration of a Timoshenko beam. In [10], the vibration analysis of beams traversed by distributed moving masses was studied. Gbadeyan and Dada [11] also studied, recently, the effect of linearly varying distributed moving load on beams. Abiala[1] studied the effects of lateral loads on the dynamic response of beams using finite element method, where he showed that the velocity of the moving load and span length had significant affects on the dynamic response of the beam.

Furthermore, most of these studies on moving load problems are limited to Euler-Bernoulli beam elements with the assumption that the transverse shear strain is equal to zero. When the transverse shear strain is not equal to zero the beam in question becomes a Timoshenko beam. The dynamic analysis of Timoshenko beam using finite element method has been studied for a while. Lou, Dai and Zeng[18] presented a finite element formulation of a Timoshenko beam, where the equation of motion was derived from the variational approach and the equation was solved by the step-by-step integration method. Thomas and Abbas[26] presented a finite element model with nodal degrees of freedom which can satisfy all the forces and natural boundary conditions of a Timoshenko beam.

In the context discussed so far, this paper therefore, focuses on the dynamic behaviour of a tapered Timoshenko beam that is subjected to uniform partially distributed moving loads. Specifically, the elastic properties of the beam such as the cross-sectional area and moment of inertia which are usually assumed constants are hereby expressed as functions of the spatial variables. The dynamic responses of the non-uniform beams subjected to uniform partially distributed moving loads were then analysed using the finite element technique with quadratic Langrange interpolation functions. Firstly the non-uniform continuous beam was replaced by a non-continuous (discrete) system made up of beam elements. The semi-discrete, time dependent elemental and overall stiffness, mass, and centripetal matrices as well as the elemental and overall load vectors were then derived. Newmark method was used to obtain the desired responses. The key points of interest in this paper were to evaluate the effect of the following parameters: (i) the speed of the moving load; (ii) the length of the moving load; (iii) the span length of the beam; (iv) different boundary conditions of the beam; (v) the time history on the mid-span and end of the beam; and (vi) the load-beam mass ratio on the response of the beam.

2. MATHEMATICAL PROBLEM STATEMENT

We consider the problem of determining the behaviour of a non-uniform Timoshenko beam carrying a load moving at a specified speed.

The governing equation is given as:

$$\frac{\partial}{\partial x}\left[kGA\left(-\frac{\partial u}{\partial x}+w\right)\right]+\rho A\frac{\partial^2 u}{\partial t^2}-q(x,t)=0$$

$$-\frac{\partial}{\partial x}\left[EI\frac{\partial w}{\partial x}\right]+\rho I\frac{\partial^2 w}{\partial t^2}+kGA\left[-\frac{\partial u}{\partial x}+w\right]+\rho_q I_q\frac{d^2 w}{dt^2}D=0 \quad (1)$$

$u(x,t)$ is the deflection of the beam axis, and $w(x,t)$ is the rotation of its cross-section. ρ and ρ_q are the respective densities of the beam and the load; while I and I_q are the corresponding moments of inertia of their cross-sectional areas, respectively. $A(x)$ is the cross-sectional area of the beam; E - the elastic modulus; G - the shear modulus; k - the shear coefficient; $q(x,t)$ - the distributed load; t is time, and x is the position coordinate in the axial direction $[x\epsilon(0,L)]$.

The boundary conditions are: (simply-supported)

$$u(0,t) = u(L,t) = 0 \ ; \ EI \left.\frac{\partial w}{\partial x}\right|_{x=0} = EI \left.\frac{\partial w}{\partial x}\right|_{x=L} = 0 \tag{2}$$

The associated initial conditions are:

$$u(x,0) = u_t(x,0) = 0 \ ; \ w(x,0) = w_t(x,0) = 0 \tag{3}$$

For uniformly distributed load $q(x,t)$, we have

$$q(x,t) = \frac{1}{\varepsilon}\left[-pg - p\left(\frac{d^2u}{dt^2}\right)\right] D \tag{4}$$

The factor D and total derivatives in (1) and (4) are:

$$D = H\left[x - \xi + \frac{\varepsilon}{2}\right] - H\left[x - \xi - \frac{\varepsilon}{2}\right] ; \ \frac{\varepsilon}{2} \le t \le \frac{L}{v}$$

$$\frac{d^2u}{dt^2} = \frac{\partial^2 u}{\partial t^2} + 2v\frac{\partial^2 u}{\partial x \partial t} + v^2\frac{\partial^2 u}{\partial x^2}$$

$$\frac{d^2w}{dt^2} = \frac{\partial^2 w}{\partial t^2} + 2v\frac{\partial^2 w}{\partial x \partial t} + v^2\frac{\partial^2 w}{\partial x^2} \tag{5}$$

p is the mass of the load, and g is the acceleration due to gravity, ε is the load's length, ξ is the distance covered by the load, v is the moving speed of the load, and $H(x)$ is the Heaviside function.

Now, using (4) and (5) in (1), we obtain

$$-\frac{\partial}{\partial x}\left[kGA\left(\frac{\partial u}{\partial x} + w\right)\right] + \rho A\frac{\partial^2 u}{\partial t^2} - \frac{1}{\varepsilon}\left[-pg - p\left(\frac{\partial^2 u}{\partial t^2}\right.\right.$$

$$\left.\left.+2v\frac{\partial^2 u}{\partial x \partial t} + v^2\frac{\partial^2 u}{\partial x^2}\right)\right]\left[H\left(x - \xi + \frac{\varepsilon}{2}\right) - H\left(x - \xi - \frac{\varepsilon}{2}\right)\right] = 0$$

$$-\frac{\partial}{\partial x}\left[EI\frac{\partial w}{\partial x}\right] + \rho I\frac{\partial^2 w}{\partial t^2} + kGA\left[\frac{\partial u}{\partial x} + w\right] + \rho_q I_q\left[\frac{\partial^2 w}{\partial t^2}\right.$$

$$\left.+2v\frac{\partial^2 w}{\partial x \partial t} + v^2\frac{\partial^2 w}{\partial x^2}\right]\left[H\left(x - \xi + \frac{\varepsilon}{2}\right) - H\left(x - \xi - \frac{\varepsilon}{2}\right)\right] = 0 \tag{6}$$

Also, the moment of inertia and area of beam cross-section of the beam are defined respectively as

$$I(x) = I_0\left[1 - \beta_b\frac{x}{L}\right]\left[1 - \beta_h\frac{x}{L}\right]^3$$

$$A(x) = A_0\left[1 - \beta_b\frac{x}{L}\right]\left[1 - \beta_h\frac{x}{L}\right] \tag{7}$$

By [14], I is the variable moment of inertia of the beam, and A, the variable area of beam cross-section. L is the length of the element,

β_b and β_h are functions of the taper ratios of the beam α_b and α_h respectively.

The initial boundary value problem (IBVP) describing the behaviour of a non-uniform Timoshenko beam, traversed by uniform partially distributed moving load is governed by (6), (7), (2), and (3).

3. FINITE ELEMENT FORMULATION OF THE PROBLEM

The coupled equations (1), of the Timoshenko beam theory are time-dependent second order equations. The finite element formulation of such problems involves spatial approximation, which results in a semi-discrete system of equations in time; and time approximation, where the semi-discrete system of equations which are a set of ordinary differential equations, are further approximated to obtain a set of algebraic equations.

The semi-discrete formulation involves approximation of the spatial variation of the dependent variables, the first step of which is the construction of the weak form of equations (6) over a typical element $\Omega^e = (0, l_e)$.

With the direct integration of the expression of D in (6) given as,

$$\int_0^{l_e} f(x)\left[H\left(x-\xi+\frac{\varepsilon}{2}\right)-H\left(x-\xi-\frac{\varepsilon}{2}\right)\right]dx = \int_{\xi-\frac{\varepsilon}{2}}^{\xi+\frac{\varepsilon}{2}} f(x)dx \tag{8}$$

the weak form becomes,

$$\begin{aligned}
&\int_0^{l_e}\left[kGA\frac{dR_1}{dx}\left(\frac{\partial u}{\partial x}+w\right)+R_1\rho A\frac{\partial^2 u}{\partial t^2}\right]dx \\
+\ &\frac{pg}{\varepsilon}\int_{\xi-\frac{\varepsilon}{2}}^{\xi+\frac{\varepsilon}{2}} R_1 dx+\frac{p}{\varepsilon}\int_{\xi-\frac{\varepsilon}{2}}^{\xi+\frac{\varepsilon}{2}}\frac{\partial^2 u}{\partial t^2}R_1 dx+\frac{2pv}{\varepsilon}\int_{\xi-\frac{\varepsilon}{2}}^{\xi+\frac{\varepsilon}{2}}\frac{\partial^2 u}{\partial x\partial t}R_1 dx \\
+\ &\frac{pv^2}{\varepsilon}\int_{\xi-\frac{\varepsilon}{2}}^{\xi+\frac{\varepsilon}{2}}\frac{\partial^2 u}{\partial x^2}R_1 dx - R_1(l_e)Q_3^e + R_1(0)Q_1^e = 0
\end{aligned} \tag{9}$$

$$\begin{aligned}
&\int_0^{l_e}\left[EI\frac{dR_2}{dx}\frac{\partial w}{\partial x}+R_2\rho I\frac{\partial^2 w}{\partial t^2}+R_2 kGA\left(\frac{\partial u}{\partial x}+w\right)\right]dx \\
+\ &\rho_q I_q\int_{\xi-\frac{\varepsilon}{2}}^{\xi+\frac{\varepsilon}{2}}\frac{\partial^2 w}{\partial t^2}R_2 dx+2v\rho_q I_q\int_{\xi-\frac{\varepsilon}{2}}^{\xi+\frac{\varepsilon}{2}}\frac{\partial^2 w}{\partial x\partial t}R_2 dx
\end{aligned}$$

$$+ \quad v^2 \rho_q I_q \int_{\xi-\frac{\varepsilon}{2}}^{\xi+\frac{\varepsilon}{2}} \frac{\partial^2 w}{\partial x^2} R_2 dx - R_2(l_e) Q_4^e + R_2(0) Q_2^e = 0 \quad (10)$$

In (9) and (10), $R_1(x)$ and $R_2(x)$ are the weight functions. $Q_i^e (i = 1, 2, 3, 4)$, as defined in (11), are the shear forces and bending moments respectively at the boundaries.

$$Q_1^e = \left[kGA \left(\frac{\partial u}{\partial x} + w \right) \right]_{x=0} \quad ; \quad Q_2^e = \left[EI \frac{\partial w}{\partial x} \right]_{x=0}$$

$$Q_3^e = \left[kGA \left(\frac{\partial u}{\partial x} + w \right) \right]_{x=l_e} \quad ; \quad Q_4^e = \left[EI \frac{\partial w}{\partial x} \right]_{x=l_e} \quad (11)$$

Having constructed the weak form of (7) as (9) and (10), we next obtain the approximation functions for the variables u and w. Using the same degree of interpolation to approximate both u and w in the usual manner, results in *shear locking* [22]. To avoid locking, the present study employs an *equal* interpolation for u and w, with *reduced* integration of the shear stiffnesses. The resulting approximation functions are:

$$\left. \begin{array}{l} u(x,t) = \sum_{j=1}^{3} \Phi_j(x) u_j(t) = [\Phi] \{u\} \\ \\ w(x,t) = \sum_{j=1}^{3} \Psi_j(x) w_j(t) = [\Psi] \{w\} \end{array} \right\} j = 1, 2, 3 \quad (12)$$

where

$$\begin{array}{ll} \Phi_1(x) = \left(1 - \frac{3x}{l} + \frac{2x^2}{l^2}\right), & \Phi_2(x) = \left(\frac{4x}{l} - \frac{4x^2}{l^2}\right), \\ \Phi_3(x) = \left(-\frac{x}{l} + \frac{2x^2}{l^2}\right), & \Psi_1(x) = \left(1 - \frac{3x}{l} + \frac{2x^2}{l^2}\right), \\ \Psi_2(x) = \left(\frac{4x}{l} - \frac{4x^2}{l^2}\right), & \Psi_3(x) = \left(-\frac{x}{l} + \frac{2x^2}{l^2}\right) \end{array} \quad (13)$$

$\Phi_j(x)$ and $\Psi_j(x)$, $j = 1, 2, 3$, are the Lagrange quadratic approximation functions.

