Journal of the Nigerian Mathematical Society
Vol. 32, pp. 171-183, 2013

SOME CONSERVATIVE FORCE FIELD POTENTIALS IN SCATTERING THEORY

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ABSTRACT. In this paper we use the Lie symmetry method to calculate the universal group of symmetry transformations for the 3-dimensional time dependent Schrödinger wave equation. It is shown that this system admits a subgroup of SL(6,R) as symmetry group. Invariance properties of this group are used to construct solutions to the wave equation with a coloumbic force field potential. New solutions are thus obtained which are useful in scattering theory. Furthermore, it is shown that an application of the symmetry group to the set of states $S = \hat{H}$ (H complex Hilbert space) of the system preserves transition probabilities as well as the dynamics of the system.

Keywords and phrases: Prolongations, Symmetry Groups, Sets of States.
2010 Mathematical Subject Classification: 80A20

1. INTRODUCTION

The application of continuous group of transformations otherwise known as Lie groups to the study of systems of partial differential equations has its origin in the researches and work of Sophus Lie, over a century ago. Lie showed that one could reduce the order of an ordinary differential equation if it is invariant under a one-parameter group of point transformations. The Lie group admitted by such a differential equation can be found by a straightforward computational algorithm and involves the solution of a large number of partial differential equations of an elementary type. Early researchers found this method of limited application in the construction of the general solution to quite a number of partial differential equations encountered then. Lie"s method however came into prominence in the late 1950"s following the work of L. V. Ovsiannikov (1982), providing a theoretical foundation for a comprehensive study of the symmetry groups of differential equations. An

Received by the editors February 13, 2012; Revised: July 6, 2012; Accepted: September 10, 2012
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improved modern version of Lie"s theory has been developed by P. J. Olver (1986) and gives a ready verifiable means of obtaining the maximal group of symmetries admitted by many systems of partial differential equations, linear or nonlinear. Olver"s method has more recently been implemented in symbolic computational software such as MAXIMA, and MAPLE. This great success motivated Stephani et al (2003) and more recently Natorf and Tafel (2007) to apply Lie point symmetries to the Robinson-Trautman equation of Petrov type III a problem which has eluded any definite solution for the past fifty years, Ifidon (2011).

There is no doubt that the application of Lie group symmetries to the Schrödinger equation can lead to greater insights as far as non-relativistic quantum systems are concerned. Encouraged by this, we construct the symmetry group of the 3-dimensional time-dependent Schrödinger equation that models a simple nonrelativistic quantum system consisting of a single particle moving under the influence of a conservative force. The groups under consideration would be local Lie groups of transformations. The advantage of considering local groups is that Lie's three fundamental theorems have shown that such a group can be completely characterized in terms of the infinitesimal generators of their Lie algebra, which are relatively easy to find. Once the infinitesimal generators of the Lie algebra are known, the corresponding Lie groups can be found by exponentiation using Taylor's theorem (Gilmore, 1974). Attempts have been made to globalize Lie's transformation theory (see for example Palais, 1957) but the applications make use of only the local theory since only those group elements in a neighbourhood of the identity can in general be guaranteed to transform functions. Non-local symmetries of differential equations have also been studied for some time now (Muriel and Romero, 2001). Theoretical investigations of non-local symmetries are based on the theory of coverings in which a system of differential equations is said to cover another system of equations (called the covering system) provided its solutions give rise to solutions of the covered system. Symmetries of the covered system arise as a non-local symmetry of the covered system. This procedure however works only for differential equations that admit non-abelian Lie algebras. The next few definitions are useful. Let S denote the set of possible states of a physical system, then

Definition 1. *A group G is a symmetry group of S if for each $s \in S$ and $g \in G$, $g.s \in S$ whenever $g.s$ is defined.*

It is clear from the above definition that symmetries may be used to generate new solutions if one has found a solution. Instead of looking for the Lie group $\mathcal{G}$, we look for it's Lie algebra $\mathfrak{g}$. The corresponding group elements can then be found by exponentiation using Taylors theorem, Ifidon (2003). The infinitesimal generators of the Lie algebra is

$$\chi = \sum_{i=1}^{p} \eta^i(x,u)\frac{\partial}{\partial x_i} + \sum_{l}^{q} \varphi_l(x,u)\frac{\partial}{\partial u^l} \tag{1}$$

the procedure for finding $\eta^i(x,u)$ and $\varphi_l(x,u)$ is given explicitly in Olver (1986). We briefly outline this process. First the kth prolongation $\mathrm{Pr}^k\chi$ of the vector field χ can be calculated through the prolongation formulae

