Journal of the Nigerian Mathematical Society
Vol. 32, pp. 185-191, 2013

AN AUGMENTED CUBIC LINE SEARCH ALGORITHM FOR SOLVING HIGH-DIMENSIONAL NONLINEAR OPTIMIZATION PROBLEMS

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ABSTRACT. This paper presents a performance study of a one-dimensional search algorithm for solving general high-dimensional nonlinear optimization problems. The proposed approach is a hybrid between the conventional cubic line search algorithm and a variant of Armijo's line search algorithm. The resulting algorithm, called Augmented Cubic line search, was tested on some standard optimization problems, with a view to observing how optimization techniques for nonlinear optimization problems respond with increasing dimension. To this end, we report the successful performance of the algorithm on objective functions with 5, 000 and 10, 000 independent variables..

Keywords and phrases: Armijo's line search algorithm, High-dimensional nonlinear optimization problems, Augmented Cubic line search.
2010 Mathematical Subject Classification: 65k, 90c

1. INTRODUCTION

Many conjugate gradient methods seek to optimize an objective function, f, by choosing the best step length $\alpha = \alpha_k$ that optimizes $f(x_k + \alpha d_k)$ on the line $x = x_k + \alpha d_k$ through a descent direction d_k. Wolfe [1] stated that the α_k satisfies, at least approximately, the requirement

$$f(x_k + \alpha_k d_k) = \text{Optimize } f(x_k + \alpha d_k). \tag{1}$$

Therefore, if $\phi : \Re \to \Re$ is defined by

$$\phi(\alpha) = f(x_k + \alpha d_k), \tag{2}$$

then, one method of determining α_k is to estimate a local optimizer of ϕ, so that α_k satisfies the condition

Received by the editors March 10, 2011; Revised: April 18, 2012; Accepted: April 23, 2012
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$$\phi'(\alpha) = 0, \tag{3}$$

where ϕ' is the derivative of ϕ.
In general, (3) is a nonlinear equation and so it is difficult to be solved analytically.
Therefore, a numerical method for estimating the value of α which satisfies (3) must be used. Some researchers such as Nwaeze [2], Nwaeze and Bamigbola [3], Nwaeze [4], Hestenes [5], Turner [6], Brian et al [7] and Himmelblau [8] used various line search methods. The commonly-used line search rules are as follows.

(a) Minimization Rule. At each iteration, α_k is selected so that

$$f(x_k + \alpha_k d_k) = \min_{\alpha > 0} f(x_k + \alpha d_k) \tag{4}$$

(b) Approximate Minimization Rule. At each iteration, α_k is selected so that

$$\alpha_k = \min\left\{\alpha \mid g(x_k + \alpha d_k)^T d_k = 0,\ \alpha > 0\right\} \tag{5}$$

(c) Armijo Rule. Set scalars $s_k,\ \beta,\ L > 0,\ \sigma$.with
$s_k = -g_k^T d_k/(L||d_k||^2),\ \beta \in (0,\ 1)$and $\sigma \in (0,\ 0.5)$. Then α_k is the largest α in
$\{s_k,\ \beta s_k,\ \beta^2 s_k,\ ...\}$ such that

$$f_k - f(x_k + \alpha d_k) \geq -\sigma\alpha\, g_k^T d_k. \tag{6}$$

(d) Strong Wolfe Rule. α_k is chosen to satisfy
$f_k - f(x_k + \alpha d_k) \geq -\sigma\alpha g_k^T d_k$ and

$$|g(x_k + \alpha_k d_k)^T d_k| \leq -\beta g_k^T d_k \tag{7}$$

where $\beta \in (0,\ 1),\ f_k = f(x_k)$ and $\sigma \in (0,\ 0.5)$.
(e) Cubic line search as described in section 2.
(f) Modified Armijo Rule as described in section 3.