4. THE FINITE ELEMENT EQUATIONS

The discrete equations of motion for a typical element of the problem under consideration are obtained by employing the Rayleigh-Ritz technique. Hence we have:

$$\sum_{j=1}^{3} \left[\int_0^{l_e} kGA \frac{d\Phi_i}{dx} \frac{d\Phi_j}{dx} u_j dx + \int_0^{l_e} \rho A \Phi_i \Phi_j \frac{d^2 u_j}{dt^2} dx \right.$$

$$+\frac{p}{\varepsilon}\int_{\xi-\frac{\varepsilon}{2}}^{\xi+\frac{\varepsilon}{2}}\Phi_i\Phi_j\frac{d^2u_j}{dt^2}dx+\frac{2pv}{\varepsilon}\int_{\xi-\frac{\varepsilon}{2}}^{\xi+\frac{\varepsilon}{2}}\Phi_i\frac{d\Phi_j}{dx}\frac{du_j}{dt}dx$$

$$+\frac{pv^2}{\varepsilon}\int_{\xi-\frac{\varepsilon}{2}}^{\xi+\frac{\varepsilon}{2}}\Phi_i\frac{d^2\Phi_j}{dx^2}u_jdx+\int_0^{l_e}kGA\frac{d\Phi_i}{dx}\Psi_kw_kdx\Bigg]$$

$$+\frac{pg}{\varepsilon}\int_{\xi-\frac{\varepsilon}{2}}^{\xi+\frac{\varepsilon}{2}}\Phi_idx+Q_i=0 \tag{14}$$

$Q_i=\Phi_i(0)Q_1^e-\Phi_i(l_e)Q_3^e\ ;\ i=1,2,3.$

$$\sum_{j=1}^{3}\Bigg[\int_0^{l_e}\left[EI\frac{d\Psi_i}{dx}\frac{d\Psi_k}{dx}w_k+kGA\Psi_i\Psi_kw_k\right]dx+\int_0^{l_e}\rho I\Psi_i\Psi_k\frac{d^2w_k}{dt^2}dx$$

$$+\rho_qI_q\int_{\xi-\frac{\varepsilon}{2}}^{\xi+\frac{\varepsilon}{2}}\Psi_i\Psi_k\frac{d^2w_k}{dt^2}dx+2v\rho_qI_q\int_{\xi-\frac{\varepsilon}{2}}^{\xi+\frac{\varepsilon}{2}}\Psi_i\frac{d\Psi_k}{dx}\frac{dw_k}{dt}dx$$

$$+v^2\rho_qI_q\int_{\xi-\frac{\varepsilon}{2}}^{\xi+\frac{\varepsilon}{2}}\Psi_i\frac{d^2\Psi_k}{dx^2}w_kdx+\int_0^{l_e}kGA\Psi_i\frac{d\Phi_j}{dx}u_jdx\Bigg]+Q_\iota=0 \tag{15}$$

$Q_j=\Psi_j(0)Q_2^e-\Psi_j(l_e)Q_4^e\ ;\ j=4,5,6.$

Equations (14) and (15) are re-written as:

$$\sum_{j=1}^{3}\left[K_{ij}^*+K_{ij}^{**}\right]u_j+\sum_{k=1}^{3}K_{ik}w_k+\sum_{j=1}^{3}\left[M_{ij}^*\qquad+M_{ij}^{**}\right]\ddot{u}_j$$

$$+\sum_{j=1}^{3}C_{ij}\dot{u}_j+F_i^1=0 \tag{16}$$

$$\sum_{k=1}^{3}\left[K_{ik}^*+K_{ik}^{**}\right]w_k+\sum_{j=1}^{3}K_{ij}u_j+\sum_{k=1}^{3}\left[M_{ik}^*+M_{ik}^{**}\right]\ddot{w}_k$$

$$+\sum_{k=1}^{3}C_{ik}\dot{w}_k+F_i^2=0 \tag{17}$$

where in (16)

$$K_{ij}^*=\int_0^{l_e}kGA\Phi_i'\Phi_j'dx;M_{ij}^*=\int_0^{l_e}\rho A\Phi_i\Phi_jdx;$$

$$M_{ij}^{**}=\frac{p}{\varepsilon}\int_{\xi-\frac{\varepsilon}{2}}^{\xi+\frac{\varepsilon}{2}}\Phi_i\Phi_jdx;C_{ij}=\frac{2pv}{\varepsilon}\int_{\xi-\frac{\varepsilon}{2}}^{\xi+\frac{\varepsilon}{2}}\Phi_i\Phi_j'dx$$

$$K_{ij}^{**} = \frac{pv^2}{\varepsilon}\int_{\xi-\frac{\varepsilon}{2}}^{\xi+\frac{\varepsilon}{2}} \Phi_i\Phi_j'' dx; K_{ik} = \int_0^{l_e} kGA\Phi_i'\Psi_k dx;$$

$$F_i^1 = \frac{pg}{\varepsilon}\int_{\xi-\frac{\varepsilon}{2}}^{\xi+\frac{\varepsilon}{2}} \Phi_i dx + Q_i \ ; \quad i = 1,2,3. \tag{18}$$

and in (17)

$$K_{ik}^{*} = \int_0^{l_e} \left(EI\Psi_i'\Psi_k' + kGA\Psi_i\Psi_k\right) dx; M_{ik}^{*} = \int_0^{l_e} \rho I\Psi_i\Psi_k dx;$$

$$M_{ik}^{**} = \rho_q I_q \int_{\xi-\frac{\varepsilon}{2}}^{\xi+\frac{\varepsilon}{2}} \Psi_i\Psi_k dx; \ C_{ik} = 2v\rho_q I_q \int_{\xi-\frac{\varepsilon}{2}}^{\xi+\frac{\varepsilon}{2}} \Psi_i\Psi_k' dx;$$

$$K_{ik}^{**} = v^2\rho_q I_q \int_{\xi-\frac{\varepsilon}{2}}^{\xi+\frac{\varepsilon}{2}} \Psi_i\Psi_k'' dx; \quad K_{ij} = \int_0^{l_e} kGA\Psi_i\Phi_j' dx;$$

$$F_i^2 = Q_\iota \qquad i = 1,2,3 \ ; \ \iota = 4,5,6. \tag{19}$$

Where the $(')$ and $('')$ in (18) and (19) denotes first and second derivatives w.r.t. x respectively.
Equations (16) and (17) can be put compactly as

$$\sum_{j=1}^{3} K_{ij}^{11}u_j + \sum_{j=1}^{3} K_{ij}^{12}w_j + \sum_{j=1}^{3} M_{ij}^{11}\ddot{u}_j + \sum_{j=1}^{3} C_{ij}^{11}\dot{u}_j + F_i^1 = 0$$

$$\sum_{j=1}^{3} K_{ij}^{21}u_j + \sum_{j=1}^{3} K_{ij}^{22}w_j + \sum_{j=1}^{3} M_{ij}^{22}\ddot{w}_j + \sum_{j=1}^{3} C_{ij}^{22}\dot{w}_j + F_i^2 = 0 \tag{20}$$

this in matrix form becomes:

$$\begin{bmatrix} [K^{11}] & [K^{12}] \\ [K^{21}] & [K^{22}] \end{bmatrix} \begin{Bmatrix} \{u\} \\ \{w\} \end{Bmatrix} + \begin{bmatrix} [M^{11}] & [0] \\ [0] & [M^{22}] \end{bmatrix} \begin{Bmatrix} \{\ddot{u}\} \\ \{\ddot{w}\} \end{Bmatrix}$$

$$+ \begin{bmatrix} [C^{11}] & [0] \\ [0] & [C^{22}] \end{bmatrix} \begin{Bmatrix} \{\dot{u}\} \\ \{\dot{w}\} \end{Bmatrix} + \begin{Bmatrix} \{F^1\} \\ \{F^2\} \end{Bmatrix} \tag{21}$$

Or simply

$$[K]\{v(t)\} + [M]\{\ddot{v}(t)\} + [C]\{\dot{v}(t)\} + \{F\} = 0 \tag{22}$$

Equation (22) is a semi-discrete system of equations in time. Where $[K]$ is the element stiffness matrix; $[M]$ is the element mass matrix; $[C]$ is the element centripetal matrix; $\{F\}$ is the sum of the element applied force vector $\{f\}$ and the element internal generalized force of $\{Q\}$ (boundary term vector); and v is a vector of generalized displacements $\{u, w\}^T$. It is the desired finite element model for a typical element of the tapered (non-uniform) Timoshenko beam under the present study.

5. THE ELEMENT STIFFNESS MATRIX

From the stiffness expression of (18) and (19); and using (7) and (13), the entries of the stiffness coefficient matrix are obtained:
For $[K^{11}]$:

$$k_{ij}^{e} = \int_0^{l_e} \phi_{ij} \left\{ kGA_{e-1} \left[1 - \beta_b^e \frac{x}{l_e}\right] \left[1 - \beta_h^e \frac{x}{l_e}\right] \right\} dx + \left(\frac{pv^2}{\varepsilon}\right) p_{ij}$$
$$i, j = 1, 2, 3. \tag{23}$$

For $[K^{12}]$:

$$k_{mn}^{e} = \int_0^{l_e} d_{ij} kGA_{e-1} \left[1 - \beta_b^e \frac{x}{l_e}\right] \left[1 - \beta_h^e \frac{x}{l_e}\right] dx$$
$$i, j, m = 1, 2, 3; \quad n = 4, 5, 6. \tag{24}$$

For $[K^{21}]$:

$$k_{mn}^{e} = \int_0^{l_e} g_{ij} kGA_{e-1} \left[1 - \beta_b^e \frac{x}{l_e}\right] \left[1 - \beta_h^e \frac{x}{l_e}\right] dx$$
$$i, j, n = 1, 2, 3; \quad m = 4, 5, 6. \tag{25}$$

For $[K^{22}]$:

$$k_{mn}^{e} = \int_0^{l_e} \alpha_{ij} \left\{ EI_{e-1} \left[1 - \beta_b^e \frac{x}{l_e}\right] \left[1 - \beta_h^e \frac{x}{l_e}\right]^3 \right\}$$
$$+\sigma_{ij} \left\{ kGA_{e-1} \left[1 - \beta_b^e \frac{x}{l_e}\right] \left[1 - \beta_h^e \frac{x}{l_e}\right] \right\} dx + \left(v^2 \rho_q I_q\right) q_{ij}$$
$$i, j = 1, 2, 3; \quad m, n = 4, 5, 6. \tag{26}$$

Where, for (23) to (26),

$$\phi_{11} = \left[\frac{9}{l_e^2} - \frac{24x}{l_e^3} + \frac{16x^2}{l_e^4}\right], \ \phi_{12} = \left[-\frac{12}{l_e^2} + \frac{40x}{l_e^3} - \frac{32x^2}{l_e^4}\right]$$
$$\phi_{13} = \left[\frac{3}{l_e^2} - \frac{16x}{l_e^3} + \frac{16x^2}{l_e^4}\right], \phi_{21} = \left[-\frac{12}{l_e^2} + \frac{40x}{l_e^3} - \frac{32x^2}{l_e^4}\right]$$
$$\phi_{22} = \left[\frac{16}{l_e^2} - \frac{64x}{l_e^3} + \frac{64x^2}{l_e^4}\right], \ \phi_{23} = \left[-\frac{4}{l_e^2} + \frac{24x}{l_e^3} - \frac{32x^2}{l_e^4}\right]$$
$$\phi_{31} = \left[\frac{3}{l_e^2} - \frac{16x}{l_e^3} + \frac{16x^2}{l_e^4}\right], \ \phi_{32} = \left[-\frac{4}{l_e^2} + \frac{24x}{l_e^3} - \frac{32x^2}{l_e^4}\right]$$
$$\phi_{33} = \left[\frac{1}{l_e^2} - \frac{8x}{l_e^3} + \frac{16x^2}{l_e^4}\right] \tag{27}$$

$$p_{11} = \frac{pv^2}{\varepsilon} \left[\left[\frac{4\eta}{l_e^2} - \frac{6\eta^2}{l_e^3} + \frac{8\eta^3}{3l_e^4}\right] - \left[\frac{4\mu}{l_e^2} - \frac{6\mu^2}{l_e^3} + \frac{8\mu^3}{3l_e^4}\right]\right]$$

$$p_{12} = \frac{pv^2}{\varepsilon}\left[\left[-\frac{8\eta}{l_e^2} + \frac{12\eta^2}{l_e^3} - \frac{16\eta^3}{3l_e^4}\right] - \left[-\frac{8\mu}{l_e^2} + \frac{12\mu^2}{l_e^3} - \frac{16\mu^3}{3l_e^4}\right]\right]$$

$$p_{13} = \frac{pv^2}{\varepsilon}\left[\left[\frac{2\eta^2}{l_e^2} - \frac{4\eta^3}{l_e^3} + \frac{2\eta^4}{l_e^4}\right] - \left[\frac{2\mu^2}{l_e^2} - \frac{4\mu^3}{l_e^3} + \frac{2\mu^4}{l_e^4}\right]\right]$$