$$\mathrm{Pr}^k\chi = \chi + \sum_l \sum_J \left(\Psi^l\right)^J \frac{\partial}{\partial u^l_J} \tag{2}$$

where the J-sum over all partitions $J \equiv (j_1, j_2, ..., j_n)$ with $0 < \sum j_i \leq k$ and

$$\left(\Psi^l\right)^J = D^J\left(\Upsilon^l - \sum_i u^J_i \xi^i\right) + \sum_i u^l_{J,i}\xi^i \tag{3}$$

here

$$u_{J,i} = \frac{\partial u_J}{\partial x^i}, \quad u_J = \partial_J u, \qquad D^J = D_1^{j_1} \odot D_2^{j_2} \odot ...D_n^{j_n} \tag{4}$$

and

$$D_i = \frac{\partial}{\partial x^i} + \sum_l \sum_J u^l_{J,i}\frac{\partial}{\partial u^l_J} \qquad 0 \leq \sum j_i \leq k \tag{5}$$

is the derivative operator.

Theorem 0.1. *Suppose $\Delta(x, u^{(k)}) = 0$ is a system of partial differential equations. Then $\mathcal{G}$ is a symmetry group of the equation $\Delta = 0$ iff*

$$Pr^k\chi\Delta(x, u^{(k)}) = 0 \tag{6}$$

whenever $\Delta(x, u^{(k)}) = 0$

for proof see Nwachuku and Ifidon (1991).

Definition 2. *Let G be such that*

$$g_1 \cdot (g_2 \cdot s) = (g_1 g_2) \cdot s \quad g_1, g_2 \in G$$

$$e \cdot s = s$$

where e is the identity element in G, then G is called a group of symmetries for S.

In quantum mechanics the set of states $S = \hat{H}$ is the set of all rays $\hat{\phi}\{\lambda\phi, \quad \lambda \in C\}$ where $\hat{\phi}$ is a nonzero vector in the complex Hilbert space H.

Next we show that if G is a local transformation group then the action of G on the set of states $\{\hat{H}\}$ preserves transitional probabilities.

Theorem 0.2. *Given a ray $\hat{\phi} \in \hat{H}$, if G is a Lie group of symmetries for H, then the probability of going from the state $g \cdot \phi$ to the state $g \cdot \psi$ is the same as that of going from ϕ to ψ for all $g \in G$ and $\phi, \psi \in H$.*

Proof. Consider a ray $\hat{\phi} \in \hat{H}$. It's trajectory may be calculated by computing solutions to the equation.

$$\frac{\partial}{\partial t}(\phi_1) = iH(\phi_1) \tag{7}$$

where H is a self adjoint operator called the Hamiltonian of the system.

Thus a ray $\hat{\phi} \in \hat{H}$ may be uniquely determined by a pair of points

$$(x, \phi) \in \mathbb{R}^P \times H.$$

Here $x \in \mathbb{R}^d$ represents the real line probability measures ϕ assigns to the various self adjoint operators in H as it evolves in time. If G acts regularly on $\mathbb{R}^d \times H$, then since G is a local transformation group, we have for every $g \in G$, close to the identity

$$g(x, \phi) = (\Lambda_g, x, \lambda_g \phi) = (\tilde{x}, \tilde{\phi}) = g \cdot \phi$$

where (Λ_g, λ_g) are the C^∞ composition maps of G. Thus G can be viewed as acting to change our frame of reference since $g \cdot \phi$ is the old state ϕ viewed form anew frame of reference. Since acting to change the frame of reference does not change the state vector, we have a transformation in which norms are preserved. We can conclude therefore that if G is a local transformation group, then the action of G on the set of states $\hat{H}$ preserve transition probabilities in H. $\square$

Next we find out which Lie group G leaves (7) invariant. In the sequel, we shall assume that our system consists of a single spinless particle moving in a conservative force potential $v = v(x, y, z)$ so that the Hilbert space is $H = L^2(\mathbb{R}^3, \eta) : \eta$ Lesbesgue measure. A state vector ϕ would be represented by a wave function $\Psi(x, y, z; t)$ satisfying $\frac{\partial \Psi}{\partial t} = -iH\Psi$ with

$$H = -\frac{\nabla^2}{2} + V \tag{8}$$

where ∇^2 denotes the Laplacian. (8) is the time dependent Schrödinger equation.

