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VIBRATIONS OF A SIMPLY SUPPORTED PLATE UNDER MOVING MASSES AND RESTING ON PASTERNAK ELASTIC FOUNDATION WITH STIFFNESS VARIATION

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ABSTRACT. In this investigation, the dynamic behaviour of simply supported rectangular plate carrying moving masses and resting on Pasternak elastic foundation with stiffness variation is considered. In order to solve the governing fourth order partial differential equation, a technique based on separation of variables is used to reduce the equation with variable and singular coefficients to a sequence of second order ordinary differential equations. The modified method of Struble and the integral transformations are then employed for the solutions of the reduced equations. Numerical results in plotted curves are then presented. The results show that as the value of the rotatory inertia correction factor R_o increases, the response amplitudes of the plate decrease and for fixed value of R_o, the displacements of the simply supported rectangular plates resting on Pasternak elastic foundations with stiffness variation decrease as the foundation modulus F_o increases. It is also shown that as the value of the shear modulus G_o increases the displacement amplitudes of the plate decrease. For fixed R_o, F_o and G_o, the transverse deflections of the rectangular plates under the actions of moving masses are higher than those when only the force effects of the moving load are considered. This implies that resonance is reached earlier in moving mass problem than in moving force problem. Furthermore, the result shows that the critical speed increases as G_o, F_o and R_o increase, this implies that risk is reduced.

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1. INTRODUCTION

Plates resting on subgrades form key components in a wide range of industrial applications, such as bridges, highway pavements, decking slabs and road ways. Such structures are constantly acted upon

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by moving masses, hence, the problem of analyzing the dynamic response of elastic structures under the action of moving masses continues to motivate a variety of investigations [1-6].
The behaviour of plate structures under moving load, in general, is rather complex more so when the inertia effect of the moving load is taken into consideration [1]. Thus, most of the research works available in literature are those in which this effect has been neglected. This is due, at least in part, to the great amount of computational labour, which is required both to set up and to solve the necessary equations. One important problem that arises when the inertia effects of the masses are considered is the singularity which occurs in the inertia terms of the governing differential equation of motion.
Generally, the dynamical problems of structures under moving loads and resting on a foundation are complex. The complexity increases if the foundation stiffness varies along the span of the structure. Aside the problem of singularity brought in by the inclusion of the inertia effects of the moving load, the coefficients of the governing fourth order partial differential equation are no longer constant but variable. Earlier researchers into beam member on variable elastic foundation include Franklin and Scott [7] who presented a closed-form solution to a linear variation of the foundation modulus using contour-integrals. Closely following this, Lentini [8] presented a finite difference method to solve the problem where the foundation stiffness varies along x (the special coordinate) as a power of x. These works, though useful, considered the loads acting on the beams to be static (not moving). Recently, Oni and Awodola [9] extended the works of these previous authors to investigate the dynamic response to moving concentrated masses of uniform Rayleigh beams resting on variable Winkler elastic foundation. Oni and Awodola [10] again considered the dynamic response under a moving load of an elastically supported non-prismatic Bernoulli-Euler beam on variable elastic foundation. The technique was based on the generalized Galerkin's method and integral transformations.
The foundation model based on Winkler's approximation model is very common in literature, whereas, in such an important Engineering problem as the vibration of plates resting on elastic foundation, a more accurate Two-Parameter (Pasternak) foundation model which in addition to subgrade modulus incorporates the

shear effect of the foundation should be used rather than the Winkler's approximation model. Eisenberger and Clastornik [11] presented two methods for the solution of beams on variable two-parameter elastic foundation. Also, Gbadeyan and Oni [12] studied the dynamic analysis of an elastic plate continuously supported by an elastic Pasternak foundation traversed by an arbitrary number of concentrated masses. In their work, they assumed that both the foundation modulus and the shear modulus are constants.
In all these investigations, extension of the theory to cover two-dimensional (plate) problem in which the plate is resting on Pasternak elastic foundation with stiffness variation has not been effected. This study is therefore concerned with the behaviour of simply supported rectangular plate under the action of concentrated moving masses and resting on Pasternak elastic foundation with stiffness variation.

2. GOVERNING EQUATION

The dynamic transverse displacement W(x,y,t) of a rectangular plate when it is resting on a Pasternak elastic foundation with stiffness variation and traversed by several moving concentrated masses is governed by the fourth order partial differential equation given by

$$D\nabla^4 W(x,y,t) + \mu \frac{\partial^2 W(x,y,t)}{\partial t^2} + P_G(x,y,t) = \mu R_0 \frac{\partial^2}{\partial t^2}\nabla^2 W(x,y,t) + P(x,y,t) \quad (1)$$

where

$$D = \frac{Eh^2}{12(1-v)} \quad (2)$$

is the bending rigidity of the plate, ∇^2 is the two-dimensional Laplacian operator, W(x,y,t) is the transverse displacement, h is the plate's thickness, E is the Young's Modulus, v is the Poisson's ratio ($v < 1$), μ is the mass per unit area of the plate, R_0 is the Rotatory inertia correction factor, P_G(x,y,t) is the foundation reaction, P(x,y,t) is the Moving load, x and y are respectively the spatial coordinates in x and y directions and t is the time coordinate.

The relation between the foundation reaction and the lateral deflection W(x,y,t) is given by [11]

$$P_G(x,y,t) = F(x)W(x,y,t) - \frac{\partial}{\partial x}\left[G(x)\frac{\partial}{\partial x}W(x,y,t)\right] - \frac{\partial}{\partial y}\left[G(x)\frac{\partial}{\partial y}W(x,y,t)\right] \quad (3)$$

where F(x) and G(x) are the two variable parameters of the elastic foundation. Specifically, F(x) is the variable foundation stiffness and G(x) is the variable shear modulus.

When the effect of the mass of the moving load on the response of the plate is taken into consideration, the external moving surface load takes on the form

$$P(x,y,t) = P_f(x,y,t)\left[1 - \frac{\Delta^*}{g}W(x,y,t)\right] \tag{4}$$

where $P_f(x,y,t)$ is the continuous moving force, Δ^* is the substantive acceleration operator and g is the acceleration due to gravity.

The moving force acting on the plate is defined as [6]

$$P_f(x,y,t) = \sum_{i=1}^{N} M_i g\delta(x - c_i t)\,\delta(y - s) \tag{5}$$

where $\delta(.)$ is the Dirac – Delta function.

The operator Δ^* used in equation (4) for masses traveling with constant velocity and in an arbitrary path in the x – y plane is defined as

$$\Delta^* = \frac{\partial^2}{\partial t^2} + 2c_i\frac{\partial^2}{\partial x\partial t} + c_i^2\frac{\partial^2}{\partial x^2} \tag{6}$$

As an example in this problem, a variable elastic foundation stiffness of the form [11, 13]

$$F(x) = F_0(4x - 3x^2 + x^3) \tag{7}$$

where F_0 is the foundation constant, and a variable shear modulus of the form

$$G(x) = G_0(12 - 13x + 6x^2 - x^3) \tag{8}$$

where G_0 is a constant are considered.

Thus, substituting (3), (4), (5), (6), (7) and (8) into (1), one obtains

$$\begin{aligned} D\,\nabla^4 W(x,y,t) + \mu\frac{\partial^2 W(x,y,t)}{\partial t^2} = \quad & \mu R_0\left[\tfrac{\partial^4}{\partial t^2\partial x^2} + \tfrac{\partial^4}{\partial t^2\partial y^2}\right]W(x,y,t) \\ & -F_0\left[4x - 3x^2 + x^3\right]W(x,y,t) \\ & +G_0[-13 + 12x - 3x^2]\tfrac{\partial}{\partial x}W(x,y,t) \\ & +G_0[12 - 13x + 6x^2 - x^3]\left[\tfrac{\partial^2}{\partial x^2} + \tfrac{\partial^2}{\partial y^2}\right]W(.) \\ & +\textstyle\sum_{i=1}^{N}[M_i g\delta(x - c_i t)\delta(y - s) \\ & -M_i\left(\tfrac{\partial^2}{\partial t^2} + 2c_i\tfrac{\partial^2}{\partial t\partial x} + c_i^2\tfrac{\partial^2}{\partial x^2}\right) \\ & \times W(x,y,t)\delta(x - c_i t)\delta(y - s)] \end{aligned} \tag{9}$$

The initial conditions, without any loss of generality, is taken as

$$W(x,y,t)\big|_{t=0} = 0 = \left.\frac{\partial W(x,y,t)}{\partial t}\right|_{t=0} \tag{10}$$

3. ANALYTICAL APPROXIMATE SOLUTION

This section seeks to obtain the analytical solution to the problem of the dynamic response of a rectangular plate resting on Pasternak elastic foundation with stiffness variation and subjected to arbitrary support conditions. The method of analysis involves expressing the Dirac – Delta function as a Fourier cosine series. A technique [14] based on separation of variables is used to tackle the fourth order partial differential equation governing the motion of the plate and reduce it to a set of coupled second order ordinary differential equations. Then, the modified asymptotic method of Struble in conjunction with the techniques of integral transformation and convolution theory are then employed to obtain the closed form solution of the resulting second order ordinary differential equations.