$$p_{21} = \frac{pv^2}{\varepsilon}\left[\left(\frac{8\eta^2}{l_e^3} - \frac{16\eta^3}{3l_e^4}\right) - \left(\frac{8\mu^2}{l_e^3} - \frac{16\mu^3}{3l_e^4}\right)\right]$$

$$p_{22} = \frac{pv^2}{\varepsilon}\left[\left(-\frac{18\eta^2}{l_e^3} + \frac{32\eta^3}{3l_e^4}\right) - \left(-\frac{18\mu^2}{l_e^3} + \frac{32\mu^3}{3l_e^4}\right)\right]$$

$$p_{23} = \frac{pv^2}{\varepsilon}\left[\left(\frac{16\eta^3}{3l_e^3} - \frac{4\eta^4}{l_e^4}\right) - \left(\frac{16\mu^3}{3l_e^3} - \frac{4\mu^4}{l_e^4}\right)\right]$$

$$p_{31} = \frac{pv^2}{\varepsilon}\left[\left(-\frac{2\eta^2}{l_e^3} + \frac{8\eta^3}{3l_e^4}\right) - \left(-\frac{2\mu^2}{l_e^3} + \frac{8\mu^3}{3l_e^4}\right)\right]$$

$$p_{32} = \frac{pv^2}{\varepsilon}\left[\left(\frac{4\eta^2}{l_e^3} - \frac{16\eta^3}{3l_e^4}\right) - \left(\frac{4\mu^2}{l_e^3} - \frac{16\mu^3}{3l_e^4}\right)\right]$$

$$p_{33} = \frac{pv^2}{\varepsilon}\left[\left(-\frac{4\eta^3}{3l_e^3} + \frac{2\eta^4}{l_e^4}\right) - \left(-\frac{4\mu^3}{3l_e^3} + \frac{2\mu^4}{l_e^4}\right)\right] \tag{28}$$

$$d_{11} = \left[-\frac{3}{l_e} + \frac{13x}{l_e^2} - \frac{18x^2}{l_e^3} + \frac{8x^3}{l_e^4}\right], d_{12} = \left[-\frac{12x}{l_e^2} + \frac{28x^2}{l_e^3} - \frac{16x^3}{l_e^4}\right]$$

$$d_{13} = \left[\frac{3x}{l_e^2} - \frac{10x^2}{l_e^3} + \frac{8x^3}{l_e^4}\right], d_{21} = \left[\frac{4}{l_e} - \frac{20x}{l_e^2} + \frac{32x^2}{l_e^3} - \frac{16x^3}{l_e^4}\right]$$

$$d_{22} = \left[\frac{16x}{l_e^2} - \frac{48x^2}{l_e^3} + \frac{32x^3}{l_e^4}\right],\ d_{23} = \left[-\frac{4x}{l_e^2} + \frac{16x^2}{l_e^3} - \frac{16x^3}{l_e^4}\right]$$

$$d_{31} = \left[-\frac{1}{l_e} + \frac{7x}{l_e^2} - \frac{14x^2}{l_e^3} + \frac{8x^3}{l_e^4}\right],\ d_{32} = \left[-\frac{4x}{l_e^2} + \frac{20x^2}{l_e^3} - \frac{16x^3}{l_e^4}\right]$$

$$d_{33} = \left[\frac{x}{l_e^2} - \frac{6x^2}{l_e^3} + \frac{8x^3}{l_e^4}\right] \tag{29}$$

$$g_{11} = \left[-\frac{3}{l_e} + \frac{13x}{l_e^2} - \frac{18x^2}{l_e^3} + \frac{8x^3}{l_e^4}\right] g_{12} = \left[\frac{4}{l_e} - \frac{20x}{l_e^2} + \frac{32x^2}{l_e^3} - \frac{16x^3}{l_e^4}\right]$$

$$g_{13} = \left[-\frac{1}{l_e} + \frac{7x}{l_e^2} - \frac{14x^2}{l_e^3} + \frac{8x^3}{l_e^4}\right] g_{21} = \left[-\frac{12x}{l_e^2} + \frac{28x^2}{l_e^3} - \frac{16x^3}{l_e^4}\right]$$

$$g_{22} = \left[\frac{16x}{l_e^2} - \frac{48x^2}{l_e^3} + \frac{32x^3}{l_e^4}\right] g_{23} = \left[-\frac{4x}{l_e^2} + \frac{20x^2}{l_e^3} - \frac{16x^3}{l_e^4}\right]$$

$$g_{31} = \left[\frac{3x}{l_e^2} - \frac{10x^2}{l_e^3} + \frac{8x^3}{l_e^4}\right] g_{32} = \left[-\frac{4x}{l_e^2} + \frac{16x^2}{l_e^3} - \frac{16x^3}{l_e^4}\right]$$

$$g_{33} = \left[\frac{x}{l_e^2} - \frac{6x^2}{l_e^3} + \frac{8x^3}{l_e^4}\right] \tag{30}$$

$$\alpha_{11} = \left[\frac{9}{l_e^2} - \frac{24x}{l_e^3} + \frac{16x^2}{l_e^4}\right], \alpha_{12} = \left[-\frac{12}{l_e^2} + \frac{40x}{l_e^3} - \frac{32x^2}{l_e^4}\right]$$
$$\alpha_{13} = \left[\frac{3}{l_e^2} - \frac{16x}{l_e^3} + \frac{16x^2}{l_e^4}\right], \alpha_{21} = \left[-\frac{12}{l_e^2} + \frac{40x}{l_e^3} - \frac{32x^2}{l_e^4}\right]$$
$$\alpha_{22} = \left[\frac{16}{l_e^2} - \frac{64x}{l_e^3} + \frac{64x^2}{l_e^4}\right], \alpha_{23} = \left[-\frac{4}{l_e^2} + \frac{24x}{l_e^3} - \frac{32x^2}{l_e^4}\right]$$
$$\alpha_{31} = \left[\frac{3}{l_e^2} - \frac{16x}{l_e^3} + \frac{16x^2}{l_e^4}\right], \alpha_{32} = \left[-\frac{4}{l_e^2} + \frac{24x}{l_e^3} - \frac{32x^2}{l_e^4}\right]$$
$$\alpha_{33} = \left[\frac{1}{l_e^2} - \frac{8x}{l_e^3} + \frac{16x^2}{l_e^4}\right] \tag{31}$$

$$\sigma_{11} = \left[1 - \frac{6x}{l_e} + \frac{11x^2}{l_e^2} - \frac{6x^3}{l_e^3}\right] \sigma_{12} = \left[\frac{4x}{l_e} - \frac{16x^2}{l_e^2} + \frac{20x^3}{l_e^3} - \frac{8x^4}{l_e^4}\right]$$
$$\sigma_{13} = \left[-\frac{x}{l_e} + \frac{5x^2}{l_e^2} - \frac{8x^3}{l_e^3} + \frac{4x^4}{l_e^4}\right] \sigma_{21} = \left[\frac{4x}{l_e} - \frac{16x^2}{l_e^2} + \frac{20x}{l_e^3} - \frac{8x^4}{l_e^4}\right]$$
$$\sigma_{22} = \left[\frac{16x^2}{l_e^2} - \frac{32x^3}{l_e^3} + \frac{16x^4}{l_e^4}\right] \sigma_{23} = \left[-\frac{4x^2}{l_e^2} + \frac{12x^3}{l_e^3} - \frac{8x^4}{l_e^4}\right]$$
$$\sigma_{31} = \left[-\frac{x}{l_e} + \frac{5x^2}{l_e^2} - \frac{8x^3}{l_e^3} + \frac{4x^4}{l_e^4}\right] \sigma_{32} = \left[-\frac{4x^2}{l_e^2} + \frac{12x^3}{l_e^3} - \frac{8x^4}{l_e^4}\right]$$
$$\sigma_{33} = \left[\frac{x^2}{l_e^2} - \frac{4x^3}{l_e^3} + \frac{4x^4}{l_e^4}\right] \tag{32}$$

$$q_{11} = \frac{pv^2}{\varepsilon}\left[\left(\frac{4\eta}{l_e^2} - \frac{6\eta^2}{l_e^3} + \frac{8\eta^3}{3l_e^4}\right) - \left(\frac{4\mu}{l_e^2} - \frac{6\mu^2}{l_e^3} + \frac{8\mu^3}{3l_e^4}\right)\right]$$
$$q_{12} = \frac{pv^2}{\varepsilon}\left[\left[-\frac{8\eta}{l_e^2} + \frac{12\eta^2}{l_e^3} - \frac{16\eta^3}{3l_e^4}\right] - \left[-\frac{8\mu}{l_e^2} + \frac{12\mu^2}{l_e^3} - \frac{16\mu^3}{3l_e^4}\right]\right]$$
$$q_{13} = \frac{pv^2}{\varepsilon}\left[\left(\frac{2\eta^2}{l_e^2} - \frac{4\eta^3}{l_e^3} + \frac{2\eta^4}{l_e^4}\right) - \left(\frac{2\mu^2}{l_e^2} - \frac{4\mu^3}{l_e^3} + \frac{2\mu^4}{l_e^4}\right)\right]$$
$$q_{21} = \frac{pv^2}{\varepsilon}\left[\left(\frac{8\eta^2}{l_e^3} - \frac{16\eta^3}{3l_e^4}\right) - \left(\frac{8\mu^2}{l_e^3} - \frac{16\mu^3}{3l_e^4}\right)\right]$$
$$q_{22} = \frac{pv^2}{\varepsilon}\left[\left(-\frac{18\eta^2}{l_e^3} + \frac{32\eta^3}{3l_e^4}\right) - \left(-\frac{18\mu^2}{l_e^3} + \frac{32\mu^3}{3l_e^4}\right)\right]$$
$$q_{23} = \frac{pv^2}{\varepsilon}\left[\left(\frac{16\eta^3}{3l_e^3} - \frac{4\eta^4}{l_e^4}\right) - \left(\frac{16\mu^3}{3l_e^3} - \frac{4\mu^4}{l_e^4}\right)\right]$$

$$q_{31} = \frac{pv^2}{\varepsilon}\left[\left(-\frac{2\eta^2}{l_e^3} + \frac{8\eta^3}{3l_e^4}\right) - \left(-\frac{2\mu^2}{l_e^3} + \frac{8\mu^3}{3l_e^4}\right)\right]$$
$$q_{32} = \frac{pv^2}{\varepsilon}\left[\left(\frac{4\eta^2}{l_e^3} - \frac{16\eta^3}{3l_e^4}\right) - \left(\frac{4\mu^2}{l_e^3} - \frac{16\mu^3}{3l_e^4}\right)\right]$$
$$q_{33} = \frac{pv^2}{\varepsilon}\left[\left(-\frac{4\eta^3}{3l_e^3} + \frac{2\eta^4}{l_e^4}\right) - \left(-\frac{4\mu^3}{3l_e^3} + \frac{2\mu^4}{l_e^4}\right)\right] \tag{33}$$

where for (28) and (33), $\eta = \xi - \frac{\varepsilon}{2},\ \mu = \xi + \frac{\varepsilon}{2}$

6. THE ELEMENT MASS MATRIX

From the mass expressions of (18) and (19); and using (7) and (13), the entries of the mass coefficient matrix are obtained:
For $[M^{11}]$:

$$m_{ij}^e = \int_0^{l_e} o_{ij}\left\{\rho A_{e-1}\left[1 - \beta_b^e\frac{x}{l_e}\right]\left[1 - \beta_h^e\frac{x}{l_e}\right]\right\}dx + \left(\frac{p}{\varepsilon}\right)\chi_{ij} \qquad i,j = 1,2,3. \tag{34}$$

For $[M^{22}]$:

$$m_{mn}^e = \int_0^{l_e} n_{ij}\left\{\rho I_{e-1}\left[1 - \beta_b^e\frac{x}{l_e}\right]\left[1 - \beta_h^e\frac{x}{l_e}\right]^3\right\}dx + (\rho_q I_q)\,\vartheta_{ij} \qquad i,j = 1,2,3;\quad m,n = 4,5,6. \tag{35}$$