The rest of this paper is organised as follows, in sec.2. the infinitesimal criteria for G- invariance of a system of partial differential equations is adapted in the derivation of the one-parameter group of symmetries for the system (8). It has been shown (Theorem 0.2) that if G is a Lie group of symmetries for the set of states $\hat{H}$ then G preserves transition probabilities in $\hat{H}$. Furthermore, we show that (8) admits the 13 one-parameter Lie group which are generators of the group $SL(6, \mathbb{R})$ in the representation state as a symmetric group. Thus we conclude that $SL(6, \mathbb{R})$ is the maximal group of symmetries for non-relativistic quantum systems which preserves transition probabilities as well as the dynamics of the system. The fact that the generators of this group are integrals of motion lead to a number of conservation laws in quantum mechanics. The conservation of energy, linear and angular momentum are well known conservation laws, which are consistent with our formulation. Other conservation laws are similarly derived. Group invariant solutions to the Schrödinger equation which are useful in scattering theory are constructed for various values of the potential in sec 3. The scale invariant solutions give a state of the system for which exact values for all three components of the angular momentum can be specified. In sec 4 we give an overview of the advantages of considering Lie groups.

2. DERIVATION OF THE INFINITESIMAL GENERATORS OF THE GROUP

Given a local group of transformation G acting on $N \times M$ the space of the independent and the dependent variable (x, u), there is induced an action of G on the space $N \times M^{(k)}$ consisting of points $(x, u^{(k)})$ where $u^{(k)}$ represents derivatives of the dependent variables of order $\leq k$ given by $\partial_j u = \frac{\partial^j u}{\partial_{x_p}^{j_1} \ldots \partial_{x_p}^{j_p}}$,$0 \leq |J| \leq k, \quad |J| = j_i + j_2 + \cdots + j_p$ See (Nwachuku and Ifidon 1991).

This induced action of G, called the $k-$th prolongation of G, can easily be obtained from the corresponding prolonged infinitesimal generators $P_r^{(k)}\chi$ of the group which are vector fields on $N \times M^{(k)}$ and have a relatively simple expression. The generators χ of the group are vector fields on $N \times M$ given by $\chi = \sum_{i=1}^{p} \zeta^i(x, u)\frac{\partial}{\partial x_i} + \sum_{i=1}^{q} \varphi_i(x, u)\frac{\partial}{\partial u^i}$ where p and q represents the number of independent variable in the space $N \times M$ respectively.

The corresponding expression for the prolonged vector field is then

$$P_r^k \chi = \chi + \sum_{i=1}^{q} \sum_{J} \varphi_i^r(x, u^{(k)}) \frac{\partial}{\partial u_J^i}$$

with

$$\phi_l^J = D^J \left(\phi^l - \sum_{i=1}^{p} u_i^j \xi_l + \sum_{i=1}^{p} u_{J,i}^l \xi_l \right)$$

where

$$D^J = D_1^{j_1} \circ D_2^{j_2} \circ \cdots \circ D_n^{j_n}$$

is the total derivative operator. The infinitesimal criteria for invariance of a system of partial differential equations $\nabla(x, u^{(k)}) = 0$ under the action of G states that G is a symmetry group of the system $\nabla(x, u^{(k)}) = 0$ if and only if for every infinitesimal generators χ of G.

$$P_l^{(k)} \chi \nabla(x, u^{(k)}) = 0 \tag{9}$$

whenever

$$\nabla(x, u^{(k)}) = 0$$

for the case of the Schrödinger equation

$$\nabla(x, u^{(k)}) = \Psi_1 + i(-\nabla^2)\Psi \tag{10}$$

A typical vector field on $\mathbb{R}^4 \times \mathbb{R}^2$ with coordinates (x, y, z, t, v, Ψ) is given by

$$\chi = \xi \partial_x + \eta \partial_y + \lambda \partial_z + \gamma \partial + \Phi \partial_x + \varphi \partial_\Psi \tag{11}$$

where the coefficients $\{\xi, \eta, \lambda, \gamma, \phi, \Phi\}$ are arbitrary functions of x, y, z, v and Ψ. The corresponding second prolongation of χ is

$$P_r^{(2)} \chi = \chi + \sum_J \theta^J \partial_{\Psi_J} + \sum_J \Lambda^J \partial_{v_j} \tag{12}$$

where the $J-$sum is over all partitions

$$J = \{(1,0,0,0)(0,1,0,0)(0,0,1,0)(0,0,0,1)(1,1,0,0)(1,0,1,0)$$
$$(1,0,0,1)(0,1,1,0)(0,1,0,1)(0,0,1,1)(2,0,0,0)(0,2,0,0)$$
$$(0,0,2,0)(0,0,0,2)\}$$