In order to solve equation (9), in the first instance, the deflection is written in the form [14]

$$W(x,y,t)=\sum_{n=1}^{\infty}\phi_n(x,y)T_n(t) \tag{11}$$

where ϕ_n are the known eigenfunctions of the plate with the same boundary conditions. The ϕ_n have the form of

$$\nabla^4\phi_n-\omega_n^4\phi_n=0 \tag{12}$$

where

$$\omega_n^4=\frac{\Omega_n^2\mu}{D} \tag{13}$$

Ω_n, n = 1, 2, 3, ..., are the natural frequencies of the dynamical system and $T_n(t)$ are amplitude functions which have to be calculated.

In order to solve the equation (9), it is rewritten as

$$\begin{aligned}\frac{D}{\mu}\nabla^4W(x,y,t)+\frac{\partial^2W(x,y,t)}{\partial t^2} = {} & R_0\left[\frac{\partial^4}{\partial t^2\partial x^2}+\frac{\partial^4}{\partial t^2\partial y^2}\right]W(x,y,t)\\ & -\frac{F_0}{\mu}\left[4x-3x^2+x^3\right]W(x,y,t)\\ & +\frac{G_0}{\mu}[-13+12x-3x^2]\frac{\partial}{\partial x}W(x,y,t)\\ & +\frac{G_0}{\mu}[12-13x+6x^2-x^3]\\ & \times\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]W\\ & +\sum_{i=1}^{N}[\frac{M_ig}{\mu}\delta(x-c_it)\delta(y-s)\\ & -\frac{M_i}{\mu}\left(\frac{\partial^2}{\partial t^2}+2c_i\frac{\partial^2}{\partial t\partial x}+c_i^2\frac{\partial^2}{\partial x^2}\right)\\ & \times W(x,y,t)\delta(x-c_it)\delta(y-s)]\end{aligned} \tag{14}$$

The right hand side of equation (14) is written in the form of a series to have

$$\begin{aligned}\sum_{n=1}^{\infty}\phi_n(x,y)B_n(t) = \; & R_0\left[\tfrac{\partial^4}{\partial t^2\partial x^2}+\tfrac{\partial^4}{\partial t^2\partial y^2}\right]W(x,y,t)-\tfrac{F_0}{\mu}\left[4x-3x^2+x^3\right]W(x,y,t)\\ & +\tfrac{G_0}{\mu}\left[\left(-13+12x-3x^2\right)\tfrac{\partial}{\partial x}W(x,y,t)\left(12-13x+6x^2-x^3\right)\right.\\ & \times\left(\tfrac{\partial^2}{\partial x^2}+\tfrac{\partial^2}{\partial y^2}\right)W(x,y,t)\;\Big]+\textstyle\sum_{i=1}^{N}\left[\tfrac{M_i g}{\mu}\delta(x-c_it)\delta(y-s)\right.\\ & \left.-\tfrac{M_i}{\mu}\left(\tfrac{\partial^2}{\partial t^2}+2c_i\tfrac{\partial^2}{\partial t\partial x}+c_i^2\;\tfrac{\partial^2}{\partial x^2}\right)W(x,y,t)\delta(x-c_it)\delta(y-s)\right]\end{aligned} \tag{15}$$

Substituting equation (11) into equation (15) we have

$$\begin{aligned}\sum_{n=1}^{\infty}\phi_n(x,y)B_n(t) \;=\; & \sum_{n=1}^{\infty}\{R_0\left[\phi_{n,xx}(x,y)T_{n,tt}(t)+\phi_{n,yy}(x,y)T_{n,tt}(t)\right]\\ & -\frac{F_0}{\mu}\left[4x-3x^2+x^3\right]\phi_n(x,y)T_n(t)\\ & +\frac{G_0}{\mu}\left[\;\left(-13+12x-3x^2\right)\;\phi_{n,x}(x,y)T_n(t)\right.\\ & +\left(12-13x+6x^2-x^3\right)\;(\phi_{n,xx}(x,y)T_n(t)+\\ & \;\phi_{n,yy}(x,y)T_n(t))\;\Big]+\sum_{i=1}^{N}\left[\frac{M_i g}{\mu}\delta(x-c_it)\delta(y-s)\right.\\ & -\frac{M_i}{\mu}\left(\phi_n(x,y)T_{n,tt}(t)+2c_i\phi_{n,x}(x,y)T_{n,t}(t)\right.\\ & \left.\left.+c_i^2\phi_{n,xx}(x,y)T_n(t)\;\right)\delta(x-c_it)\delta(y-s)\;\right]\;\}\end{aligned} \tag{16}$$

where

$$\begin{aligned}&\phi_{n,x}(x,y)\quad\text{implies}\quad\frac{\partial\phi_n(x,y)}{\partial x},\quad\phi_{n,xx}(x,y)\quad\text{implies}\quad\frac{\partial^2\phi_n(x,y)}{\partial x^2},\\ &\phi_{n,y}(x,y)\quad\text{implies}\quad\frac{\partial\phi_n(x,y)}{\partial y},\quad\phi_{n,yy}(x,y)\quad\text{implies}\quad\frac{\partial^2\phi_n(x,y)}{\partial y^2},\\ &T_{n,t}(t)\quad\text{implies}\quad\frac{dT_n(t)}{dt}\;\text{ and }\;T_{n,tt}(t)\quad\text{implies}\quad\frac{d^2T_n(t)}{dt^2}\end{aligned} \tag{17}$$

Multiplying both sides of equation (16) by ϕ_p(x,y) and integrating on area A of the plate, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\int_A\phi_n(x,y)\phi_p(x,y)B_n(t)dA \;=\; & \sum_{n=1}^{\infty}\int_A\{R_0\left[\phi_{n,xx}(x,y)\phi_p(x,y)T_{n,tt}(t)\right.\\ & \left.+\phi_{n,yy}(x,y)\phi_p(x,y)T_{n,tt}(t)\right]\\ & -\frac{F_0}{\mu}\left[4x-3x^2+x^3\right]\phi_n(x,y)\phi_p(x,y)T_n(t)\\ & +\frac{G_0}{\mu}\left[\left(-13+12x-3x^2\right)\;\phi_{n,x}(x,y)\phi_p(x,y)T_n\right.\\ & +\left(12-13x+6x^2-x^3\right)\;(\phi_{n,xx}(x,y)\\ & \left.\times\phi_p(x,y)T_n(t)+\phi_{n,yy}(x,y)\phi_p(x,y)T_n(t))\right]\end{aligned} \tag{18}$$

$$
\begin{aligned}
&+\sum_{i=1}^{N}\Big[\ \frac{M_i g}{\mu}\phi_p(x,y)\delta(x-c_it)\delta(y-s)\\
&-\frac{M_i}{\mu}\left(\phi_n(x,y)\phi_p(x,y)T_{n,tt}(t)\right.\\
&+2c_i\phi_{n,x}(x,y)\phi_p(x,y)T_{n,t}(t)\\
&\left.+c_i^2\phi_{n,xx}(x,y)\phi_p(x,y)T_n(t)\ \right)\delta(x-c_it)\\
&\delta(y-s)\ \Big]\ \Big\}\,dA
\end{aligned}
$$

Considering the orthogonality of ϕ_n(x,y), we have

$$
\begin{aligned}
B_n(t) = \;& \tfrac{1}{P^*}\textstyle\sum_{n=1}^{\infty}\int_A \{R_0\left[\phi_{n,xx}(x,y)\phi_p(x,y)T_{n,tt}(t)\right.\\
&\left.+\phi_{n,yy}(x,y)\phi_p(x,y)T_{n,tt}(t)\right]\\
&-\tfrac{F_0}{\mu}\left[4x-3x^2+x^3\right]\phi_n(x,y)\phi_p(x,y)T_n(t)\\
&+\tfrac{G_0}{\mu}\left[\left(-13+12x-3x^2\right)\phi_{n,x}(x,y)\phi_p(x,y)T_n(t)\right.\\
&+\left(12-13x+6x^2-x^3\right)(\phi_{n,xx}(x,y)\phi_p(x,y)T_n(t)\\
&\left.+\phi_{n,yy}(x,y)\phi_p(x,y)T_n(t))\right]\\
&+\textstyle\sum_{i=1}^{N}\big[\ \tfrac{M_i g}{\mu}\phi_p(x,y)\delta(x-c_it)\delta(y-s)\\
&-\tfrac{M_i}{\mu}\left(\phi_n(x,y)\phi_p(x,y)T_{n,tt}(t)\right.\\
&+2c_i\phi_{n,x}(x,y)\phi_p(x,y)T_{n,t}(t)\\
&\left.+c_i^2\phi_{n,xx}(x,y)\phi_p(x,y)T_n(t)\ \right)\delta(x-c_it)\delta(y-s)\ \big]\ \}\ dA
\end{aligned}
\tag{19}
$$