Where, for (34) and (35),

$$o_{11} = \left[1 - \frac{6x}{l_e} + \frac{11x^2}{l_e^2} - \frac{6x^3}{l_e^3}\right], o_{12} = \left[\frac{4x}{l_e} - \frac{16x^2}{l_e^2} + \frac{20x^3}{l_e^3} - \frac{8x^4}{l_e^4}\right]$$
$$o_{13} = \left[-\frac{x}{l_e} + \frac{5x^2}{l_e^2} - \frac{8x^3}{l_e^3} + \frac{4x^4}{l_e^4}\right], o_{21} = \left[\frac{4x}{l_e} - \frac{16x^2}{l_e^2} + \frac{20x^3}{l_e^3} - \frac{8x^4}{l_e^4}\right]$$
$$o_{22} = \left[\frac{16x^2}{l_e^2} - \frac{32x^3}{l_e^3} + \frac{16x^4}{l_e^4}\right], o_{23} = \left[-\frac{4x^2}{l_e^2} + \frac{12x^3}{l_e^3} - \frac{8x^4}{l_e^4}\right]$$
$$o_{31} = \left[-\frac{x}{l_e} + \frac{5x^2}{l_e^2} - \frac{8x^3}{l_e^3} + \frac{4x^4}{l_e^4}\right], o_{32} = \left[-\frac{4x^2}{l_e^2} + \frac{12x^3}{l_e^3} - \frac{8x^4}{l_e^4}\right]$$
$$o_{33} = \left[\frac{x^2}{l_e^2} - \frac{4x^3}{l_e^3} + \frac{4x^4}{l_e^4}\right] \tag{36}$$

$$\chi_{11} = \frac{p}{\varepsilon}\left[\left[\eta - \frac{3\eta^2}{l_e} + \frac{11\eta^3}{3l_e^2} - \frac{3\eta^4}{2l_e^3}\right] - \left[\eta - \frac{3\mu^2}{l_e} + \frac{11\mu^3}{3l_e^2} - \frac{3\mu^4}{2l_e^3}\right]\right]$$

$$\chi_{12} = \frac{p}{\varepsilon}\left[\left[\frac{2\eta^2}{l_e} - \frac{16\eta^3}{3l_e^2} + \frac{5\eta^4}{l_e^3} - \frac{8\eta^5}{5l_e^4}\right] - \left[\frac{2\mu^2}{l_e} - \frac{16\mu^3}{3l_e^2} + \frac{5\mu^4}{l_e^3} - \frac{8\mu^5}{5l_e^4}\right]\right]$$

$$\chi_{13} = \frac{p}{\varepsilon}\left[\left[-\frac{\eta^2}{2l_e} + \frac{5\eta^3}{3l_e^2} - \frac{2\eta^4}{l_e^3} + \frac{4\eta^5}{5l_e^4}\right] - \left[-\frac{\mu^2}{2l_e} + \frac{5\mu^3}{3l_e^2} - \frac{2\mu^4}{l_e^3} + \frac{4\mu^5}{5l_e^4}\right]\right]$$

$$\chi_{21} = \frac{p}{\varepsilon}\left[\left[\frac{2\eta^2}{l_e} - \frac{16\eta^3}{3l_e^2} + \frac{5\eta^4}{l_e^3} - \frac{8\eta^5}{5l_e^4}\right] - \left[\frac{2\mu^2}{l_e} - \frac{16\mu^3}{3l_e^2} + \frac{5\mu^4}{l_e^3} - \frac{8\mu^5}{5l_e^4}\right]\right]$$

$$\chi_{22} = \frac{p}{\varepsilon}\left[\left[\frac{16\eta^3}{3l_e^2} - \frac{8\eta^4}{l_e^3} + \frac{16\eta^5}{5l_e^4}\right] - \left[\frac{16\mu^3}{3l_e^2} - \frac{8\mu^4}{l_e^3} + \frac{16\mu^5}{5l_e^4}\right]\right] \quad (37)$$

$$\chi_{23} = \frac{p}{\varepsilon}\left[\left[-\frac{4\eta^3}{3l_e^2} + \frac{3\eta^4}{l_e^3} - \frac{8\eta^5}{5l_e^4}\right] - \left[-\frac{4\eta^3}{3l_e^2} + \frac{3\eta^4}{l_e^3} - \frac{8\eta^5}{5l_e^4}\right]\right]$$

$$\chi_{31} = \frac{p}{\varepsilon}\left[\left[-\frac{\eta^2}{2l_e} + \frac{5\eta^3}{3l_e^2} - \frac{2\eta^4}{l_e^3} + \frac{4\eta^5}{5l_e^4}\right] - \left[-\frac{\mu^2}{2l_e} + \frac{5\mu^3}{3l_e^2} - \frac{2\mu^4}{l_e^3} + \frac{4\mu^5}{5l_e^4}\right]\right]$$

$$\chi_{32} = \frac{p}{\varepsilon}\left[\left[-\frac{4\eta^3}{3l_e^2} + \frac{3\eta^4}{l_e^3} - \frac{8\eta^5}{5l_e^4}\right] - \left[-\frac{4\mu^3}{3l_e^2} + \frac{3\mu^4}{l_e^3} - \frac{8\mu^5}{5l_e^4}\right]\right]$$

$$\chi_{33} = \frac{p}{\varepsilon}\left[\left[\frac{\eta^3}{3l_e^2} - \frac{\eta^4}{l_e^3} + \frac{4\eta^5}{5l_e^4}\right] - \left[\frac{\mu^3}{3l_e^2} - \frac{\mu^4}{l_e^3} + \frac{4\mu^5}{5l_e^4}\right]\right] \quad (38)$$

$$n_{11} = \left[1 - \frac{6x}{l_e} + \frac{11x^2}{l_e^2} - \frac{6x^3}{l_e^3}\right], n_{12} = \left[\frac{4x}{l_e} - \frac{16x^2}{l_e^2} + \frac{20x^3}{l_e^3} - \frac{8x^4}{l_e^4}\right]$$

$$n_{13} = \left[-\frac{x}{l_e} + \frac{5x^2}{l_e^2} - \frac{8x^3}{l_e^3} + \frac{4x^4}{l_e^4}\right], n_{21} = \left[\frac{4x}{l_e} - \frac{16x^2}{l_e^2} + \frac{20x}{l_e^3} - \frac{8x^4}{l_e^4}\right]$$

$$n_{22} = \left[\frac{16x^2}{l_e^2} - \frac{32x^3}{l_e^3} + \frac{16x^4}{l_e^4}\right], n_{23} = \left[-\frac{4x^2}{l_e^2} + \frac{12x^3}{l_e^3} - \frac{8x^4}{l_e^4}\right]$$

$$n_{31} = \left[-\frac{x}{l_e} + \frac{5x^2}{l_e^2} - \frac{8x^3}{l_e^3} + \frac{4x^4}{l_e^4}\right], n_{32} = \left[-\frac{4x^2}{l_e^2} + \frac{12x^3}{l_e^3} - \frac{8x^4}{l_e^4}\right]$$

$$n_{33} = \left[\frac{x^2}{l_e^2} - \frac{4x^3}{l_e^3} + \frac{4x^4}{l_e^4}\right] \quad (39)$$

$$\vartheta_{11} = \frac{p}{\varepsilon}\left[\left[\eta - \frac{3\eta^2}{l_e} + \frac{11\eta^3}{3l_e^2} - \frac{3\eta^4}{2l_e^3}\right] - \left[\eta - \frac{3\mu^2}{l_e} + \frac{11\mu^3}{3l_e^2} - \frac{3\mu^4}{2l_e^3}\right]\right]$$

$$\vartheta_{12} = \frac{p}{\varepsilon}\left[\left[\frac{2\eta^2}{l_e} - \frac{16\eta^3}{3l_e^2} + \frac{5\eta^4}{l_e^3} - \frac{8\eta^5}{5l_e^4}\right] - \left[\frac{2\mu^2}{l_e} - \frac{16\mu^3}{3l_e^2} + \frac{5\mu^4}{l_e^3} - \frac{8\mu^5}{5l_e^4}\right]\right]$$

$$\vartheta_{13} = \frac{p}{\varepsilon}\left[\left[-\frac{\eta^2}{2l_e} + \frac{5\eta^3}{3l_e^2} - \frac{2\eta^4}{l_e^3} + \frac{4\eta^5}{5l_e^4}\right] - \left[-\frac{\mu^2}{2l_e} + \frac{5\mu^3}{3l_e^2} - \frac{2\mu^4}{l_e^3} + \frac{4\mu^5}{5l_e^4}\right]\right]$$

$$\vartheta_{21} = \frac{p}{\varepsilon}\left[\left[\frac{2\eta^2}{l_e} - \frac{16\eta^3}{3l_e^2} + \frac{5\eta^4}{l_e^3} - \frac{8\eta^5}{5l_e^4}\right] - \left[\frac{2\mu^2}{l_e} - \frac{16\mu^3}{3l_e^2} + \frac{5\mu^4}{l_e^3} - \frac{8\mu^5}{5l_e^4}\right]\right]$$

$$\vartheta_{22} = \frac{p}{\varepsilon}\left[\left[\frac{16\eta^3}{3l_e^2} - \frac{8\eta^4}{l_e^3} + \frac{16\eta^5}{5l_e^4}\right] - \left[\frac{16\mu^3}{3l_e^2} - \frac{8\mu^4}{l_e^3} + \frac{16\mu^5}{5l_e^4}\right]\right] \quad (40)$$

$$\vartheta_{23} = \frac{p}{\varepsilon}\left[\left[-\frac{4\eta^3}{3l_e^2} + \frac{3\eta^4}{l_e^3} - \frac{8\eta^5}{5l_e^4}\right] - \left[-\frac{4\eta^3}{3l_e^2} + \frac{3\eta^4}{l_e^3} - \frac{8\eta^5}{5l_e^4}\right]\right]$$

$$\vartheta_{31} = \frac{p}{\varepsilon}\left[\left[-\frac{\eta^2}{2l_e} + \frac{5\eta^3}{3l_e^2} - \frac{2\eta^4}{l_e^3} + \frac{4\eta^5}{5l_e^4}\right] - \left[-\frac{\mu^2}{2l_e} + \frac{5\mu^3}{3l_e^2} - \frac{2\mu^4}{l_e^3} + \frac{4\mu^5}{5l_e^4}\right]\right]$$

$$\vartheta_{32} = \frac{p}{\varepsilon}\left[\left[-\frac{4\eta^3}{3l_e^2} + \frac{3\eta^4}{l_e^3} - \frac{8\eta^5}{5l_e^4}\right] - \left[-\frac{4\mu^3}{3l_e^2} + \frac{3\mu^4}{l_e^3} - \frac{8\mu^5}{5l_e^4}\right]\right]$$

$$\vartheta_{33} = \frac{p}{\varepsilon}\left[\left[\frac{\eta^3}{3l_e^2} - \frac{\eta^4}{l_e^3} + \frac{4\eta^5}{5l_e^4}\right] - \left[\frac{\mu^3}{3l_e^2} - \frac{\mu^4}{l_e^3} + \frac{4\mu^5}{5l_e^4}\right]\right] \quad (41)$$

where for (37), (38), (40) and (41), $\eta = \xi - \frac{\varepsilon}{2},\ \mu = \xi + \frac{\varepsilon}{2}$

7. THE ELEMENT CENTRIPETAL MATRIX

From the centripetal expressions of (18) and (19); and using (13), the entries of the centripetal coefficient matrix are obtained:
For $[C^{11}]$:

$$c_{ij}^e = \frac{2pv}{\varepsilon}\left(\gamma_{ij}\right) \qquad i,j = 1,2,3. \quad (42)$$

For $[C^{22}]$:

$$c_{mn}^e = 2v\rho_q I_q\left(\beta_{ij}\right) \qquad m,n = 4,5,6. \quad (43)$$