Substitution of (11) and (12) in (9) yields

$$i\theta^{(0,0,0,1)} + \theta^{(2,0,0,0)} + \theta^{(0,2,0,0)} + \theta^{(0,0,3,0)} - v\Phi - \Psi\varphi = 0$$

subject to

$$i\Psi + \Psi_{xx} + \Psi_{yy} + \Psi_{zz} - v\psi \tag{13}$$

where

$$\theta^{(0,0,0,1)} = \begin{cases} \Phi_t - \Psi_x\xi_t - \Psi_y\eta_t - \Psi_x\lambda_t + \Psi_z(\lambda_\Psi - \gamma_t) \\ -\Psi_t\Psi_x\xi_\Psi - \Psi_t\Psi_y\eta_\Psi - \Psi_t\Psi_z\lambda_\Psi - \Psi_t^2\gamma_\Psi \end{cases}$$

$$\theta^{(0,0,0,1)} = \begin{cases} \Phi_{xx} - \Psi_x(2\gamma_{x\Psi} - \xi_{xx}) - \Psi_y\eta_{xx} - \Psi_z\lambda_{xx} - \Psi_l\gamma_{xx} \\ +\Psi_{xx}(\Phi_\Psi - 2\xi_x) - 2\Psi_{xx}\lambda_x - 2\Psi_{xx}\lambda_x - 2\Psi_{tx}\gamma_x \\ +\Psi_x^2(\phi_{\Psi\Psi} - 2\xi_{\Psi x}) - 2\Psi_x\Psi_y\eta_{\Psi x} \\ +2v_x\Phi_{xx}2\Psi_x v_x(\phi_{v\Psi} - \xi_{vx})2\psi_x v_x(\varphi_{v\Psi} - \xi_{vx}) \\ +2\Psi_x\Psi_x\lambda_{x\Psi} - 2\Psi_x\Psi_t\gamma_{\Psi x} - 2\Psi_x\Psi_z\lambda_{x\Psi} - \\ 2\Psi_z\Psi_t\gamma_{x\Psi} - v_x\Psi_y\eta_{xv} - 2\Psi_z v_x\lambda_{xv} - 2v_x\Psi_l\gamma_{xv} - \\ 3\Psi_x\Psi_{xx}\xi_\Psi\Psi_{xx}\Psi_y\eta_\Psi - \Psi_{xx}\Psi_z\lambda\Psi + \Psi_{xx}\Psi_t\gamma\Psi - \\ 2\Psi_{xx}v_x\xi_v - 2\Psi_x\Psi_{xy}\eta_v - 2\Psi_x\Psi_{xz}\lambda_\Psi - 2v_x\Psi_{xz}\eta_v - \\ 2\Psi_x\Psi_{xt}\gamma_\Psi - \Psi_x^3\xi_{\Psi\Psi} - \Psi_x^2\Psi_y\eta_{\Psi\Psi} - \Psi_x^2\Psi_z\lambda_{\Psi\Psi} \\ -\Psi_x^2\Psi_l\gamma_{\Psi\Psi} - 2\Psi_x^2 v_x\xi_{v\Psi} - 2v_x\Psi_x\Psi_y\eta_{v\Psi} \\ -\Psi_x\Psi_z v_x\lambda_{v\Psi} - 2v\Psi_x\Psi_t\gamma_{v\Psi} + v_x^2\Phi_{vv} \\ -\Psi_x v_x^2\xi_{vv} - v_x^2\Psi_y\eta_{vv} - v_x^2\Psi_t\gamma_{vv} + v_{xx}\Phi_v \\ -\Psi_x v_{xx}\xi_r - v_{xx}\Psi_y\eta_y + \Psi_z v_{xx}\lambda_v - v_{xx}\Psi_{tv} \end{cases}$$