where $P^* = \int_A \phi_p^2 dA$ Using (19), equation (14), taking into account (11) and (12), can be written as

$$
\begin{aligned}
\phi_n(x,y)\left[\frac{D\omega_n^4}{\mu}T_n(t)+T_{n,tt}(t)\right] =\;& \frac{\phi_n(x,y)}{P^*}\sum_{q=1}^{\infty}\int_A\{R_0\left[\phi_{q,xx}(x,y)\phi_p(x,y)T_{q,tt}(t)\right.\\
&\left.+\ \phi_{q,yy}(x,y)\phi_p(x,y)T_{q,tt}(t)\right]\\
&-\frac{F_0}{\mu}\left[4x-3x^2+x^3\right]\phi_q(x,y)\phi_p(x,y)T_q(t)\\
&+\frac{G_0}{\mu}\left[\left(-13+12x-3x^2\right)\phi_{q,x}(x,y)\phi_p(x,y)\right.\\
&\times T_q(t)+\left(12-13x+6x^2-x^3\right)(\phi_{q,xx}(x,y)\\
&\left.\times\phi_p(x,y)T_q(t)+\phi_{q,yy}(x,y)\phi_p(x,y)T_q(t))\right]\\
&+\sum_{i=1}^{N}\Big[\ \frac{M_i g}{\mu}\phi_p(x,y)\delta(x-c_it)\delta(y-s)\\
&-\frac{M_i}{\mu}\left(\phi_q(x,y)\phi_p(x,y)T_{q,tt}(t)\ +2c_i\phi_{q,x}(x,y)\phi_p\right.\\
&\left.\times(x,y)T_{q,t}(t)+c_i^2\phi_{q,xx}(x,y)\phi_p(x,y)T_q(t)\ \right)\\
&\times\delta(x-c_it)\delta(y-s)\Big]\}\ dA
\end{aligned}
\tag{20}
$$

Equation (20) must be satisfied for arbitrary x, y and this is possible only when

$$
\begin{aligned}
T_{n,tt}(t)+\frac{D\omega_n^4}{\mu}T_n(t) =\;& \frac{1}{P^*}\sum_{q=1}^{\infty}\int_A\{R_0\left[\phi_{q,xx}(x,y)\phi_p(x,y)T_{q,tt}(t)\right.\\
&\left.+\phi_{q,yy}(x,y)\phi_p(x,y)T_{q,tt}(t)\right]
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{F_0}{\mu}\left[4x-3x^2+x^3\right]\phi_q(x,y)\phi_p(x,y)T_q(t)\\
&+\frac{G_0}{\mu}\left[\left(-13+12x-3x^2\right)\phi_{q,x}(x,y)\phi_p(x,y)T_q(t)\right.\\
&+\left(12-13x+6x^2-x^3\right)(\phi_{q,xx}(x,y)\phi_p(x,y)T_q(t) \qquad (21)\\
&\left.+\phi_{q,yy}(x,y)\phi_p(x,y)T_q(t))\right]\\
&+\sum_{i=1}^{N}\Big[\;\frac{M_i g}{\mu}\phi_p(x,y)\delta(x-c_i t)\delta(y-s)\\
&-\frac{M_i}{\mu}\left(\phi_q(x,y)\phi_p(x,y)T_{q,tt}(t)\right.\\
&+2c_i\phi_{q,x}(x,y)\phi_p(x,y)T_{q,t}(t)\\
&\left.+c_i^2\phi_{q,xx}(x,y)\phi_p(x,y)T_q(t)\;\right)\delta(x-c_i t)\delta(y-s)\;\Big]\;\Big\}\;dA
\end{aligned}
$$

The system in equation (21) is a set of coupled ordinary differential equations.
Considering the property of the Dirac-Delta function and expressing it in the Fourier cosine series as

$$
\delta(x-c_i t)=\frac{1}{L_X}\left[1+2\sum_{j=1}^{\infty}\cos\frac{j\pi c_i t}{L_X}\cos\frac{j\pi x}{L_X}\right] \qquad (22)
$$

and

$$
\delta(y-s)=\frac{1}{L_Y}\left[1+2\sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}\cos\frac{k\pi y}{L_Y}\right] \qquad (23)
$$

equation (21) becomes

$$
\begin{aligned}
\sum_{i=1}^{N}\frac{M_i g}{P^*\mu}\phi_p(c_i t,s) \quad = \quad & \frac{d^2T_n(t)}{dt^2}+\alpha_n^2T_n(t)-\frac{1}{P^*}\sum_{q=1}^{\infty}\left\{R_0P_1^*\frac{d^2T_q(t)}{dt^2}\right.\\
&-\left[\frac{F_0}{\mu}P_{2A}^*-\frac{G_0}{\mu}P_{2B}^*\right]T_q(t)\\
&-\sum_{i=1}^{N}\frac{M_i}{L_XL_Y\mu}\left[2\left(\frac{P_3^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_3^{**}(k)\right.\right.\\
&+\sum_{j=1}^{\infty}\cos\frac{j\pi c_i t}{L_X}P_3^{***}(j) \qquad (24)\\
&\left.+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi c_i t}{L_X}\cos\frac{k\pi s}{L_Y}P_3^{****}(j,k)\right)\frac{d^2T_q(t)}{dt^2}\\
&+4c_i\left(\frac{P_4^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_4^{**}(k)+\sum_{j=1}^{\infty}\cos\frac{j\pi c_i t}{L_X}P_4^{***}(j)\right.\\
&\left.+\;2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi c_i t}{L_X}\cos\frac{k\pi s}{L_Y}P_4^{****}(j,k)\right)\frac{dT_q(t)}{dt}
\end{aligned}
$$

$$+2c_i^2\left(\frac{P_5^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_5^{**}(k)+\sum_{j=1}^{\infty}\cos\frac{j\pi c_i t}{L_X}P_5^{***}(j)\right.$$
$$\left.+\ 2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi c_i t}{L_X}\cos\frac{k\pi s}{L_Y}P_5^{****}(j,k)\right)T_q(t)\Bigg]\Bigg\}$$

where

- $\alpha_n^2 = \frac{D\omega_n^4}{\mu}$,
- $P_1^* = \int_0^{L_X}\int_0^{L_Y}\left[\phi_{n,xx}(x,y)+\phi_{n,yy}(x,y)\right]\phi_p(x,y)dydx$,
- $P_2^* = \int_0^{L_X}\int_0^{L_Y}\left[4x-3x^2+x^3\right]\phi_n(x,y)\,\phi_p(x,y)dydx$,
- $P_3^* = \int_0^{L_X}\int_0^{L_Y}\phi_n(x,y)\phi_p(x,y)dydx$,
- $P_3^{**}(k) = \int_0^{L_X}\int_0^{L_Y}\cos\frac{k\pi y}{L_Y}\phi_n(x,y)\phi_p(x,y)dydx$,
- $P_3^{***}(j) = \int_0^{L_X}\int_0^{L_Y}\cos\frac{j\pi x}{L_X}\phi_n(x,y)\phi_p(x,y)dydx$,
- $P_3^{****}(j,k) = \int_0^{L_X}\int_0^{L_Y}\cos\frac{j\pi x}{L_X}\cos\frac{k\pi y}{L_Y}\phi_n(x,y)\,\phi_p(x,y)dydx$,
- $P_4^* = \int_0^{L_X}\int_0^{L_Y}\phi_{n,x}(x,y)\,\phi_p(x,y)dydx$,
- $P_4^{**}(k) = \int_0^{L_X}\int_0^{L_Y}\cos\frac{k\pi y}{L_Y}\phi_{n,x}(x,y)\phi_p(x,y)dydx$,
- $P_4^{***}(j) = \int_0^{L_X}\int_0^{L_Y}\cos\frac{j\pi x}{L_X}\phi_{n,x}(x,y)\phi_p(x,y)dydx$,
- $P_4^{****}(j,k) = \int_0^{L_X}\int_0^{L_Y}\cos\frac{j\pi x}{L_X}\cos\frac{k\pi y}{L_Y}\,\phi_{n,x}(x,y)\,\phi_p(x,y)dydx$,
- $P_5^* = \int_0^{L_X}\int_0^{L_Y}\phi_{n,xx}(x,y)\,\phi_p(x,y)\,dy\,dx$,
- $P_5^{**}(k) = \int_0^{L_X}\int_0^{L_Y}\cos\frac{k\pi y}{L_Y}\,\phi_{n,xx}(x,y)\,\phi_p(x,y)\,dy\,dx$,
- $P_5^{***}(j) = \int_0^{L_X}\int_0^{L_Y}\cos\frac{j\pi x}{L_X}\phi_{n,xx}(x,y)\,\phi_p(x,y)\,dy\,dx$
- $P_5^{****}(j,k) = \int_0^{L_X}\int_0^{L_Y}\cos\frac{j\pi x}{L_X}\cos\frac{k\pi y}{L_Y}\,\phi_{n,xx}(x,y)\,\phi_p(x,y)\,dy\,dx$,
- $P_{2A}^* = 4h_1 - 3h_2 + h_3$
- $P_{2B}^* = -13h_4 + 12h_5 - 3h_6 + 12(h_7+h_8) - 13(h_9+h_{10}) + 6(h_{11}+h_{12}) - (h_{13}+h_{14})$,
- $h_1 = \int_0^{L_Y}\int_0^{L_X} x\phi_n(x,y)\phi_p(x,y)dxdy$,
- $h_2 = \int_0^{L_Y}\int_0^{L_X} x^2\phi_n(x,y)\phi_p(x,y)dxdy$,
- $h_3 = \int_0^{L_Y}\int_0^{L_X} x^3\phi_n(x,y)\phi_p(x,y)dxdy$,
- $h_4 = \int_0^{L_Y}\int_0^{L_X} \phi_{n,x}(x,y)\phi_p(x,y)dxdy$,
- $h_5 = \int_0^{L_Y}\int_0^{L_X} x\phi_{n,x}(x,y)\phi_p(x,y)dxdy$,
- $h_6 = \int_0^{L_Y}\int_0^{L_X} x^2\phi_{n,x}(x,y)\phi_p(x,y)dxdy$,
- $h_7 = \int_0^{L_Y}\int_0^{L_X} \phi_{n,xx}(x,y)\phi_p(x,y)dxdy$,
- $h_8 = \int_0^{L_Y}\int_0^{L_X} \phi_{n,yy}(x,y)\phi_p(x,y)dxdy$,
- $h_9 = \int_0^{L_Y}\int_0^{L_X} x\phi_{n,xx}(x,y)\phi_p(x,y)dxdy$,
- $h_{10} = \int_0^{L_Y}\int_0^{L_X} x\phi_{n,yy}(x,y)\phi_p(x,y)dxdy$,
- $h_{11} = \int_0^{L_Y}\int_0^{L_X} x^2\phi_{n,xx}(x,y)\phi_p(x,y)dxdy$,