Where for (42) and (43)

$$\gamma_{11} = \left[-\frac{3x}{l_e} + \frac{13x^2}{2l_e^2} - \frac{6x^3}{l_e^3} + \frac{2x^4}{l_e^4}\right], \gamma_{12} = \left[\frac{4x}{l_e} - \frac{10x^2}{l_e^2} + \frac{32x^3}{3l_e^3} - \frac{4x^4}{l_e^4}\right]$$

$$\gamma_{13} = \left[-\frac{x}{l_e} + \frac{7x^2}{2l_e^2} - \frac{14x^3}{3l_e^3} + \frac{8x^3}{l_e^4}\right], \gamma_{21} = \left[-\frac{6x^2}{l_e^2} + \frac{28x^3}{3l_e^3} - \frac{4x^4}{l_e^4}\right]$$

$$\gamma_{22} = \left[\frac{8x^2}{l_e^2} - \frac{16x^3}{l_e^3} + \frac{8x^4}{l_e^4}\right], \gamma_{23} = \left[-\frac{2x^2}{l_e^2} + \frac{20x^3}{3l_e^3} - \frac{4x^4}{l_e^4}\right]$$

$$\gamma_{31} = \left[\frac{3x^2}{2l_e^2} - \frac{10x^3}{3l_e^3} + \frac{2x^4}{l_e^4}\right], \gamma_{32} = \left[-\frac{2x^2}{l_e^2} + \frac{16x^3}{3l_e^3} - \frac{4x^4}{l_e^4}\right]$$

$$\gamma_{33} = \left[\frac{x^2}{2l_e^2} - \frac{2x^3}{l_e^3} + \frac{2x^4}{l_e^4}\right] \quad (44)$$

$$\beta_{11} = \left[-\frac{3x}{l_e} + \frac{13x^2}{2l_e^2} - \frac{6x^3}{l_e^3} + \frac{2x^4}{l_e^4}\right], \beta_{12} = \left[\frac{4x}{l_e} - \frac{10x^2}{l_e^2} + \frac{32x^3}{3l_e^3} - \frac{4x^4}{l_e^4}\right]$$

$$\beta_{13} = \left[-\frac{x}{l_e} + \frac{7x^2}{2l_e^2} - \frac{14x^3}{3l_e^3} + \frac{8x^3}{l_e^4}\right], \beta_{21} = \left[-\frac{6x^2}{l_e^2} + \frac{28x^3}{3l_e^3} - \frac{4x^4}{l_e^4}\right]$$

$$\beta_{22} = \left[\frac{8x^2}{l_e^2} - \frac{16x^3}{l_e^3} + \frac{8x^4}{l_e^4}\right], \beta_{23} = \left[-\frac{2x^2}{l_e^2} + \frac{20x^3}{3l_e^3} - \frac{4x^4}{l_e^4}\right]$$
$$\beta_{31} = \left[\frac{3x^2}{2l_e^2} - \frac{10x^3}{3l_e^3} + \frac{2x^4}{l_e^4}\right], \beta_{32} = \left[-\frac{2x^2}{l_e^2} + \frac{16x^3}{3l_e^3} - \frac{4x^4}{l_e^4}\right]$$
$$\beta_{33} = \left[\frac{x^2}{2l_e^2} - \frac{2x^3}{l_e^3} + \frac{2x^4}{l_e^4}\right] \tag{45}$$

8. THE ELEMENT LOAD VECTOR

From the load expressions of (18) and (19); and using (13), the entries of the load coefficient vector are obtained:
For $\{F^1\}$:

$$f_i^e = \frac{pg}{\varepsilon}(\kappa_i) + Q_i \quad i = 1, 2, 3. \tag{46}$$

For $\{F^2\}$:

$$f_j^e = Q_j \qquad j = 4, 5, 6. \tag{47}$$

where

$$\kappa_1 = \left[\left(\eta - \frac{3\eta^2}{2l_e} + \frac{2\eta^3}{3l_e^2}\right) - \left(\mu - \frac{3\mu^2}{2l_e} + \frac{2\mu^3}{3l_e^2}\right)\right]$$
$$\kappa_2 = \left[\left(\frac{2\eta^2}{l_e} - \frac{4\eta^3}{3l_e^2}\right) - \left(\frac{2\mu^2}{l_e} - \frac{4\mu^3}{3l_e^2}\right)\right]$$
$$\kappa_3 = \left[\left(-\frac{\eta^2}{2l_e} + \frac{2\eta^3}{3l_e^2}\right) - \left(-\frac{\mu^2}{2l_e} + \frac{2\mu^3}{3l_e^2}\right)\right]$$
$$\eta = \xi - \frac{\varepsilon}{2}, \ \mu = \xi + \frac{\varepsilon}{2} \tag{48}$$

The specification of $Q_i, i = 1, 2, ..., 6$ depends on the associated boundary conditions for a particular moving load problem.

9. ASSEMBLY OF ELEMENT EQUATIONS

The assembling of the various matrices and vector corresponding to each element are done bearing in mind that for every beam element, there are two degrees of freedom at each end nodes.

For an n-element tapered Timoshenko beam, there are $(n + 1)$ global nodes and a total of $2(n+1)$ global generalized displacements and $2(n + 1)$ generalized internal forces.

For convenience, we harmonize the displacement terms and rotation terms by re-labelling them as follows:

$$u_1^e = v_1^e, \ w_1^e = v_2^e, \ u_2^e = v_3^e, \ w_2^e = v_4^e, \ u_3^e = v_5^e, \ w_3^e = v_6^e \tag{49}$$

The continuity of the primary variables implies:

$$v_1^e = V_1,\ v_5^e = v_1^{e+1} = V_3,\ v_5^{e+1} = V_5$$
$$v_2^e = V_2,\ v_6^e = v_2^{e+1} = V_4,\ v_6^{e+1} = V_6 \tag{50}$$

Where v_i^e are the elements degrees of freedom and V_i are the global degrees of freedom.

The equilibrium of the generalized forces at the connecting nodes between elements e and $e+1$ requires that:

$$Q_5^e + Q_1^{e+1} = 0$$
$$Q_6^e + Q_2^{e+1} = 0 \tag{51}$$

Consequently, the assembled stiffness matrix for two elements connected in series for the present study is:

$$[K] = \begin{bmatrix} k_{11}^1 & k_{12}^1 & k_{15}^1 & k_{16}^1 & 0 & 0 \\ k_{21}^1 & k_{22}^1 & k_{25}^1 & k_{26}^1 & 0 & 0 \\ k_{51}^1 & k_{52}^1 & k_{55}^1 + k_{11}^2 & k_{56}^1 + k_{12}^2 & k_{15}^2 & k_{16}^2 \\ k_{61}^1 & k_{62}^1 & k_{65}^1 + k_{21}^2 & k_{66}^1 + k_{22}^2 & k_{25}^2 & k_{26}^2 \\ 0 & 0 & k_{51}^2 & k_{52}^2 & k_{55}^2 & k_{56}^2 \\ 0 & 0 & k_{61}^2 & k_{62}^2 & k_{65}^2 & k_{66}^2 \end{bmatrix} \tag{52}$$

The mass matrix, the centripetal matrix, and the load vector for the present study are similarly obtained:

$$[M] = \begin{bmatrix} m_{11}^1 & m_{12}^1 & 0 & 0 & 0 & 0 \\ m_{21}^1 & m_{22}^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_{55}^1 + m_{11}^2 & m_{56}^1 + m_{12}^2 & 0 & 0 \\ 0 & 0 & m_{65}^1 + m_{21}^2 & m_{66}^1 + m_{22}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_{55}^2 & m_{56}^2 \\ 0 & 0 & 0 & 0 & m_{65}^2 & m_{66}^2 \end{bmatrix} \tag{53}$$

$$[C] = \begin{bmatrix} c_{11}^1 & c_{12}^1 & 0 & 0 & 0 & 0 \\ c_{21}^1 & c_{22}^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{55}^1 + c_{11}^2 & c_{56}^1 + c_{12}^2 & 0 & 0 \\ 0 & 0 & c_{65}^1 + c_{21}^2 & c_{66}^1 + c_{22}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55}^2 & c_{56}^2 \\ 0 & 0 & 0 & 0 & c_{65}^2 & c_{66}^2 \end{bmatrix} \tag{54}$$

$$[F] = \begin{bmatrix} f_1^1 \\ 0 \\ f_3^1 + f_1^2 \\ 0 \\ f_3^2 \\ 0 \end{bmatrix} + \begin{bmatrix} Q_1^1 \\ Q_2^1 \\ Q_5^1 + Q_1^2 \\ Q_6^1 + Q_2^2 \\ Q_5^2 \\ Q_6^2 \end{bmatrix} \tag{55}$$

The equation of motion governing the dynamic behaviour of a tapered Timoshenko beam under a uniform partially distributed moving load becomes:

$$\begin{bmatrix} k_{11}^1 & k_{12}^1 & k_{15}^1 & k_{16}^1 & 0 & 0 \\ k_{21}^1 & k_{22}^1 & k_{25}^1 & k_{26}^1 & 0 & 0 \\ k_{51}^1 & k_{52}^1 & k_{55}^1 + k_{11}^2 & k_{56}^1 + k_{12}^2 & k_{15}^2 & k_{16}^2 \\ k_{61}^1 & k_{62}^1 & k_{65}^1 + k_{21}^2 & k_{66}^1 + k_{22}^2 & k_{25}^2 & k_{26}^2 \\ 0 & 0 & k_{51}^2 & k_{52}^2 & k_{55}^2 & k_{56}^2 \\ 0 & 0 & k_{61}^2 & k_{62}^2 & k_{65}^2 & k_{66}^2 \end{bmatrix} \begin{bmatrix} V_1(t) \\ V_2(t) \\ V_3(t) \\ V_4(t) \\ V_5(t) \\ V_6(t) \end{bmatrix}$$

$$+ \begin{bmatrix} m_{11}^1 & m_{12}^1 & 0 & 0 & 0 & 0 \\ m_{21}^1 & m_{22}^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_{55}^1 + m_{11}^2 & m_{56}^1 + m_{12}^2 & 0 & 0 \\ 0 & 0 & m_{65}^1 + m_{21}^2 & m_{66}^1 + m_{22}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_{55}^2 & m_{56}^2 \\ 0 & 0 & 0 & 0 & m_{65}^2 & m_{66}^2 \end{bmatrix} \begin{bmatrix} \ddot{V}_1(t) \\ \ddot{V}_2(t) \\ \ddot{V}_3(t) \\ \ddot{V}_4(t) \\ \ddot{V}_5(t) \\ \ddot{V}_6(t) \end{bmatrix}$$

$$+ \begin{bmatrix} c_{11}^1 & c_{12}^1 & 0 & 0 & 0 & 0 \\ c_{21}^1 & c_{22}^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{55}^1 + c_{11}^2 & c_{56}^1 + c_{12}^2 & 0 & 0 \\ 0 & 0 & c_{65}^1 + c_{21}^2 & c_{66}^1 + c_{22}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55}^2 & c_{56}^2 \\ 0 & 0 & 0 & 0 & c_{65}^2 & c_{66}^2 \end{bmatrix} \begin{bmatrix} \dot{V}_1(t) \\ \dot{V}_2(t) \\ \dot{V}_3(t) \\ \dot{V}_4(t) \\ \dot{V}_5(t) \\ \dot{V}_6(t) \end{bmatrix}$$

$$= \begin{bmatrix} f_1^1 \\ 0 \\ f_3^1 + f_1^2 \\ 0 \\ f_3^2 \\ 0 \end{bmatrix} + \begin{bmatrix} Q_1^1 \\ Q_2^1 \\ Q_5^1 + Q_1^2 \\ Q_6^1 + Q_2^2 \\ Q_5^2 \\ Q_6^2 \end{bmatrix} \tag{56}$$

Or simply as:

$$[K]\,V(t) + [C]\,\dot{V}(t) + [M]\,\ddot{V}(t) = F \tag{57}$$

10. IMPOSITION OF BOUNDARY CONDITIONS

In (57), there are 6 equations and 14 unknowns. Application of the associated boundary conditions will make the number of equations equal the number of unknowns and yield a unique solution.