$$\theta^{(0,2,0,0)} = \begin{cases} \Phi_{yy} - \Psi_x\xi_{yy} + \Psi_y(2\Phi_{\Psi y} - \eta_{yy}) - \Psi_y\lambda_{yy} - \Psi_z\lambda_{yy} \\ -2\Psi_x\Psi_y\xi_{\Psi y} + \Psi_y^2(\Phi_{\Psi\Psi} - 2\xi_{\Psi x}) - 2\Psi_y 2\Psi_z\lambda_{\Psi y} \\ -2\Psi_t\Psi_y\gamma_{\Psi y} - 2\Psi_{xy}\xi_y + \Psi_{yy}^2(\phi_\Psi - 2\xi_y) - 2\Psi_{zy}\lambda_y \\ -2\Psi_{ty}\gamma_y + 2v_y\Phi_{vy} - 2\Psi_x v_y\xi_{xy} + 2\Psi_y v_y(\Phi_{v\Psi} - \eta_{xy}) \\ -2\Psi_z v_y\lambda_{xy} - 2\Psi_t v_y\gamma_{vy} - 2\Psi_x\Psi_y^2\xi_{\Psi\Psi} - \Psi_y^3\eta_{\Psi\Psi} \\ -\Psi_y^2\Psi_z\lambda_{\Psi\Psi} - \Psi_t\Psi_y^2\gamma_{\Psi\Psi} - 2\Psi_y\Psi_{xy}\xi_\Psi - 3\Psi_y\Psi_{yy}\eta_\Psi \\ -2\Psi_y\Psi_{xy}\lambda_\Psi - 2\Psi_y\Psi_{ty}\gamma_\Psi - 2\Psi_x\Psi_y v_y\xi_{v\Psi} - 2v_y\Psi_y^2\Phi_{\Psi v} \\ -2\Psi_y\Psi_z v_y\lambda_{v\Psi} - 2\Psi_y\Psi_t\gamma_{\Psi v} - 2\Psi_{xy}v_y\xi_v - \Psi_x\Psi_{yy}\xi_\Psi \\ -\Psi_{yy}\Psi_z\lambda_\Psi - \Psi_{yy}\Psi_t\gamma_\Psi - 2\Psi_{yy}v_y\eta_y - 2\Psi_{xy}v_y\lambda_v \\ -v_y\Psi_{ty}\gamma_v + y_y^2\Psi_{vv} - v_y^2\Psi_x\xi_{vv} - v_y^2\Psi_y\eta_{vv} - \Psi_z v_y^2\lambda_{vv} \\ -\Psi_t v_y^2\gamma_{vv} - \Psi_z v_{yy}\lambda_v - \Psi_t v_{yy}\gamma_v \end{cases}$$

$$\theta^{(0,0,0,2)} = \begin{cases} \Phi_{zz} - \Psi_x\xi_{zz} - \Psi_y\eta_{zz}\Psi_z(2\Phi_{\Psi z} - \lambda_{zz}) - \Psi_t\gamma_{zz} \\ -2\Psi_y\Psi_z\xi_{\Psi z} + \Psi_z^2(\Phi_\Psi - 2\lambda_{\Psi z} - 2\lambda_{\Psi z}) - 2\Psi_z\Psi_t\gamma_{\Psi z} \\ -2\Psi_{xz}\xi_z - 2\Psi_{yz}\eta_z + \Psi_{zz}(\Phi_\Psi - 2\lambda_z) - 2\Psi_{zt}\gamma_z \\ +2v_z\Phi_{vz} - 2v_z\Psi_x\xi_{xz} - 2v_z\Psi_y\eta_{vz}v_z(\Phi_{v\Psi} - \lambda_{vz}) \\ -2v_z\Psi_t\gamma_{vz} - \Psi_z^2\Psi_x\xi_{\Psi\Psi} - \Psi_z^2\Psi_y\eta_{\Psi\Psi} - \Psi_z^3\lambda_{\Psi\Psi} \\ -\Psi_z^2\Psi_t\gamma_{\Psi\Psi} - 2\Psi_z\Psi_{xz}\xi_\Psi - 2\Psi_z\Psi_{yz}\eta_\Psi - 3\Psi_z\Psi_{zz}\lambda_\Psi \\ -2\Psi_x\Psi_{zt}\gamma_\Psi - 2v_z\Psi_z\Psi_x\xi_{\Psi v} - 2\Psi_z v_z\Psi_y\eta_{v\Psi} - 2v\Psi_{xz}\xi_v \\ -2\Psi_z\Psi_y\eta_{v\Psi} - 2v_z\Psi_{xz}\xi_v - 2v_z\Psi_{yz}\eta_v - \Psi_{zz}\Psi_x\xi_\Psi \\ -\Psi_{zz}\Psi_y\eta_\Psi - \Psi_{zz}\Psi_t\gamma_\Psi - 2v_z\Psi_{zz}\lambda_v - 2v_z\Psi_{zt}\gamma_v \\ +v_z^2\Phi_{yy} - v_z^2\Psi_{vv} - \xi_{vv} - v_z^2\Psi_y\eta_{vv} - v_z^2\Psi_z\lambda_{vv} \\ -v_z^2\Psi_t\eta_{vv} + v_{zz}\Phi_v - v_{vv}\Phi_v - v_{zz}\Psi_x\xi_v \\ -v_{zz}\Psi_y\eta_v - v_{zz}\Psi_z\lambda_v - v_{zz}\Psi_t\gamma_v \end{cases}$$