- $h_{12} = \int_0^{L_Y} \int_0^{L_X} x^2\phi_{n,yy}(x,y)\phi_p(x,y)dxdy$,
- $h_{13} = \int_0^{L_Y} \int_0^{L_X} x^3\phi_{n,xx}(x,y)\phi_p(x,y)dxdy$ and
- $h_{14} = \int_0^{L_Y} \int_0^{L_X} x^3\phi_{n,yy}(x,y)\phi_p(x,y)dxdy$.

The second order coupled differential equation (24) is the transformed equation governing the problem of a rectangular plate on a Pasternak elastic foundation with stiffness variation.
ϕ_n(x,y) are assumed to be the products of the functions ψ_{ni}(x) and ψ_{nj}(y) which are the beam functions in the directions of x and y axes respectively [15, 16]. That is

$$\phi_n(x,y) = \psi_{ni}(x)\psi_{nj}(y) \tag{25}$$

these beam functions can be defined respectively, as

$$\psi_{ni}(x) = \sin\frac{\Omega_{ni}x}{L_X} + A_{ni}\cos\frac{\Omega_{ni}x}{L_X} + B_{ni}\sinh\frac{\Omega_{ni}x}{L_X} + C_{ni}\cosh\frac{\Omega_{ni}x}{L_X} \tag{26}$$

and

$$\psi_{nj}(x) = \sin\frac{\Omega_{nj}y}{L_Y} + A_{nj}\cos\frac{\Omega_{nj}y}{L_Y} + B_{nj}\sinh\frac{\Omega_{nj}y}{L_Y} + C_{nj}\cosh\frac{\Omega_{nj}y}{L_Y} \tag{27}$$

where A$_{ni}$, A$_{nj}$, B$_{ni}$, B$_{nj}$, C$_{ni}$ and C$_{nj}$ are constants determined by the boundary conditions. Ω_{ni} and Ω_{nj} are called the mode frequencies.
In order to solve equation (24) we shall consider only one mass M traveling with uniform velocity c along the line y = s. Thus for the single mass M equation (24) reduces to

$$\begin{aligned}
\frac{Mg}{P^*\mu}\Psi_{pi}(ct)\Psi_{pj}(s) \quad = \quad & \frac{d^2T_n(t)}{dt^2} + \alpha_n^2 T_n(t) - \frac{1}{P^*}\sum_{q=1}^{\infty}\Bigg\{R_0P_1^*\frac{d^2T_q(t)}{dt^2} - \frac{G_0}{\mu}\Bigg[\frac{F_0}{G_0}P_{2A}^* \\
& -P_{2B}^*\Big]T_q(t)\Gamma^0\Bigg[2\Bigg(\frac{P_3^*}{2} + \sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}\ P_3^{**}(k) \\
& + \sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_3^{***}(j) \\
& + 2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_3^{****}(j,k)\Bigg)\frac{d^2T_q(t)}{dt^2} \\
& + 4c\Bigg(\frac{P_4^*}{2} + \sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_4^{**}(k) + \sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_4^{***}(j) \\
& + 2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_4^{****}(j,k)\Bigg)\frac{dT_q(t)}{dt} \\
& + 2c^2\Bigg(\frac{P_5^*}{2} + \sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_5^{**}(k) + \sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_5^{***}(j) \\
& + 2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_5^{****}(j,k)\Bigg)T_q(t)\Bigg]\Bigg\}
\end{aligned} \tag{28}$$

where

$$\Gamma^0 = \frac{M}{L_X L_Y \mu} \tag{29}$$

Equation (28) is now the fundamental equation of our problem when the rectangular plate has arbitrary end support conditions. We shall then solve the equation (28) when the plate has simple supports at all its edges.

For the elastic rectangular plate resting on a variable Pasternak elastic foundation and having simple supports at all its edges, the deflection and bending moment vanish at all the edges. Thus

$$W(0, y, t) = 0, \qquad W(L_X, y, t) = 0 \tag{30}$$

$$W(x, 0, t) = 0, \qquad W(x, L_Y, t) = 0 \tag{31}$$

$$\frac{\partial^2 W(0, y, t)}{\partial x^2} = 0, \qquad \frac{\partial^2 W(L_X, y, t)}{\partial x^2} = 0 \tag{32}$$

$$\frac{\partial^2 W(x, 0, t)}{\partial y^2} = 0, \qquad \frac{\partial^2 W(x, L_Y, t)}{\partial y^2} = 0 \tag{33}$$

For the normal modes

$$\Psi_{ni}(0) = 0, \qquad \Psi_{ni}(L_X) = 0 \tag{34}$$

$$\Psi_{nj}(0) = 0, \qquad \Psi_{nj}(L_Y) = 0 \tag{35}$$

$$\frac{\partial^2 \Psi_{ni}(0)}{\partial x^2} = 0, \qquad \frac{\partial^2 \Psi_{ni}(L_X)}{\partial x^2} = 0 \tag{36}$$

$$\frac{\partial^2 \Psi_{nj}(0)}{\partial y^2} = 0, \qquad \frac{\partial^2 \Psi_{nj}(L_Y)}{\partial y^2} = 0 \tag{37}$$

Therefore

$$A_{ni} = 0, B_{ni} = 0, C_{ni} = 0 \text{ and } \Omega_{ni} = n_i \pi \tag{38}$$

$$A_{nj} = 0, B_{nj} = 0, C_{nj} = 0 \text{ and } \Omega_{nj} = n_j \pi \tag{39}$$

Similarly,

$$A_{pi} = 0, B_{pi} = 0, C_{pi} = 0, \text{ and } \Omega_{pi} = p_i \pi \tag{40}$$

$$A_{pj} = 0, B_{pj} = 0, C_{pj} = 0 \text{ and } \Omega_{pj} = p_j \pi \tag{41}$$