For a clamped Timoshenko beam, the boundary conditions imply that at global nodes 1 and 3, the unknowns are equal to zero, that

is:

$$u_1^1 \equiv v_1^1 = V_1 = 0,\ w_1^1 \equiv v_2^1 = V_2 = 0,$$

$$u_3^2 \equiv v_5^2 = V_5 = 0,\ w_3^2 \equiv v_6^2 = V_6 = 0 \tag{58}$$

By (51), at global node 2, the connecting nodes, there are no externally applied shear forces and bending moment. Hence,

$$Q_5^1 + Q_1^2 = 0;\ Q_6^1 + Q_2^2 = 0 \tag{59}$$

Using (51, (58) and (59) in (57), the system of equation for a two-element clamped tapered Timoshenko beam under a uniform partially distributed moving load becomes:

$$\begin{bmatrix} k_{11}^1 & k_{12}^1 & k_{15}^1 & k_{16}^1 & 0 & 0 \\ k_{21}^1 & k_{22}^1 & k_{25}^1 & k_{26}^1 & 0 & 0 \\ k_{51}^1 & k_{52}^1 & k_{55}^1 + k_{11}^2 & k_{56}^1 + k_{12}^2 & k_{15}^2 & k_{16}^2 \\ k_{61}^1 & k_{62}^1 & k_{65}^1 + k_{21}^2 & k_{66}^1 + k_{22}^2 & k_{25}^2 & k_{26}^2 \\ 0 & 0 & k_{51}^2 & k_{52}^2 & k_{55}^2 & k_{56}^2 \\ 0 & 0 & k_{61}^2 & k_{62}^2 & k_{65}^2 & k_{66}^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ V_3 \\ V_4 \\ 0 \\ 0 \end{bmatrix}$$

$$+ \begin{bmatrix} m_{11}^1 & m_{12}^1 & 0 & 0 & 0 & 0 \\ m_{21}^1 & m_{22}^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_{55}^1 + m_{11}^2 & m_{56}^1 + m_{12}^2 & 0 & 0 \\ 0 & 0 & m_{65}^1 + m_{21}^2 & m_{66}^1 + m_{22}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_{55}^2 & m_{56}^2 \\ 0 & 0 & 0 & 0 & m_{65}^2 & m_{66}^2 \end{bmatrix} \begin{bmatrix} \ddot{V}_1 \\ \ddot{V}_2 \\ \ddot{V}_3 \\ \ddot{V}_4 \\ \ddot{V}_5 \\ \ddot{V}_6 \end{bmatrix}$$

$$+ \begin{bmatrix} c_{11}^1 & c_{12}^1 & 0 & 0 & 0 & 0 \\ c_{21}^1 & c_{22}^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{55}^1 + c_{11}^2 & c_{56}^1 + c_{12}^2 & 0 & 0 \\ 0 & 0 & c_{65}^1 + c_{21}^2 & c_{66}^1 + c_{22}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55}^2 & c_{56}^2 \\ 0 & 0 & 0 & 0 & c_{65}^2 & c_{66}^2 \end{bmatrix} \begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{V}_3 \\ \dot{V}_4 \\ \dot{V}_5 \\ \dot{V}_6 \end{bmatrix}$$

$$= \begin{bmatrix} f_1^1 \\ 0 \\ f_3^1 + f_1^2 \\ 0 \\ f_3^2 \\ 0 \end{bmatrix} + \begin{bmatrix} Q_1^1 \\ Q_2^1 \\ 0 \\ 0 \\ Q_5^2 \\ Q_6^2 \end{bmatrix} \tag{60}$$

For a simply supported Timoshenko beam, the boundary conditions imply that

$$u_1^1 \equiv v_1^1 = V_1 = 0,\ u_3^2 \equiv v_5^2 = V_5 = 0\ ;\ Q_2^1 = Q_6^2 = 0 \tag{61}$$

Using (51) and (61) in (57), the system of equation for a two-element simply supported tapered Timoshenko beam under a uniform partially distributed moving load becomes:

$$\begin{bmatrix} k_{11}^1 & k_{12}^1 & k_{15}^1 & k_{16}^1 & 0 & 0 \\ k_{21}^1 & k_{22}^1 & k_{25}^1 & k_{26}^1 & 0 & 0 \\ k_{51}^1 & k_{52}^1 & k_{55}^1 + k_{11}^2 & k_{56}^1 + k_{12}^2 & k_{15}^2 & k_{16}^2 \\ k_{61}^1 & k_{62}^1 & k_{65}^1 + k_{21}^2 & k_{66}^1 + k_{22}^2 & k_{25}^2 & k_{26}^2 \\ 0 & 0 & k_{51}^2 & k_{52}^2 & k_{55}^2 & k_{56}^2 \\ 0 & 0 & k_{61}^2 & k_{62}^2 & k_{65}^2 & k_{66}^2 \end{bmatrix} \begin{bmatrix} 0 \\ V_2 \\ V_3 \\ V_4 \\ 0 \\ V_6 \end{bmatrix}$$

$$+ \begin{bmatrix} m_{11}^1 & m_{12}^1 & 0 & 0 & 0 & 0 \\ m_{21}^1 & m_{22}^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_{55}^1 + m_{11}^2 & m_{56}^1 + m_{12}^2 & 0 & 0 \\ 0 & 0 & m_{65}^1 + m_{21}^2 & m_{66}^1 + m_{22}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_{55}^2 & m_{56}^2 \\ 0 & 0 & 0 & 0 & m_{65}^2 & m_{66}^2 \end{bmatrix} \begin{bmatrix} \ddot{V}_1 \\ \ddot{V}_2 \\ \ddot{V}_3 \\ \ddot{V}_4 \\ \ddot{V}_5 \\ \ddot{V}_6 \end{bmatrix}$$

$$+ \begin{bmatrix} c_{11}^1 & c_{12}^1 & 0 & 0 & 0 & 0 \\ c_{21}^1 & c_{22}^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{55}^1 + c_{11}^2 & c_{56}^1 + c_{12}^2 & 0 & 0 \\ 0 & 0 & c_{65}^1 + c_{21}^2 & c_{66}^1 + c_{22}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55}^2 & c_{56}^2 \\ 0 & 0 & 0 & 0 & c_{65}^2 & c_{66}^2 \end{bmatrix} \begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{V}_3 \\ \dot{V}_4 \\ \dot{V}_5 \\ \dot{V}_6 \end{bmatrix}$$

$$= \begin{bmatrix} f_1^1 \\ 0 \\ f_3^1 + f_1^2 \\ 0 \\ f_3^2 \\ 0 \end{bmatrix} + \begin{bmatrix} Q_1^1 \\ 0 \\ 0 \\ 0 \\ Q_5^2 \\ 0 \end{bmatrix} \tag{62}$$

However, for the cantilever Timoshenko beam, the boundary conditions indicates,

$$V_1 = V_2 = 0\ ;\quad Q_5^2 = Q_6^2 = 0 \tag{63}$$

Again, using (51) and (63) in (57), the system of equation for a two-element cantilever tapered Timoshenko beam under a uniform

partially distributed moving load becomes:

$$\begin{bmatrix} k_{11}^1 & k_{12}^1 & k_{15}^1 & k_{16}^1 & 0 & 0 \\ k_{21}^1 & k_{22}^1 & k_{25}^1 & k_{26}^1 & 0 & 0 \\ k_{51}^1 & k_{52}^1 & k_{55}^1 + k_{11}^2 & k_{56}^1 + k_{12}^2 & k_{15}^2 & k_{16}^2 \\ k_{61}^1 & k_{62}^1 & k_{65}^1 + k_{21}^2 & k_{66}^1 + k_{22}^2 & k_{25}^2 & k_{26}^2 \\ 0 & 0 & k_{51}^2 & k_{52}^2 & k_{55}^2 & k_{56}^2 \\ 0 & 0 & k_{61}^2 & k_{62}^2 & k_{65}^2 & k_{66}^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{bmatrix}$$

$$+ \begin{bmatrix} m_{11}^1 & m_{12}^1 & 0 & 0 & 0 & 0 \\ m_{21}^1 & m_{22}^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_{55}^1 + m_{11}^2 & m_{56}^1 + m_{12}^2 & 0 & 0 \\ 0 & 0 & m_{65}^1 + m_{21}^2 & m_{66}^1 + m_{22}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_{55}^2 & m_{56}^2 \\ 0 & 0 & 0 & 0 & m_{65}^2 & m_{66}^2 \end{bmatrix} \begin{bmatrix} \ddot{V}_1 \\ \ddot{V}_2 \\ \ddot{V}_3 \\ \ddot{V}_4 \\ \ddot{V}_5 \\ \ddot{V}_6 \end{bmatrix}$$

$$+ \begin{bmatrix} c_{11}^1 & c_{12}^1 & 0 & 0 & 0 & 0 \\ c_{21}^1 & c_{22}^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{55}^1 + c_{11}^2 & c_{56}^1 + c_{12}^2 & 0 & 0 \\ 0 & 0 & c_{65}^1 + c_{21}^2 & c_{66}^1 + c_{22}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55}^2 & c_{56}^2 \\ 0 & 0 & 0 & 0 & c_{65}^2 & c_{66}^2 \end{bmatrix} \begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{V}_3 \\ \dot{V}_4 \\ \dot{V}_5 \\ \dot{V}_6 \end{bmatrix}$$

$$= \begin{bmatrix} f_1^1 \\ 0 \\ f_3^1 + f_1^2 \\ 0 \\ f_3^2 \\ 0 \end{bmatrix} + \begin{bmatrix} Q_1^1 \\ Q_2^1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{64}$$

The systems in (60), (62), and (64) comprises of 6 equations and 6 unknowns, which can be solved to yield unique solutions.

11. SOLUTION OF ELEMENT EQUATIONS

Any of the system of second order differential equations (60), (62) or (64), which can be written as:

$$[K]\,V(t) + [C]\,\dot{V}(t) + [M]\,\ddot{V}(t) = F \tag{65}$$

is solved using the Newmark method.

The Newmark equations are [22]:

$$V_{s+1} = V_s + \Delta t \dot{V}_s + \left(\frac{1}{2} - \beta\right)(\Delta t)^2 \ddot{V}_s + \beta(\Delta t)^2 \ddot{V}_{s+1} \tag{66}$$

$$\dot{V}_{s+1} = \dot{V}_s + (1-\alpha)\,\Delta t\ddot{V}_s + \alpha\Delta t\ddot{V}_{s+1} \tag{67}$$

The parameters β and α in (66) and (67) define the variation of acceleration over a time step and determine the stability and accuracy characteristics of the method. Apart from the special cases such as the Galerkin method and the backward difference methods, α has a constant value of $\frac{1}{2}$ for all other cases, including the cases considered in the present study. As a result, (67) becomes

$$\dot{V}_{s+1} = \dot{V}_s + \frac{\Delta t}{2}\left(\ddot{V}_s + \ddot{V}_{s+1}\right) \tag{68}$$

where the notations used are as defined in (69):

$$V_s = V \ at\ time\ t_s\ ;\ V_{s+1} = V\ at\ time\ t_{s+1}\ ;\ \Delta t = t_{s+1} - t_s \tag{69}$$

The substitution of (68) into (66), applying it to (65), and collecting like terms results in:

$$\hat{K}_{s+1}V_{s+1} = \hat{F}_{s,s+1} \tag{70}$$

where

$$\begin{aligned}
\hat{K}_{s+1} &= (K_{s+1} + a_3M_{s+1} + a_6C_{s+1}) \\
\hat{F}_{s,s+1} &= F_{s+1} + M_{s+1}\left(a_3V_s + a_4\dot{V}_s + a_5\ddot{V}_s\right) \\
&+ C_{s+1}\left(a_6V_s + a_7\dot{V}_s + a_8\ddot{V}_s\right) \\
a_3 &= \frac{1}{\beta(\Delta t)^2},\ a_4 = a_3\Delta t,\ a_5 = \frac{1}{2\beta} - 1
\end{aligned} \tag{71}$$

The $\ddot{V}_0$ needed to start the computation is obtained from (65) as:

$$\ddot{V}_0 = M^{-1}\left(F_0 - C\dot{V}_0 - KV_0\right) \tag{72}$$

with the assumption that the applied force F at $t = 0$, is zero. That is, $F_0 = 0$.

Using equation (70) and (71), the displacement vector (deflection and rotation) V_{s+1}, at time t_{s+1} can be obtained from the previously determined earlier values of the displacement, velocity, and acceleration vectors $V_s, \dot{V}_s$, and $\ddot{V}_s$ obtained at time t_s.

At the end of each time step, the new velocity vector $\dot{V}_{s+1}$ and acceleration vector $\ddot{V}_{s+1}$ are computed using (66) and (68) as,

$$\begin{aligned}
\ddot{V}_{s+1} &= a_3\left(V_{s+1} - V_s\right) - a_4\dot{V}_s - a_5\ddot{V}_s \\
\dot{V}_{s+1} &= \dot{V}_s + a_2\ddot{V}_s + a_1\ddot{V}_{s+1}
\end{aligned}$$

$$a_1 = \alpha\Delta t, /a_2 = (1-\alpha)\Delta t \tag{73}$$

Newmark's method is stable if

$$\frac{\Delta t}{T_n} \leq \frac{1}{\pi\sqrt{2}} \frac{1}{\sqrt{0.5-2\beta}} \tag{74}$$

This means that the average acceleration method ($\beta = \frac{1}{4}$) is stable for any Δt (unconditionally stable). For the linear acceleration method ($\beta = \frac{1}{6}$), (74) indicates that it is stable if

$$\frac{\Delta t}{T_n} < 0.551 \tag{75}$$

where T_n is the time period. Meaning that a shorter time step than $0.5513T_n$ must be used.