from (**??**) and (13) the coefficient function $\{\xi, \eta, \lambda, \gamma, \phi, \Phi\}$ satisfy the symmetry equations

$$2\Phi_{\psi v} - \xi_{xx} - \xi_{yy} - \xi_{zz} + iv\psi v\psi(\Psi_x - \xi_\Psi) - i\xi_t = 0 \tag{14}$$

$$2\Phi_{y\psi} - \eta_{xx} - \eta_{yy} - \eta_{zz} + iv\psi v\psi(\Psi_y - \eta_\Psi) - i\eta_t = 0 \tag{15}$$

$$2\Phi_{z\psi} - \lambda_{xx} - \lambda_{yy} - \lambda_{zz} + iv\psi v\psi(\Psi_z - \lambda_\Psi) - i\lambda_t = 0 \tag{16}$$

$$\gamma_t - 2\xi_x - i(\gamma_{xx} + \gamma_{yy} + \gamma_{zz}) - iv\psi v\psi = 0 \tag{17}$$

$$\gamma_t - 2\eta_y - i(\gamma_{xx} + \gamma_{yy} + \gamma_{zz}) - iv\psi v\psi = 0 \tag{18}$$

$$\gamma_t - 2\lambda_y - i(\gamma_{xx} + \gamma_{yy} + \gamma_{zz}) - iv\psi v\psi = 0 \tag{19}$$

$$\eta_x - \xi_y = 0 \tag{20}$$

$$\eta_\Psi = \eta_v = \lambda_\Psi = \lambda_v = \xi_\Psi = \xi_v = \gamma_\Psi = \gamma_v = \gamma_x = \gamma_y = \gamma_z = 0 \tag{21}$$

$$\Phi_{xx} + \Phi_{yy} + \Phi_{zz} - v\Phi_{zz} - \Psi\Phi + iv\Psi v\Psi_x + \gamma_{yy} + \gamma_{zz} + i\Phi_t$$
$$+v\Psi(\Phi_v - \gamma_t) + iv^2\Psi^2\gamma_\Psi = 0 \tag{22}$$

the most general solutions to the set of equations (14)-(22) are given by

$$\begin{cases} \xi = \frac{1}{2}(c_1 t + c_2)x - c_{10}z - c_{11}y + c_4 t + c_5 \\ \eta = \frac{1}{2}(c_1 t + c_2)y - c_{12}z - c_{11}y + c_6 t + c_7 \\ \lambda = \frac{1}{2}(c_1 t + c_2)z - c_{10}z - c_{12}y + c_8 t + c_9 \\ \gamma = c_1 t^2 + c_2 t + c_3 \\ \Phi = \frac{1}{2}\left(\frac{1}{4}c_1(x^2 + y^2 + z^2) + c_4 x + c_6 y + c_8 z + c_0\right)\Psi \\ \phi = -(c_1 t + c_2)v + \frac{3}{4}ic_1 \end{cases} \tag{23}$$

where $c_i, i = 0, 1, \ldots, 12$ are arbitrary real constants. Hence the infinitesimal symmetry algebra G of the Schrödinger equation (8) is of dimensions 13, and is spanned by the basis vectors

$$\begin{cases} \chi_1 = \partial_x \\ \chi_2 = \partial_y \\ \chi_3 = \partial_z \\ \chi_4 = \partial_t \\ \chi_5 = x\partial_x + y\partial_y + z\partial_z + 2t\partial_t - 2v\partial_v \\ \chi_6 = -z\partial_x + x\partial_z \\ \chi_7 = -y\partial_x + x\partial_y \\ \chi_8 = -z\partial_y + y\partial_z \\ \chi_9 = t\partial_x + \frac{i}{2}x\Psi\partial_v \\ \chi_{10} = t\partial_y + \frac{i}{2}x\Psi\partial_v \\ \chi_{11} = t\partial_z + \frac{i}{2}\Psi\partial_\Psi \\ \chi_{12} = t\partial_z + ty\partial_y + tz\partial_z + t^2\partial_z + \frac{1}{4}(x^2 + y^2 + z^2)\Psi\partial_\Psi \\ + \left(\frac{3}{2}i - 2tv\right)\partial_v \\ \chi_{13} = \frac{i}{2}\Psi\partial_\Psi \end{cases} \tag{24}$$

the generators (24) satisfy the Lie commutation relations
$[L, L] = L$, $[L, V] = V$, $[T, V] = P$, $[L, P] = P$,
$[\chi_5, V] = V$, $[\chi_5, P] = P$, $[\chi_5, T] = T$, $[\chi_5, \chi_{12}] = \chi_{12}$,
$[P, V] = \chi_{13}$, $[T, \chi_{12}] = \chi_5$, $[P, \chi_{12}] = V$
with other commutation relation vanishing. Here