The cubic line procedure, generally, provides much more rapid convergence to the desired optimum point of the objective function. The convergence of Armijo's rule is later slow when the dimensionality of the objective function is high, though, many researchers appraised Armijo's rule as a good tool for solving nonlinear equations. It implies that Armijo's algorithm could, quickly, generate a good estimate of α in equation (1) if the step length is made larger in size. In sections 2 and 3, we consider the effectiveness and accuracy of cubic and Armijo-type line search methods. In section 4, we consider the Augmented Cubic line search procedure. Numerical examples were implemented and displayed in section 5. Finally, section 6 summarizes the findings of this paper with a conclusion.

2. PRELIMINARY

2.1 Cubic line search procedure

Suppose α_1 and α_2 are distinct values of α in equation (1). Define a cubic polynomial

$$\psi(\alpha) = a\alpha^3 + b\alpha^2 + c\alpha \ + \ d, \tag{8}$$

for which

$$\psi(\alpha_i) = \phi_i, \ \ \phi_i \ = \ \phi(\alpha_i), \ \ i = 1, \ 2$$
$$\psi'(\alpha_i) = \phi'_i, \ \ \phi'_i \ = \phi'_i(\alpha_i)$$
$$\psi'(\alpha) = \tfrac{d}{d\alpha}\,\phi(\alpha)$$

The unique optimizer, $\alpha*$, of , if it exists, satisfies $\psi'(\alpha) = 0$ and $\psi''(\alpha) \ > \ 0$. It follows that $\alpha*$ satisfies
$3a\alpha^2 \ + \ 2b\alpha \ + \ c \ = 0$ and $3a\alpha \ + \ b \ \ > 0$. Thus,
$\alpha* \ = \ \frac{-b \ + \ (b^2-3ac)^{\frac{1}{2}}}{3a}$, if $a$is nonzero,
$\alpha* \ = \ \frac{-c}{2b}$, if a=0.
The conventional cubic line search procedure was built upon the following iterative technique. If $a$is nonzero,

$$\alpha_{k+1} \ = \alpha_k \ + \ \frac{-b \ + \ (b^2 - 3ac)^{\frac{1}{2}}}{3a}, \tag{9}$$

and if a=0 we have

$$\alpha_{k+1} \ = \alpha_k \ - \ \frac{c}{2b}.$$

The algorithm terminates when $\mid \alpha_{k+1} - \alpha_k \mid$ is less than a prescribed number so that α_{k+1}
becomes an adequate estimate of $\alpha*$.

However, the above algorithm sometimes, yields a complex value of $\alpha*$ when $b^2 \ < \ 3ac$. This makes the computation of $\alpha*$ to be very tedious and unreliable.

2.2 Modified Armijo line search technique

We modify Armijo's line search to take the following form.
Set scalars $s_k, \ \beta, \ L_k > 0, \ \sigma, \ \omega$ with $s_k = -g_k^T d_k/(L_k||d_k||^2), \ \beta \in (0, \ 1), \ \omega \in [0, \ 2)$ and $\sigma \in (0, \ 0.5)$. Then we choose α_k, as the largest value α from the set of values $s_k, \ \beta \ s_k, \ \beta^2 s_k, \ \beta^3 s_k, \ ...$, such

that

$$f_k - f(x_k + \alpha d_k) \geq -\sigma\alpha\, [g_k^T d_k + 0.5\alpha\omega L_k ||d_k||^2]. \tag{10}$$

$$L_k = \frac{||g_k - g_{k-1}||}{||x_k - x_{k-1}||}. \tag{11}$$

In computations we used $\sigma = 0.35$, $\beta = 0.85$and$\omega = 1$. The step size, α_k, defined by this method is larger than that defined in the original Armijo line search. It implies that it is easier to find α_k in this variant than in Armijo's rule.

2.3 Augmented Cubic line search procedure.

This paper presents an Augmented cubic line search procedure. It is a hybrid of the conventional cubic line search and the Armijo-type technique. This method is designed to complement cubic line search method when it runs into a complex value. The method starts off line search iteration with cubic method and switches onto Armijo-type procedure, in the k^{th}iteration level, whenever cubic method fails. In the $(k+1)^{th}$ iteration level, the method continues processing with the cubic scheme, if that iteration level is feasible. The Augmented Cubic line search involves the following iterative steps.