Substituting equations (38),(39) , (40) and (41) into the transformed equation (28) to obtain the transformed equation for a rectangular plate, resting on a variable Pasternak elastic foundation and having simple supports at all its edges, we have

$$\begin{aligned}
\frac{Mg}{\mu P^*}\sin\frac{p_j\pi s}{L_Y}\sin\frac{p_i\pi ct}{L_X} &= \frac{d^2T_n(t)}{dt^2}+\alpha_n^2T_n(t)\\
&-\frac{1}{P^*}\sum_{q=1}^{\infty}\Bigg\{-R_0\left[\left(\frac{n_i\pi}{L_X}\right)^2+\left(\frac{n_j\pi}{L_Y}\right)^2\right]\\
&\times I_{1a}(x)I_{1a}(y)\frac{d^2T_q(t)}{dt^2}-\frac{G_0}{\mu}\left[\frac{F_0}{G_0}\tau_{2A}^*-\tau_{2B}^*\right]\\
&I_{1a}(y)T_q(t)-\frac{M}{L_XL_Y\mu}\Big[\Big(I_{1a}(x)I_{1a}(y)\\
&+2\sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}I_{1a}(x)I_{1a}^k(y)+2\sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}I_{1a}^j(x)\\
&\times I_{1a}(y)+4\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}I_{1a}^j(x)I_{1a}^k(y)\Big) \qquad (42)\\
&\times\frac{d^2T_q(t)}{dt^2}2c\Bigg(\frac{n_i\pi}{L_X}I_{2b}(x)I_{1a}(y)\\
&+2\sum_{k=1}^{\infty}\frac{n_i\pi}{L_X}\cos\frac{k\pi s}{L_Y}I_{2b}(x)I_{1a}^k(y)\\
&+2\sum_{j=1}^{\infty}\frac{n_i\pi}{L_X}\cos\frac{j\pi ct}{L_X}I_{2b}^j(x)I_{1a}(y)\\
&+4\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{n_i\pi}{L_X}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}I_{2b}^j(x)I_{1a}^k(y)\Bigg)\frac{dT_q(t)}{dt}\\
&-c^2\Bigg(\left(\frac{n_i\pi}{L_X}\right)^2I_{1a}(x)I_{1a}(y)\\
&+2\sum_{k=1}^{\infty}\left(\frac{n_i\pi}{L_X}\right)^2\cos\frac{k\pi s}{L_Y}I_{1a}(x)I_{1a}^k(y)\\
&+2\sum_{j=1}^{\infty}\left(\frac{n_i\pi}{L_X}\right)^2\cos\frac{j\pi ct}{L_X}I_{1a}^j(x)I_{1a}(y)\\
&+4\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{n_i\pi}{L_X}\right)^2\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}I_{1a}^j(x)I_{1a}^k(y)\Bigg)\\
&\times T_q(t)\Big]\Bigg\}
\end{aligned}$$

where

- $I_{1a}(x)=\int_0^{L_X}\sin\theta_{ni}x\sin\theta_{pi}x\,dx$,
- $I_{1a}(y)=\int_0^{L_Y}\sin\theta_{nj}y\sin\theta_{pj}y\,dy$
- $I_{1a}^j(x)=\int_0^{L_X}\cos\frac{j\pi x}{L_X}\sin\theta_{ni}x\sin\theta_{pi}x\,dx$,
- $I_{1a}^k(y)=\int_0^{L_Y}\cos\frac{k\pi y}{L_Y}\sin\theta_{nj}y\sin\theta_{pj}y\,dy$
- $I_{1a}^*(x)=\int_0^{L_X}x\sin\theta_{ni}x\sin\theta_{pi}x\,dx$,
- $I_{1a}^{**}(x)=\int_0^{L_X}x^2\sin\theta_{ni}x\sin\theta_{pi}x\,dx$,

- $I_{1a}^{***}(x) = \int_0^{L_X} x^3 \sin\theta_{ni}x \sin\theta_{pi}x\, dx,$
- $I_{2b}(x) = \int_0^{L_X} \cos\theta_{ni}x \sin\theta_{pi}x\, dx,$
- $I_{2b}^{*}(x) = \int_0^{L_X} x\cos\theta_{ni}x \sin\theta_{pi}x\, dx,$
- $I_{2b}^{**}(x) = \int_0^{L_X} x^2\cos\theta_{ni}x \sin\theta_{pi}x\, dx,$
- $\tau_{2A}^{*} = 4I_{1a}^{*}(x) - 3I_{1a}^{**}(x) + I_{1a}^{***}(x)$
- $\begin{aligned}\tau_{2B}^{*} \quad = \quad & [-13I_{2b}(x) + 12I_{2b}^{*}(x) - 3I_{2b}^{**}(x)]\,\theta_{ni} \\ & + [-12I_{1a}(x) + 13I_{1a}^{*}(x) - 6I_{1a}^{**}(x) + I_{1a}^{***}(x)] \\ & (\theta_{ni}^2 + \theta_{nj}^2)\end{aligned}$
- $\theta_{ni} = \frac{\Omega_{ni}}{L_x}$ *and* $\theta_{nj} = \frac{\Omega_{nj}}{L_Y}$

Further simplification and rearrangement of (42), taking into account (38), (39), (40) and (41), yields

$$\begin{aligned}\frac{Mg}{P^*\mu}\sin\frac{p_j\pi s}{L_Y}\sin\frac{p_i\pi ct}{L_X} \quad = \quad & \frac{d^2T_n(t)}{dt^2} + \alpha_n^2 T_n(t) \\ & -\frac{1}{P^*}\sum_{q=1}^{\infty}\left\{\frac{-R_0L_XL_Y\pi^2}{4}\left(\frac{q^2}{L_X^2} + \frac{q^2}{L_Y^2}\right)\frac{d^2T_q(t)}{dt^2}\right. \\ & -\frac{G_0L_Y}{2\mu}\left[\frac{F_0}{G_0}\tau_{2A}^{*} - \tau_{2B}^{*}\right]T_q(t) - \Gamma\left[\frac{L_XL_Y}{4}\frac{d^2T_q}{dt^2}\right. \\ & +2cL_Y\left(\frac{qp_i}{p_i^2 - q^2} + \sum_{j=1}^{\infty}\frac{q\pi}{L_X}\tau(j)\cos\frac{j\pi ct}{L_X}\right)\frac{dT_q(t)}{dt} \\ & \left.\left. -\frac{(cq\pi)^2L_Y}{4L_X}T_q(t)\right]\right\}\end{aligned} \tag{43}$$

where

$$\Gamma = \frac{M}{L_XL_Y\mu} \tag{44}$$

and

$$\tau(j) = \frac{8p_i[p_i^2 - j^2 - q^2]}{j^4 + q^4 + p_i^4 - 2[j^2p_i^2 + j^2q^2 + p_i^2q^2]} \tag{45}$$

Equation (43) is the fundamental equation of our problem when the rectangular plate resting on variable Pasternak foundation has simple support at all its edges. We shall now discuss two cases of the equation.

CASE I: SIMPLY SUPPORTED PLATE TRAVERSED BY MOVING FORCE

When $\Gamma = 0$ in equation (43), an approximate model of the system when the inertia effect of the moving mass M is neglected, we have the moving force problem associated with the system. Thus the differential equation (43) reduces to

$$\begin{aligned}&\frac{d^2T_n(t)}{dt^2} + \alpha_n^2T_n(t) - \frac{1}{P^*}\sum_{q=1}^{\infty}\left\{\frac{-R_0L_XL_Y\pi^2}{4}\left(\frac{q^2}{L_X^2} + \frac{q^2}{L_Y^2}\right)\right. \\ &\left.\times\frac{d^2T_q(t)}{dt^2} - \frac{G_0L_Y}{2\mu}\left[\frac{F_0}{G_0}\tau_{2A}^{*} - \tau_{2B}^{*}\right]T_q(t)\right\} = \frac{Mg}{P^*\mu}\sin\frac{p_j\pi s}{L_Y}\sin\frac{p_i\pi ct}{L_X}\end{aligned} \tag{46}$$

Evidently, exact analytical solution to the coupled differential equation above is impossible. Though it yields readily to numerical techniques, an analytical solution is desirable as such solutions often shed more light on vital information about the vibrating system.
In order to solve the coupled differential equation (46) we resort to an approximate analytical method which is a modification of the asymptotic method due to Struble [4, 6] and seek the modified frequency of the free system, first, due to the presence of the effect of shear modulus neglecting the rotatory inertial term. An equivalent free system operator defined by the modified frequency then replaces equation (46). Thus, equation (46) is rearranged to take the form

$$\frac{d^2T_n(t)}{dt^2} + \left[\alpha_n^2 + \Gamma^* \frac{L_Y}{2}\left(\frac{F_0}{G_0}\tau_{2A}^* - \tau_{2B}^*\right)\right] T_n(t)$$

$$+\Gamma^* \frac{L_Y}{2} \sum_{\substack{q=1 \\ q\neq n}}^{\infty} \left(\frac{F_0}{G_0}\tau_{2A}^* - \tau_{2B}^*\right) T_q(t) = K_m \sin\frac{p_j \pi s}{L_Y} \sin\frac{p_i \pi ct}{L_X} \tag{47}$$

where

$$\Gamma^* = \frac{G_0}{P^*\mu} \qquad and \qquad K_m = \frac{Mg}{P^*\mu} \tag{48}$$