$$L_j = (x \times v) = \xi_{jik}x^i\partial_k$$

$$V_j = t\nabla_j + \frac{1}{2}X^j\Psi\partial_\Psi$$

$$P_j = \nabla_j$$

$$T = \partial_t$$

It can be shown that the regular representation of this algebra is that of the thirteen dimensional subalgebra of $SL(6, \mathbb{R})$. Observe that the fact that the generators χ of this group are integrals of the motion lead to a number of conservation laws in quantum mechanics. For instance to each of the components of the generators, , is associated an observable called the angular momentum of the system. The fact that the angular momentum in a given direction is an integral of the motion is the quantum mechanical analog of the law of conservation of angular momentum. The laws of conservation of total energy and linear momentum corresponds to the generators T and P and so on. Other conservation laws an be similarly obtained. From (23) it can be seen that the vector fields χ are of the form

$$\chi = \sum_{i=1}^{p} \xi^i(x)\frac{\partial}{\partial \chi} + \phi(x,u)\frac{\partial}{\partial u}$$

which implies the symmetry group is projectile (Hammermesh, 1983). It can be shown that $[H, \chi] = 0$ for all infinitesimal generators of the group, where H is the Hamiltonian of the system. Thus the dynamics of the system is preserved.

3. APPLICATION TO THE SOLUTION OF THE SCHRÖDINGER EQUATION

Now suppose that G acts regularly on $Z \times M$, so that the quotient space Z/G can be regarded as a differentiable manifold, (Palais, 1957 as well as Pestov, 1995), then if ∇ is a system of partial differential equation defined on the space Z which has G as its symmetry group, there is a system of partial differential equations $\nabla/G \subset J^K(Z/G, p-1)$ where l is the dimensions of the orbits of G and $J^K(Z,p)$ is the extended k-jet bundle of p-sections of Z corresponding to the various partial derivatives of the dependent variables of order $\leq k$, since G leaves Δ invariant, the problem of finding the G invariant solutions to Δ is equivalent to solving the reduced system Δ/G in $p-l$ independent variables. The solutions of Δ/G when lifted back to Δ gives all the G-invariant solution of Δ. As an illustration of this we consider the scale invariant solutions, which are more representative. Other solutions may be constructed in a similar fashion. In this case, the vector field is $\chi_5 = x\partial_y + z\partial_z + 2t\partial_t - 2v\partial_y$ with corresponding one parameter group $G_5 = \exp(\lambda\chi_5)$ whose group action is

$$\begin{aligned} G_5 : (x, y, z, t, v, \Psi) \to (&\exp(\lambda), x^\lambda, \lambda, \\ &\exp(\lambda)z, \exp(2\lambda)t, \exp(2-\lambda)v, \Psi) \end{aligned} \tag{25}$$

in order that G_5 acts regularly, we must consider only the submanifold $Z = \mathbb{R}^6 = \{0\}$, which is non Hausdorff so that Z/G_5 can be realized as a 6 dimensional Torus T^6 with four exceptional points

$$q_{++} = \{x = y = z = t = 0, \nabla > 0, \Psi > 0\}$$
$$q_{+-} = \{x = y = z = t = 0, \nabla > 0, \Psi < 0\}$$
$$q_{-+} = \{x = y = z = t = 0, \nabla < 0, \Psi > 0\}$$
$$q_{--} = \{x = y = z = t = 0, \nabla < 0, \Psi < 0\}$$

corresponding to four vertical orbits. Therefore a G_5 invariant solution of the Schrödinger equation corresponds to a curve in Z/G_5 which is a solution to Δ/G_5. In order that the G_5 invariant solution be a single valued function of x, y, z, t, we concentrate on the Hursdorff submanifold $T \subset Z/G_5$. In this case the curve does not pass through $q_{++}, q_{+-}, q_{-+}, q_{--}$. Choose local coordinates

$$\xi = \frac{x^2 + y^2 + z^2}{t}; v_\xi = vt \tag{26}$$

clearly ξ and v are invariant under the group action of G_5. Treating ξ as the new independent variable and substituting (26) into (8) we see that

$$\Delta/G \equiv 4\Psi_{\xi\xi} + (6 - i\xi)\Psi_\xi - v_\xi\Psi = 0 \tag{27}$$

G_5-invariant solutions to (8) corresponding to various values of the potential v can now be found. For instance, for the potential.

$$V = \frac{-6\alpha}{x^2 + y^2 z^2}, \alpha \in \mathbb{R}; v_\xi = -\frac{6\alpha}{\xi} \tag{28}$$