12. NUMERICAL EXAMPLES

To illustrate the theory so far developed, we consider a non uniform simply supported, tapered beam according to the Timoshenko beam theory. The total length of the beam L is such that $L = 17.5m$. The beam is dicretized into ten non-uniform elements. The beam's density $\rho = 2400kgm^{-3}$, elastic modulus$E = 2.02 \times 10^{11}Nm^{-2}$, shear modulus $G = 7.7 \times 10^{10}Nm^{-2}$, and shear coefficient $K = \frac{5}{6}$. The load's density $\rho_q = 240kgm^{-3}$, moment of inertia $I_q = 0.0012m^4$, mass $p = 1062kg$, moving speed $v = 30ms^{-1}$, and length $\varepsilon = 0.2m$. The gravitational acceleration is $g = 9.8ms^{-2}$.

The length of each element is given as:

$$L_1 = 1.25m,\ L_2 = 1.25m,\ L_3 = 1.5m,\ L_4 = 1.5m,\ L_5 = 1.75m,$$
$$L_6 = 1.75m, L_7 = 2m,\ L_8 = 2m,\ L_9 = 2.25m,\ L_{10} = 2.25m \tag{76}$$

The cross-section of the beam is such that its width is uniform from end-to-end, and is given as

$$b_1 = b_2 = b_3 = b_4 = b_5 = b_6 = b_7 = b_8 = b_9 = b_{10} = 0.41m \tag{77}$$

The non-uniform (or tapered) nature of the beam is determined by its varying depth (height), which is given from left to right as:

$$h_0 = 0.52m,\ h_1 = 0.5m,\ h_2 = 0.48m,\ h_3 = 0.46m,$$
$$h_4 = 0.44m,\ h_5 = 0.42m,\ h_6 = 0.4m,\ h_7 = 0.38m,$$
$$h_8 = 0.36m,\ h_9 = 0.34m,\ h_{10} = 0.32m \tag{78}$$

The beam's cross-sectional area $A(x)$ and moment of inertia $I(x)$ are calculated using (7). For the secondary variables, bending moments at the support ends are equal to zero (simply supported boundary). They are both unknown at the ends for cantilever and

clamped beams. The solutions were obtained using Matlab programs and graphs plotted using Microsoft Excel.

The effects of the following, to the dynamic response of the present beam were investigated: i. Load's speed. ii. Load's length. iii. Beam's length. iv. Changes in boundary conditions. v. Time history of the mid-span and the end of the beam. vi. Load-Beam mass ratio.

Effect of the load's speed on the dynamic response of the beam

The effect of load's speed on the dynamic response of the non-uniform simply supported Timoshenko beam under distributed moving load was investigated with three load's speed a control speed of 30m/s, which was then reduced to 20m/s and later increased to 40m/s. It is observed that as the load's speed was increased, the maximum amplitude of deflection(u) decreased, and vice versa. The same was observed with the maximum amplitude of rotation(w), which also decreased as the speed of the load increased. The results are shown in Figures (1a) and (1b).

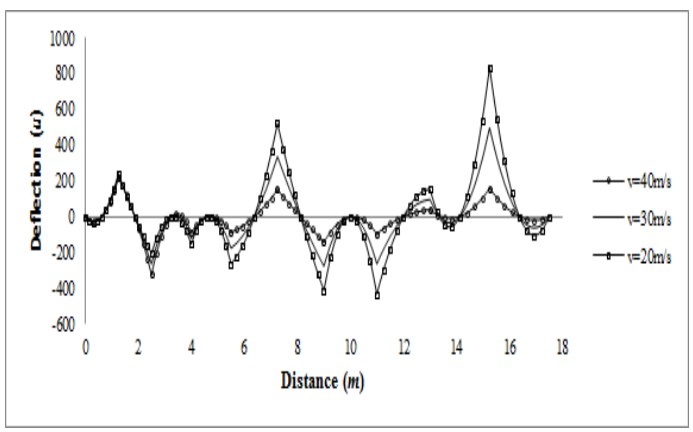

Fig. 1a. Deflection at various load speed.

Effect of the load's length on the dynamic response of the beam

To investigate the effect of load's length on the dynamic response of the non-uniform simply supported Timoshenko beam under distributed moving load, a control length of 0.2m and two other lengths, 0.3m and 0.15m. It was observed that the maximum amplitude of response of both the deflection and rotation increased as the load's length decreased, (Figures (2a) and (2b)). But when the length was further reduced to 0.1m, the trend reversed even though their amplitudes were still increasing. Figures (3a) and (3b)

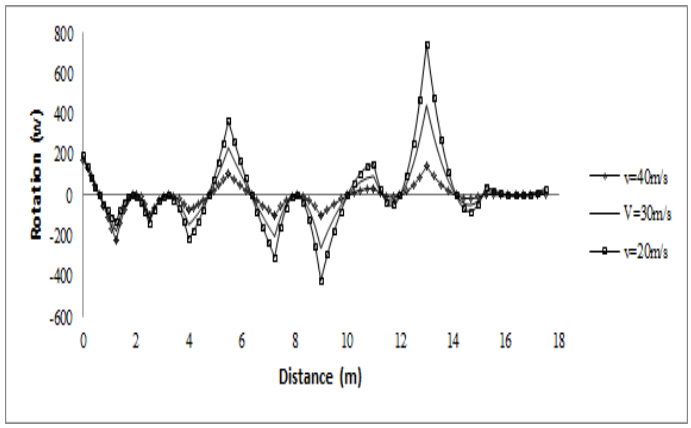

Fig. 1b. Rotation at various load speed.

show this reversed trend, which was observed for load's lengths of $\varepsilon < 0.15$.

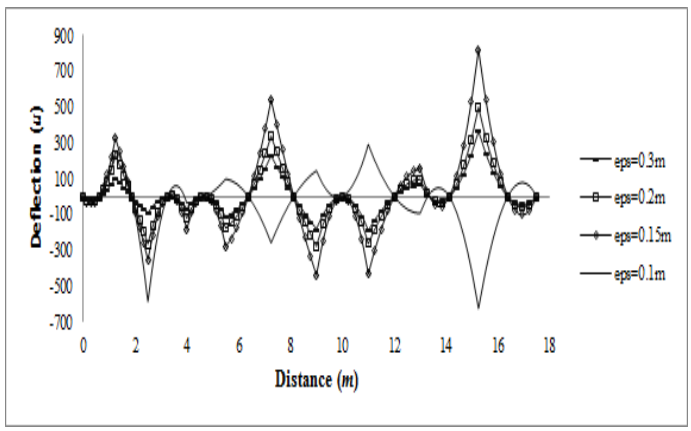

Fig. 2a. Deflection at various load length.

Effect of the beam's length on the dynamic response of the beam

With the beam's span lengths of L = 15m, L = 17.5m, and L = 20m, it was observed that the maximum amplitude of response of the deflection (u) and rotation (w) increased as the length of the simply supported non-uniform Timoshenko beam decreased. See Figures (4a), (4b) and (4c).

Effect of changes in boundary conditions on the dynamic response

The maximum amplitude of deflection and rotation were far higher with the simply supported boundary than the clamped and cantilever boundaries. The values for clamped boundary were slightly higher than for the cantilever boundary. See Figures (5a,b.c).

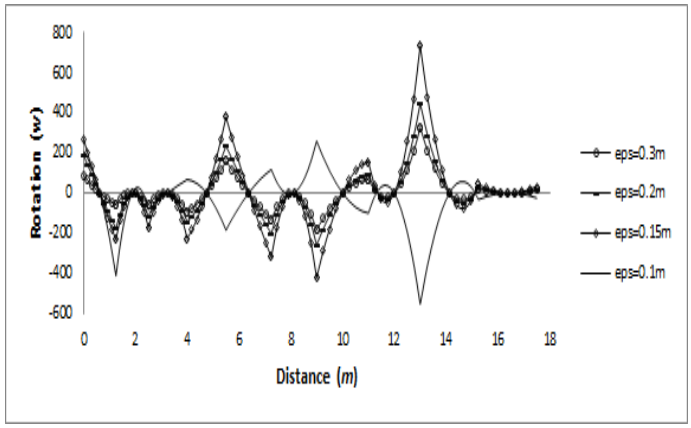

Fig. 2b. Rotation at various load length.

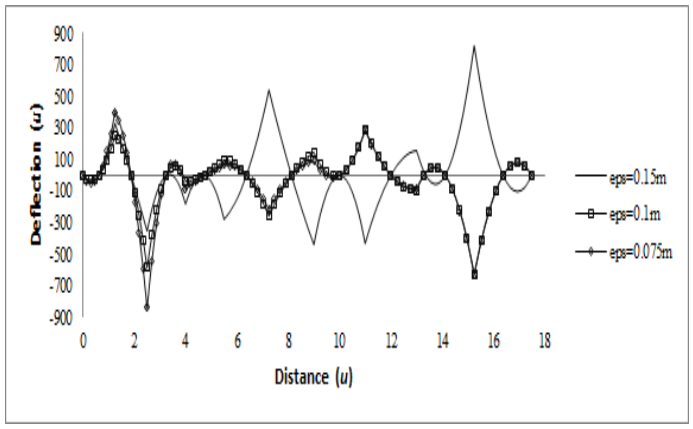

Fig. 3a. Deflection at load length $\varepsilon = 0.15m$.

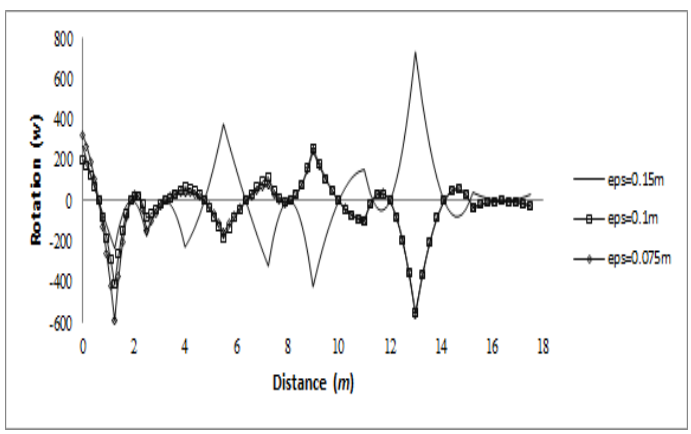

Fig. 3b. Rotation at load length $\varepsilon = 0.15m$.

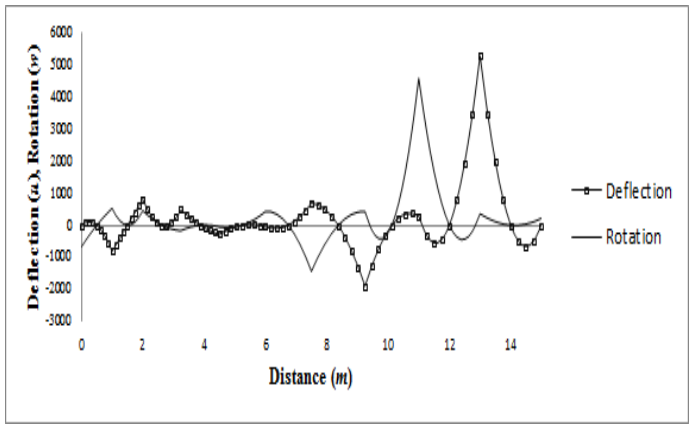

Fig. 4a. Deflection and Rotation with a beam length of 15m.

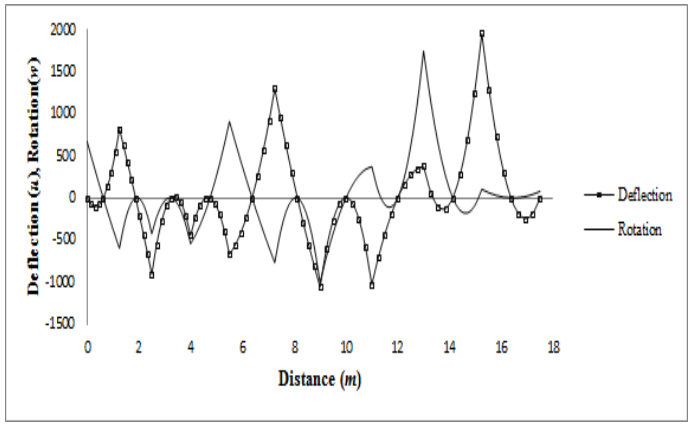

Fig. 4b. Deflection and Rotation with a beam length of 17.5m.