Consider a parameter $\lambda < 1$ for any arbitrary mass ratio Γ^*, defined as

$$\lambda = \frac{\Gamma^*}{1+\Gamma^*}$$

It can be shown that

$$\Gamma^* = \lambda + o(\lambda^2) \tag{49}$$

Since $\lambda < 1$, an asymptotic solution of the homogenous part of equation (47) can be written in the form

$$T_n(t) = A_n(t)\cos[\alpha_n t - \phi_n(t)] + \lambda T_1(t) + o(\lambda^2) \tag{50}$$

where A_n(t) and ϕ_n(t) are slowly varying functions of time.
Substituting (50) and its derivatives into (47), neglecting terms higher than o(λ) and terms which do not contribute to the variational equations describing the behaviour of A_n(t) and ϕ_n(t), one obtains

$$\gamma_{mp} = \alpha_n + \frac{\lambda\left(\frac{F_0}{G_0}\tau_{2A}^* - \tau_{2B}^*\right) L_Y}{4\alpha_n} \tag{51}$$

as the modified frequency due to the effect of the shear modulus.
Thus, the homogeneous part of equation (47) can be replaced with

$$\frac{d^2T_n(t)}{dt^2} + \gamma_{mp}^2 T_n(t) = 0 \tag{52}$$

Using (52), equation (46) can be written as

$$\frac{d^2T_n(t)}{dt^2}+\gamma_{mp}^2T_n(t)+\frac{\lambda_0L_XL_Y\pi^2}{4}\sum_{q=1}^{\infty}\left(\frac{q^2}{L_X^2}+\frac{q^2}{L_Y^2}\right)\frac{d^2T_q(t)}{dt^2}=K_m\sin\frac{p_j\pi s}{L_Y}\sin\frac{p_i\pi ct}{L_X} \tag{53}$$

where

$$\lambda_0=\frac{R_0}{P^*} \tag{54}$$

The homogeneous part of equation (53) can be written as

$$\begin{aligned}&\frac{d^2T_n(t)}{dt^2}+\frac{\gamma_{mp}^2}{1}+\frac{\lambda_0L_XL_Y\pi^2}{4}\left(\frac{n_i^2}{L_X^2}+\frac{n_j^2}{L_Y^2}\right)T_n(t)\\&+\frac{\frac{\lambda_0L_XL_Y\pi^2}{4}}{1+\frac{\lambda_0L_XL_Y\pi^2}{4}\left(\frac{n_i^2}{L_X^2}+\frac{n_j^2}{L_Y^2}\right)}\sum_{\substack{q=1\\q\neq n}}^{\infty}\left(\frac{q^2}{L_X^2}+\frac{q^2}{L_Y^2}\right)\frac{d^2T_q(t)}{dt^2}\quad=\quad 0\end{aligned} \tag{55}$$

Consider the parameter $\epsilon_0 < 1$ for any arbitrary mass ratio λ_0 defined as

$$\varepsilon_0=\frac{\lambda_0}{1+\lambda_0} \tag{56}$$

which implies

$$\lambda_0=\varepsilon_0+o(\varepsilon_0^2) \tag{57}$$

Following the same argument, (55) can be replaced with

$$\frac{d^2T_n(t)}{dt^2}+\gamma_{mpf}^2T_n(t)=0 \tag{58}$$

where

$$\gamma_{mpf}=\gamma_{mp}\left[1-\frac{\varepsilon_0L_XL_Y\pi^2}{8}\left(\frac{n_i^2}{L_X^2}+\frac{n_j^2}{L_Y^2}\right)\right] \tag{59}$$

is the modified frequency due to the presence of rotatory inertia. Therefore, the moving force problem (46) for the simply supported rectangular plate is reduced to the non-homogeneous ordinary differential equation given as

$$\frac{d^2T_n(t)}{dt^2}+\gamma_{mpf}^2T_n(t)=K_m\sin\frac{p_j\pi s}{L_Y}\sin\frac{p_i\pi ct}{L_X} \tag{60}$$

When equation (60) is solved in conjunction with the initial conditions (10), one obtains expression for $T_n(t)$. Thus in view of equation (11), one obtains

$$\begin{aligned}W(x,y,t)\quad=\quad&\sum_{ni=1}^{\infty}\sum_{nj=1}^{\infty}\frac{K_m\sin\frac{p_j\pi s}{L_Y}}{\gamma_{mpf}[\gamma_{mpf}^2-(p_i\pi c/L_X)^2]}\Big[\gamma_{mpf}\sin\frac{p_i\pi ct}{L_X}\\&-\frac{p_i\pi c}{L_X}\sin\gamma_{mpf}t\Big]\sin\frac{n_i\pi x}{L_X}\sin\frac{n_i\pi y}{L_Y}\end{aligned} \tag{61}$$

as the transverse-displacement response to a moving force of a simply supported rectangular plate on a variable Pasternak elastic foundation.

CASE II: SIMPLY SUPPORTED RECTANGULAR PLATE RESTING ON VARIABLE PASTERNAK FOUNDATION AND TRAVERSED BY A MOVING MASS

In this section we seek the solution to the entire equation (43) when no term of the equation is neglected. In order to solve this problem, as in the previous case, an exact analytical solution to this equation is impossible. Thus, we resort to the modified asymptotic method of Struble discussed in the previous case. To this end, we rearrange equation (43) to take the form

$$\begin{aligned}
&\frac{d^2T_n(t)}{dt^2} - \frac{2cL_Y\eta_0\left(\frac{n_ip_i}{p_i^2-n_i^2}+\sum_{j=1}^{\infty}\frac{n_i\pi}{L_X}\tau(j)\cos\frac{j\pi ct}{L_X}\right)}{1-\eta_0\left(\frac{L_XL_Y}{4}\right)}\frac{dT_n(t)}{dt} \qquad (62)\\
&+\frac{\gamma_{mpf}^2+\frac{\eta_0(cn_i\pi)^2L_Y}{4L_X}}{1-\eta_0\left(\frac{L_XL_Y}{4}\right)}T_n(t) - \frac{\eta_0}{1-\eta_0\left(\frac{L_XL_Y}{4}\right)}\sum_{\substack{q=1\\ q\neq n}}^{\infty}\left[\frac{L_XL_Y}{4}\frac{d^2T_q(t)}{dt^2}\right.\\
&+2cL_Y\left(\frac{qp_i}{p_i^2-q^2}+\sum_{j=1}^{\infty}\frac{q\pi}{L_X}\tau(j)\cos\frac{j\pi ct}{L_X}\right)\frac{dT_q(t)}{dt}\\
&\left.-\frac{(cq\pi)^2L_Y}{4L_X}T_q(t)\ \right] = \frac{\eta_0 gL_XL_Y}{P^*\left[1-\eta_0\left(\frac{L_XL_Y}{4}\right)\right]}\sin\frac{p_j\pi s}{L_Y}\sin\frac{p_i\pi ct}{L_X}
\end{aligned}$$

where Γ has been written as a function of the mass ratio η_0.
Thus, considering the homogeneous part of the equation (62) and going through the same arguments and analysis as in the previous case, the modified frequency corresponding to the frequency of the free system due to the presence of the moving mass is

$$\beta_{mf} = \gamma_{mpf}\left[1-\frac{\eta_0}{2}\left(1+\frac{(cn_i\pi)^2}{\gamma_{mpf}^2L_X^2}\right)\right] \qquad (63)$$

retaining terms to o(η_0) only.
Therefore, to solve the non-homogeneous equation (62), the differential operator which acts on $T_n(t)$ and $T_q(t)$ is replaced by the equivalent free system operator defined by the modified frequency β_{mf}. That is

$$\frac{d^2T_n(t)}{dt^2}+\beta_{mf}^2T_n(t) = G_g\sin\frac{n_j\pi s}{L_Y}\sin\frac{n_i\pi ct}{L_X} \qquad (64)$$

where

$$G_g = \frac{\eta_0 gL_XL_Y}{P^*} \qquad (65)$$

Clearly, equation (64) is directly analogous to equation (60). Hence when equation (64) is solved in conjunction with the initial conditions, one obtains expression for $T_n(t)$. Thus in view of equation (11), we have

$$W(x,y,t) = \sum_{ni=1}^{\infty}\sum_{nj=1}^{\infty} \frac{G_g \sin\frac{p_j \pi s}{L_Y}}{\beta_{mf}[\beta_{mf}^2 - (p_i \pi c/L_X)^2]} \left[\beta_{mf} \sin\frac{p_i \pi c t}{L_X} - \frac{p_i \pi c}{L_X} \sin\beta_{mf} t\right] \sin\frac{n_i \pi x}{L_X} \sin\frac{n_j \pi y}{L_Y} \tag{66}$$

Equation (66) represents the transverse-displacement response to a moving mass of a simply supported rectangular plate on a variable Pasternak elastic foundation.

5. DISCUSSION OF THE ANALYTICAL SOLUTIONS

It is desirable to examine the phenomenon of resonance in studying such undamped system as this.