(27) becomes

$$\frac{2}{3}x^2\Psi_{xx} + x(1 + X)\Psi_x + \alpha\Psi = 0 \tag{29}$$

where $x = \frac{i\xi}{6}$. Using the transformation (29) becomes

$$Q_{xx} + \left(-\frac{1}{4} + \frac{3}{4x'} + \frac{\frac{1}{4} - \mu^2}{x'^2}Q\right) = 0 \tag{30}$$

where $x' = -\frac{3}{2}x$ and $\mu^2 = \frac{23}{8} - 3\alpha$

(30) is the differential equation satisfied by the Kummer Confluent Hypergeometric function (Abramowits and Stegtm,1965). Therefore

$$\Psi(x, y, z, t) = \left(\frac{i}{4}\left(\frac{x^2 + y^2 + z^2}{t}\right)\right)_1$$
$$F_1\left(\mu - \frac{1}{4}, 1 + 2\mu, \frac{1}{4}\left(\frac{x^2 + y^2 + z^2}{t}\right)\right) \tag{31}$$

where $\mu^2 = \frac{23}{8} - 3\alpha$ are the G_5 invariant solutions of the Schrödinger equation (8) corresponding to the potential

$$v(x, y, z) = -\frac{6\alpha}{x^2 + y^2 + z^2}, \alpha \in \mathbb{R}$$

Other G_5-invariant solutions can be constructed for various values of v using different coordinate patches on Z/G_5. The G_5-invariant solution (31), represents a special state of the system in which an exact value for all three components of the angular momentum $Z = (\chi_6, \chi_7, \chi_8)$ can be specified, since ψ is an eigen state of each component of L with eigen value zero (Schiff, 1968). Note that since the components of Z do not commute, the system cannot in general be assigned definite values for all angular momentum components simultaneously. Also given an initial state $\phi \in (\mathbb{R}^3)$ the position probability measure over any measurable subset $\Delta \subset \mathbb{R}^3$ (Borel subset) for the state ϕ, evolving in the presence of a conservative force filed potential $v \approx \frac{1}{x^2+y^2+z^2}$ can be asymptotically obtained from (31) for large positive or negative times. This is useful in scattering theory. Next we give a general perspective on the usefulness of considering Lie group.

4. PROLONGATIONS; GENERALIZED ROTATIONS IN $\mathscr{H}$

Let $H = \{\lambda\phi\}$ be the set of all possible states of a quantum system, where ϕ is a non-zero vector in the complex Hilbert space and λ are scalars. The transformation from $\phi \to g \cdot \phi$ where g is an operation from $\mathscr{H}$ to $\mathscr{H}$ is usually referred to generalized rotation of the state vectors in $\mathscr{H}$. Usually the generalized rotations do not conserve norms in II, thus operations for which norms are conserved are useful in quantum mechanic. Now consider that set $Z = \mathbb{R}^n \times \mathscr{H}$ and where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ are real line projection valued measures, then a pair of points $(x, Q) \in Z$ determines uniquely a particular pure state of the system. Let G be a Lie group of symmetries for II and consider the group action of G on Z. if G acts regularly on Z, we may view the quotient space Z/G as a differentiable manifold. Let Z_q be q-section of Z the Z_q would be characterized by local coordinate systems which are regular on Z due to the regularity of the action of G on Z. If is the dimension of the orbits of G, then due to the regular coordinate structure of Z. we construct a subbundle (Z_q, P_G, Z_{q-1}), where $P_G = (P_1, \ldots, P_{q-1})$ are $q-1$ independent composition maps of G. Thus there is a projection of Z_{q-1}. Now since from section 1,

G preserves transition probabilities in $\mathscr{H}$, we have a generalized rotation of state vectors in $\mathscr{H}$ in which norms are preserved (Mackey 1978, Kuku et al 1985). These generalised rotations. Comprising symmetry transformation which involve stretching sealing or contraction as well as pure rotation. Constitute a change of axes in $\mathscr{H}$ without a change in the state vectors defining the system. However a change in the frame of reference would involve a change in the choice for representation. Since a particular state vector has different components when referred to different axes and these constitute the different representation of the state. Thus one expects a change in the representation of the state vectors. Our theory ascertains that under the prolonged action of g one obtains in Z_{q-1}, new representations of the state vectors in $q-1$ power components. Since Z_{q-1} comprises of state vectors whose components are defined on $\mathbb{R}^{q-1}$. Therefore for complex quantum systems, one obtains new symmetries which greatly reduces the complexity of the system.

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