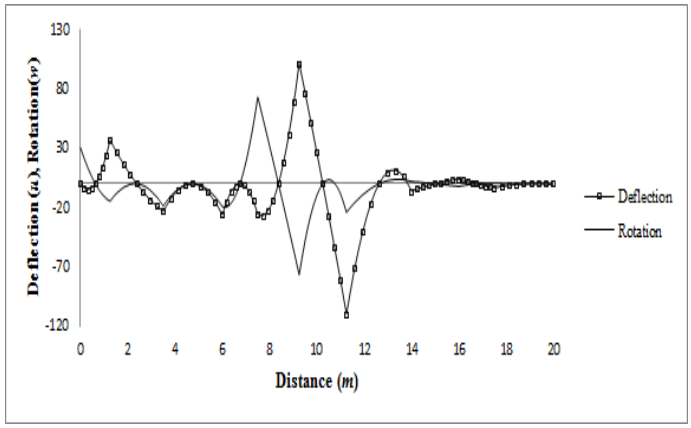

Fig. 4c. Deflection and Rotation with a beam length of 20m.

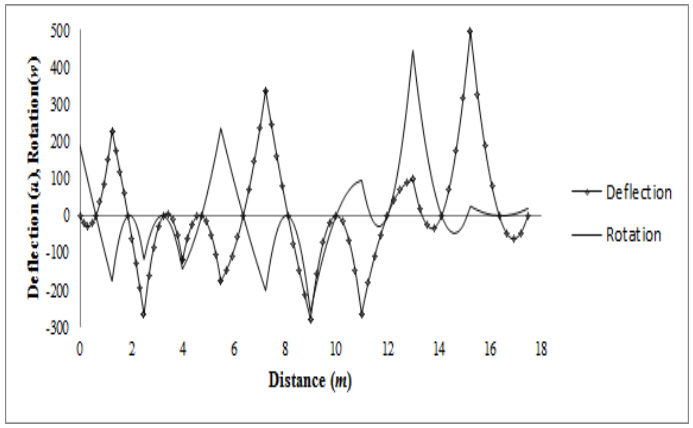

Fig. 5a. Deflection and Rotation of a simply-supported beam.

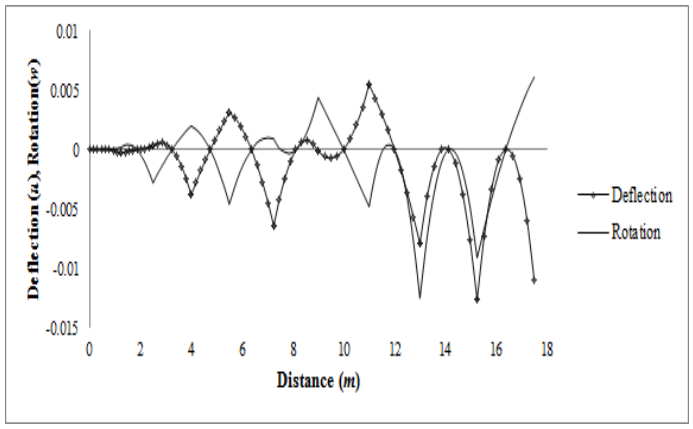

Fig. 5b. Deflection and Rotation of a cantilevered beam.

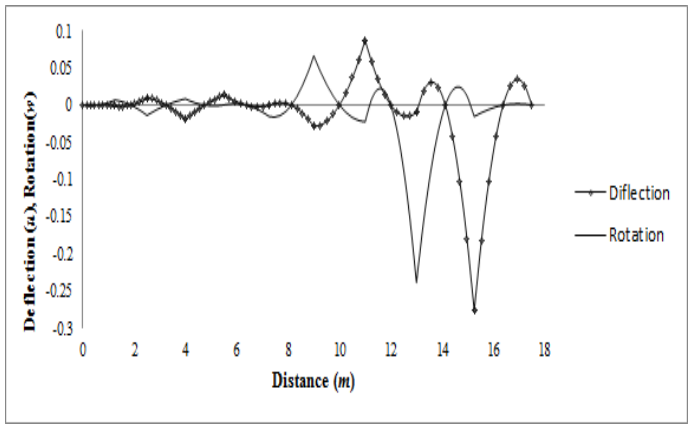

Fig. 5c. Deflection and Rotation of a clamped beam.

Time history of the mid-span and the end of the beam (dt=0.1s)

As figure (6) show, the deflection (u) and rotation (w) were not significantly different from zero until after 5 second for a simply supported boundary at the mid span, Figure (6a). The deflection was zero all through at the end of the beam, Figure (6b). As for the cantilever beam, the amplitude of both the deflection (u) and rotation (w) were not significantly different from zero until after 5 second at the end of the beam while the deflection (u) was zero and very close to zero at the mid-span of the beam, Figure (6c). The deflection (u), was zero and very close to zero at the mid-span of the beam for the clamped boundary, Figure (6d). While the rotation (w) was not significantly different from zero until after 5 second, Figure (6e).

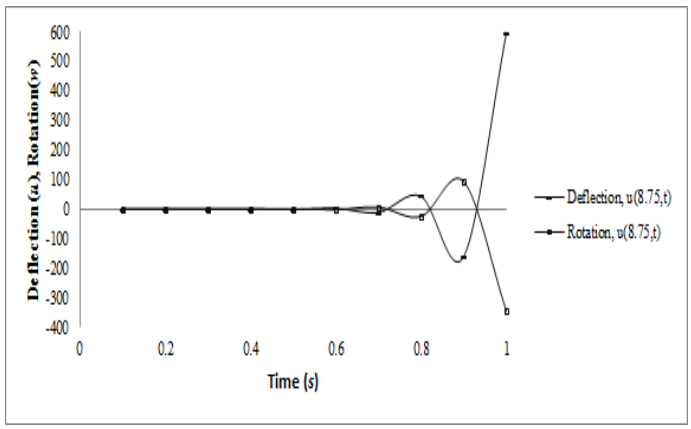

Fig. 6a. Time history at the mid-span of the simply-supported beam.

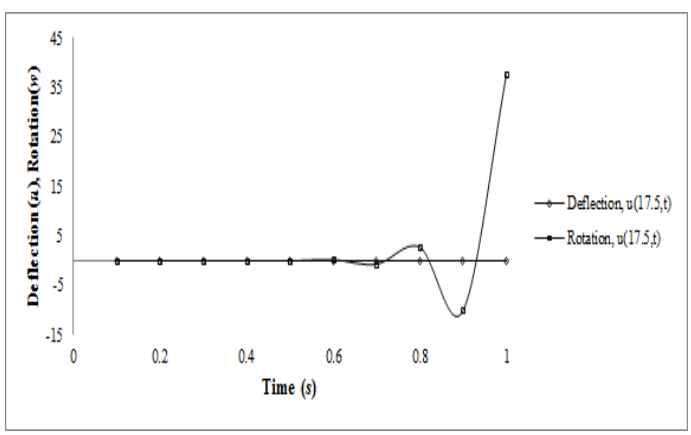

Fig. 6b. Time history at the end of the simply-supported beam.

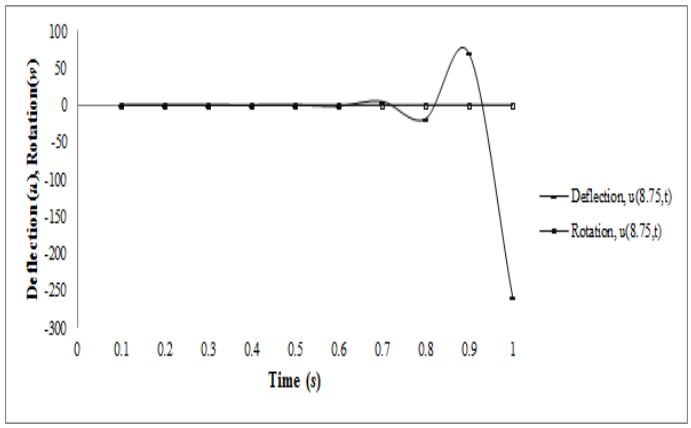

Fig. 6c. Time history at the mid-span of the clamped beam.

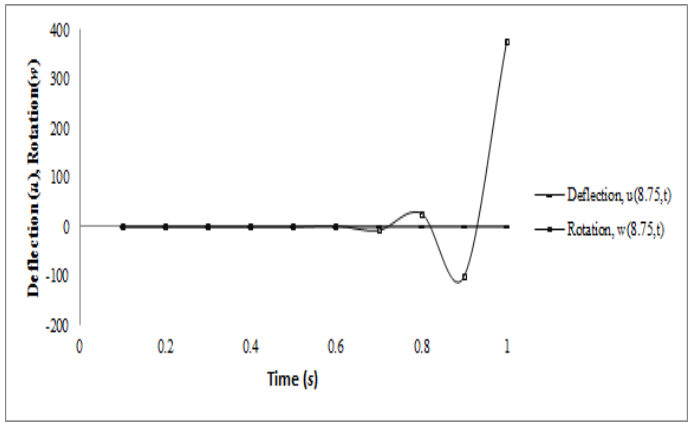

Fig. 6d. Time history at the mid-span of the cantilevered beam.

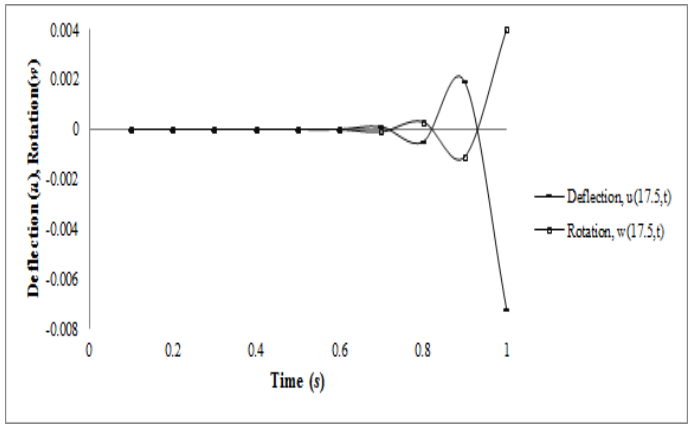

Fig. 6e. Time history at the end of the cantilevered beam.

Effect of changes in the load-beam mass ratio on the beam response

When the load-beam mass ratio was varied from an average of 0.1, 0.2, 0.3, and 0.4, the trend of deflection of the beam reversed from 0.1 to 0.2. The trend became the same from 0.2 to 0.4, but with increasing maximum amplitudes - Figure (7a). The same was observed for the trend of rotation - Figure (7b).

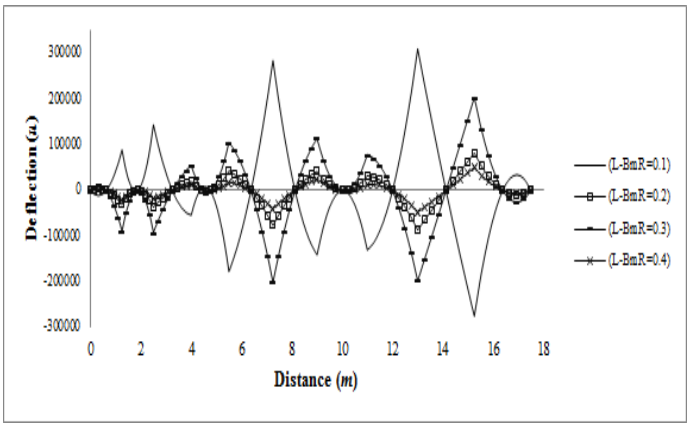

Fig. 7a. Deflection of the beam at various Load-Beam mass ratios.

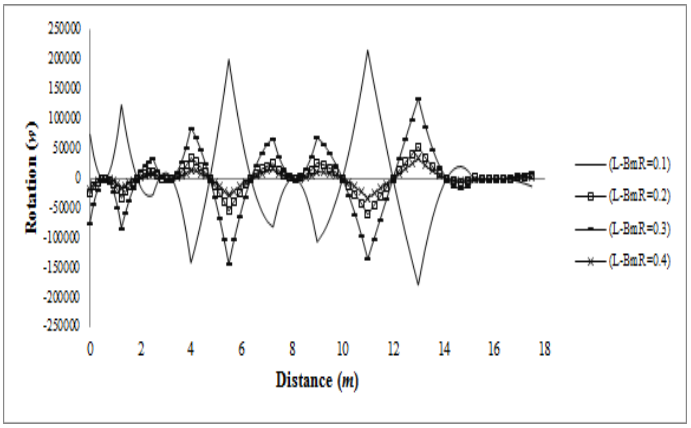

Fig. 7b. Rotation of the beam at various Load-Beam mass ratios.

13. CONCLUDING REMARKS

The dynamic response of tapered Timoshenko beams under uniform partially distributed moving loads has been presented in this paper. The finite element method as a numerical procedure using Lagrange interpolation with reduced integration element was used to obtain the finite element equations. These equations, being semi-discrete were solved using the Newmark method. The results obtained show that the load's velocity and length; as well as the boundary conditions, beam's length, and load-beam mass ratio all have significant effects on the dynamic response of the beam.

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