Equation (61) clearly shows that the simply supported rectangular plate on a variable Pasternak elastic foundation and traversed by a moving force reaches a state of resonance whenever

$$\gamma_{mpf} = \frac{p_i \pi c}{L_X} \tag{67}$$

while equation (66) shows that the same plate under the action of a moving mass experiences resonance when

$$\beta_{mf} = \frac{p_i \pi c}{L_X} \tag{68}$$

where

$$\beta_{mf} = \gamma_{mpf}\left[1 - \frac{\eta_0}{2}\left(1 + \frac{(c n_i \pi)^2}{\gamma_{mpf}^2 L_X^2}\right)\right] \tag{69}$$

Equations (68) and (69) imply that

$$\gamma_{mpf}\left[1 - \frac{\eta_0}{2}\left(1 + \frac{(c n_i \pi)^2}{\gamma_{mpf}^2 L_X^2}\right)\right] = \frac{p_i \pi c}{L_X} \tag{70}$$

clearly $\left[1 - \frac{\eta_0}{2}\left(1 + \frac{(c n_i \pi)^2}{\gamma_{mpf}^2 L_X^2}\right)\right] < 1$ for all n_i,

Consequently, for the same natural frequency, the critical speed (and the natural frequency) for the moving mass problem is smaller than that of the moving force problem. Thus, resonance is reached earlier in the moving mass system than in the moving force system.

6. NUMERICAL CALCULATIONS AND DISCUSSION OF RESULTS

In this section, calculations of practical interests in dynamics of structures are presented for all the illustrative examples. A rectangular plate of length $L_Y = 0.914$m and breadth $L_X = 0.457$m has been considered. The mass is assumed to travel at the constant velocity 0.8123m/s. Furthermore, E, S and Γ are chosen to be $2.109\text{x}10^9 \text{kg/m}^2$, 0.4m and 0.2 respectively. The results are as presented on the various graphs below for the various classes of boundary conditions considered.
Figures 5.1 and 5.2 display the effect of Rotatory inertia (R_0) on the transverse deflection of the simply supported rectangular plate for both cases of moving force and moving mass respectively. The graphs show that the response amplitudes decrease as the value of the Rotatory inertia correction factor increases.

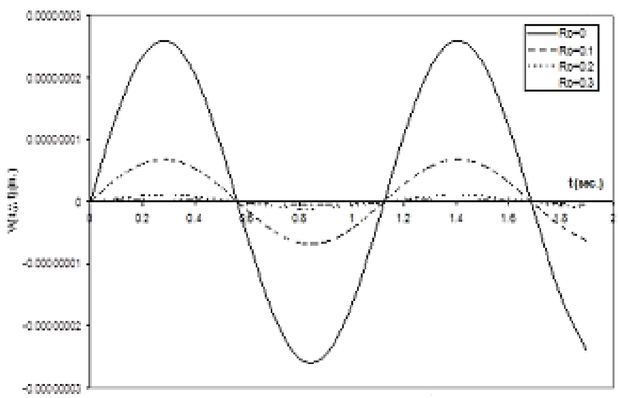

Fig. 5.1. Deflection profile of simply supported rectangular plate on variable Pasternak foundation and traversed by moving force for $F_0 = 2000000$, $G_0 = 900000$ and various values of R_0

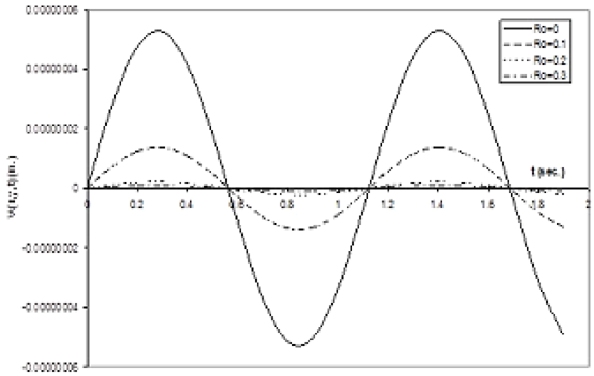

Fig. 5.2. Deflection profile of simply supported rectangular plate resting on variable Pasternak foundation and traversed by moving force for $F_0 = 2000000$, $G_0 = 900000$ and various values of R_0

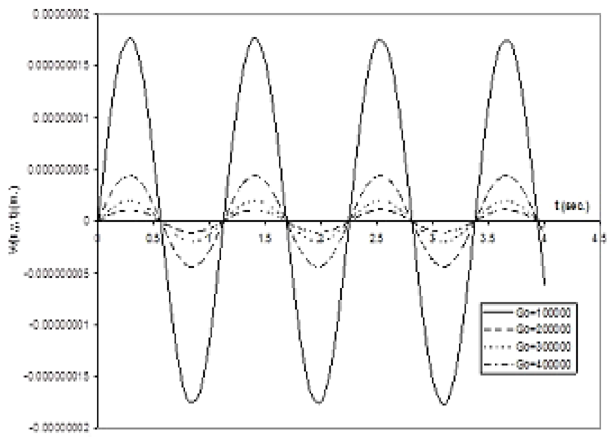

Fig. 5.3. Displacement profile of simply supported rectangular plate on variable Pasternak foundation and traversed by moving force for $F_0 = 0$, $R_0 = 0.4$ and various values of G_0.

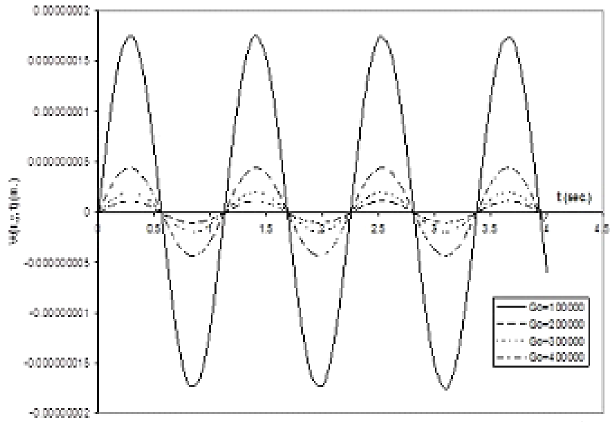

Fig. 5.4. Displacement profile of simply supported rectangular plate on variable Pasternak foundation and traversed by moving force for $F_0 = 100000$, $R_0 = 0.4$ and various values of G_0.

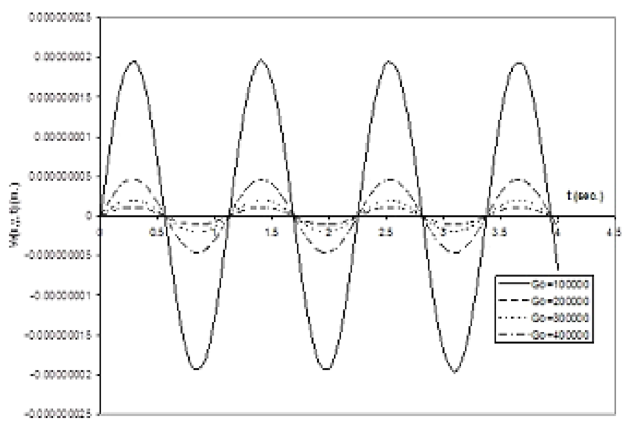

Fig. 5.5. Displacement profile of simply supported rectangular plate on variable Pasternak foundation and traversed by moving force for $F_0 = 1000000$, $R_0 = 0.4$ and various values of G_0.

Figures 5.3 – 5.5 show the deflection profile of the simply supported rectangular plate traversed by moving force. The load speed considered is lower than the critical speed. Figure 5.3 depicts the response curves of the plate for $\mathrm{F}_0 = 0$, $\mathrm{R}_0 = 0.4$ and with the subgrade's shear modulus G_0 as a parameter. The corresponding curves for

$F_0 = 100000$ N/m^3 and 1000000N/m^3 are shown in figures 5.4 and 5.5 respectively. It is evident from the graphs that, the response amplitudes decrease with an increase in the values of G_0 for fixed values of F_0 and R_0. It is also remarked at this juncture that as F_0 increases, the response amplitudes decrease in similar manner. However, the effect of G_0 is more pronounced than that of F_0.
Figures 5.6 – 5.8 display the corresponding deflection profiles for the same system under the action of moving mass. The dynamic behaviour similar to that of the moving force is obtained.

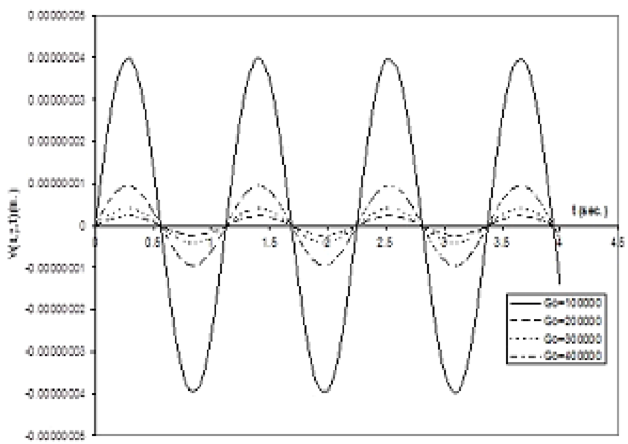

Fig. 5.6. Displacement profile of simply supported rectangular plate on variable Pasternak foundation and traversed by moving mass for $F_0 = 0$, $R_0 = 0.4$ and various values of G_0.

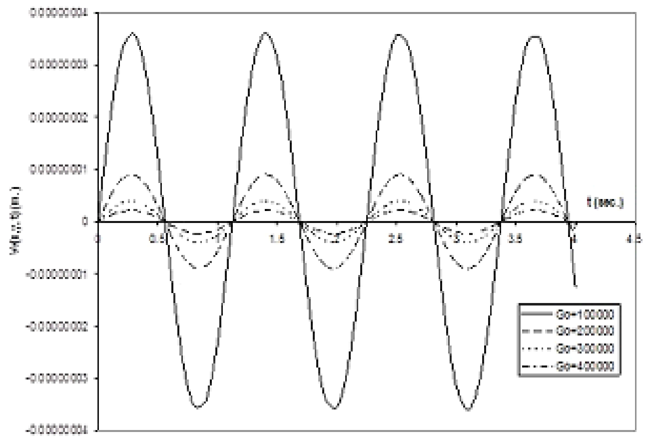

Fig. 5.7. Deflection of simply supported rectangular plate resting on variable Pasternak foundation and traversed by moving mass for $F_0 = 100000$, $R_0 = 0.4$ and various values of G_0.

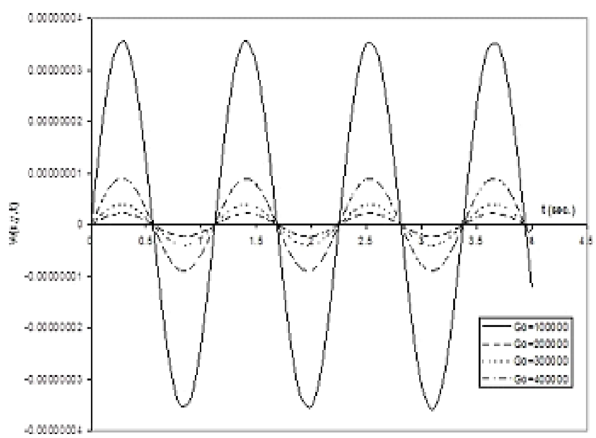

Fig. 5.8. Deflection of simply supported rectangular plate on variable Pasternak foundation and traversed by moving mass for $F_0 = 1000000$, $R_0 = 0.4$ and various values of G_0.

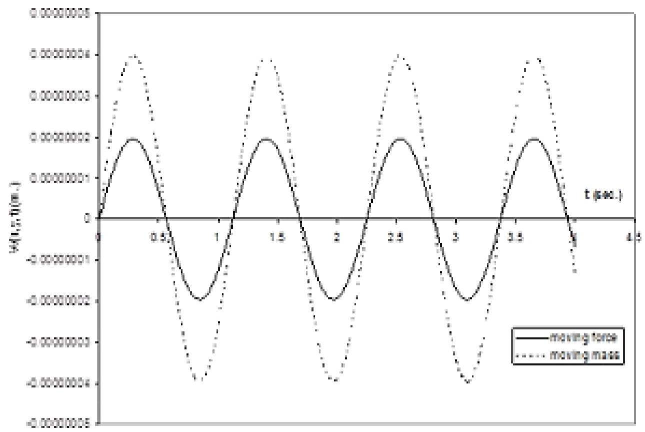

Fig. 5.9. Comparison of deflection of moving force and moving mass cases for simply supported rectangular plate on variable Pasternak foundation and traversed by moving force for $F_0 = 1000000$, $G_0 = 100000$ and $R_0 = 0.4$.

Figure 5.9 compares the displacement curves of the moving force and moving mass for a simply supported rectangular plate with $F_0 = 1000000$ N/m^3, $R_0 = 0.4$ and $G_0 = 100000$ N/m. Clearly, the response amplitude of a moving mass is greater than that of a moving force problem. However, this result holds for other choices of the values of F_0, R_0 and G_0.

Figures 5.10, 5.11 and 5.12 present the effects of G_o, F_o and R_o respectively on the critical speed of the moving load. The graphs show that as G_o, F_o and R_o increase, the critical speed increases in each case.

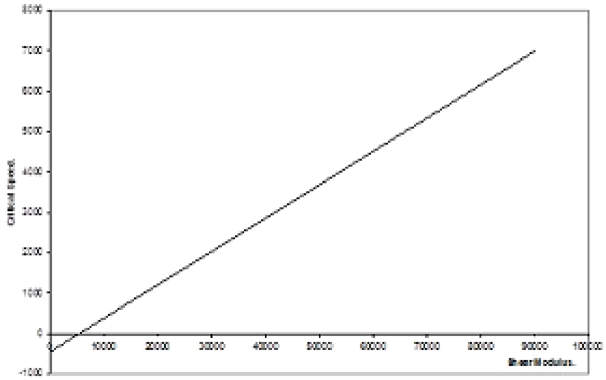

Fig. 5.10. Graph of Critical Speed against Shear Modulus.

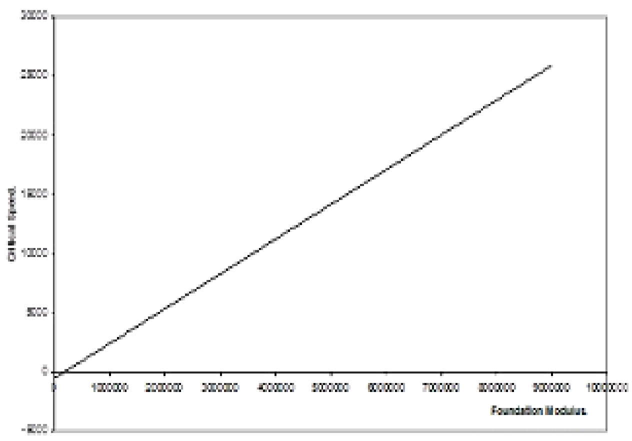

Fig. 5.11. Graph of Critical Speed against Foundation Modulus.

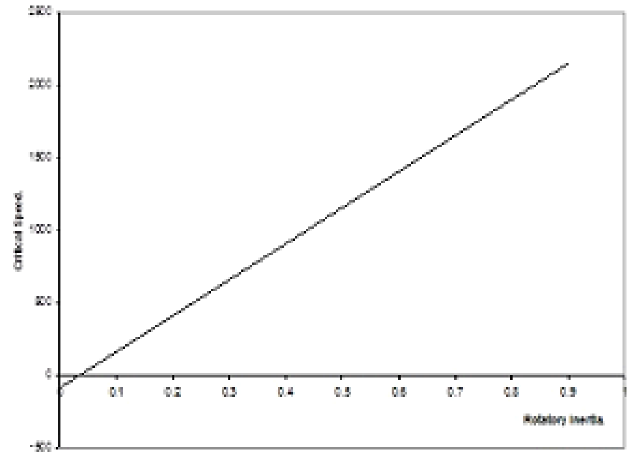

Fig. 5.12. Graph of Critical Speed against Rotatory Inertia.

7. CONCLUDING REMARKS

The objective of this work has been to study the problem of the dynamic response to moving concentrated masses of rectangular plates on variable Pasternak elastic foundations. In particular, the closed form solutions of the fourth order partial differential equations with variable and singular coefficients of the rectangular plate is obtained for both cases of moving force and moving mass. The solution technique is based on the technique of Shadnam et al [14] which was used to remove the singularity in the governing fourth order partial differential equation and to reduce it to a sequence of coupled second order differential equations. These coupled second order differential equations were then simplified using the modified Struble's asymptotic technique. The methods of integral transformation and the convolution theory are then employed to obtain the analytical solution of the two-dimensional dynamical problem.
These solutions are analyzed and resonance conditions are obtained for the problem. The analyses carried out show that the moving force solution is not an upper bound for the accurate solution of the moving mass problem and that as the rotatory inertia correction factor increases, the response amplitudes of the plates decrease for both cases of moving force and moving mass problem. When the rotatory inertia correction factor is fixed, the displacements of the simply supported rectangular plates resting on variable Pasternak elastic foundations decrease as the shear modulus increases. Also, as the foundation modulus increases, the response amplitudes of

the plates decrease, the effect of shear modulus is more noticeable than that of the foundation modulus.
It is shown further from the results that, for fixed values of rotatory inertia correction factor, foundation modulus and shear modulus, the response amplitude for the moving mass problem is greater than that of the moving force problem implying that resonance is reached earlier in moving mass problem than in moving force problem of the simply supported rectangular plate resting on variable Pasternak elastic foundation. Also, an increase in the shear modulus results in an increase in the critical speed of the moving load; this shows that risk is reduced when the shear modulus increases. The same result obtains for an increase in both foundation modulus and rotatory inertial correction factor.

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