$\alpha_{k+1} = \alpha_k + \frac{-b + (b^2 - 3ac)^{\frac{1}{2}}}{3a}$, if $a$is nonzero, $b^2 > 3ac$, $\alpha_{k+1} = \alpha_k - \frac{c}{2b}$, if a= 0 and

$\alpha_{k+1} = \alpha_k$, by Modified Armijo line search, if $b^2 < 3ac$.

The algorithm is terminated when $| \alpha_{k+1} - \alpha_k |$ is less than a prescribed number so that $\alpha_{k+1}$is an adequate estimate of $\alpha *$. Results of problem 1, in section 5, is a simple illustration of augmented cubic line search method implemented in a conjugate gradient method described in [3].

3. Numerical Examples

A java program implemented the Augmented Cubic line search method, using Nwaeze and Bamigbola, Polak Ribierre and Fletcher Reeves conjugate gradient methods (CGM), on the following optimization problems.

Problem 1: Minimize

$$f(x) = (x_1 - 1)^4 + (x_2 - 3)^2 + 4(x_3 + 5)^4; \quad x_0 = [4,\ 2,\ -1]^T$$

Table 1: Results of problem 1 where I is Iterations.

I	x_1	x_2	x_3	$f(x_1; x_2; x_3)$	α_k	$b^2 - 3ac$
1	4	2	-1	1106	4.5818e-3	1.266e16
2	3.5052	2.0092	-5.6918	41.28405	1.2225e-4	3.401e15
3	3.4843	2.0097	-5.8142	40.8275884	4.3529e-2	4.634e7
4	0.7604	2.0971	-4.9473	0.81849334	5.1617e-4	1.231e7
5	0.8115	2.0964	-4.9582	0.81771935	2.2181e-2	77536.76
6	0.8299	2.1363	-4.7328	0.76725791	3.0575e-3	6437.89
7	Cubic method fails.					-10561.5
	Armijo-type method corrects cubic method					
7	1.1235	2.9659	-5.0066	0.00139545		
	Cubic method continues					
8	1.0935	2.9966	-4.9928	0.00008799		
.	.	.	.	.		
.	.	.	.	.		
18	1.0001	2.9998	-4.9999	0.00000004		

Exact solution: $x_1 = 1,\ \ x_2 = 3,\ \ x_3 = -5,\ f(x_1,\ x_2,\ x_3) = 0.$

Problem 2: Minimize

$$F(x) = \sum_{i=1}^{n} 100(x_{2i} - x_{2i-1}^2)^2 + ((1 - x_{2i-1})^2, [x_0]_{2i} = 1$$
$$\text{and } [x_0]_{2i-1} = -1.2$$

Table 2: Results of problem 2.

CGM	n	Iterations	Time(s)	Function value
Nwaeze and Bamigbola	5000	26	103	9.374E-7
Polak Ribierre	5000	50	56.9	2.99E-5
Fletcher Reeves	5000	81	98.2	1.6119E-6

Problem 3: Minimize

$$f(x) = \sum_{i=1}^{n} [n - \sum_{j=1}^{n} \cos(x_j) + i(1 - \cos(x_i)) - \sin(x_i)]^2, [x_0]_i = \tfrac{1}{n}$$

Table 3: Results of problem 3.

CGM	n	Iterations	Time(s)	Function value
Nwaeze and Bamigbola	10,000	3	38.5	3.0E-6
Polak Ribierre	10,000	2	184	8.0E-6
Fletcher Reeves	10,000	2	184	8.0E-6

Problem 4: Minimize

$$F(x) = \sum_{i=1}^{n}(x_i - 1)^2 + [\sum_{i=1}^{n} i(x_i - 1)]^2 + [\sum_{i=1}^{n} i(x_i - 1)]^4, [x_0]_i = 1 - i/n$$

Table 4: Results of problem 4.

CGM	n	Iterations	Time(s)	Function value
Nwaeze and Bamigbola	10,000	30	109	0.00000002
Polak Ribierre	10,000	19	94.4	1.52E-8
Fletcher Reeves	10,000	21	107	0.000001

The above program results confirm that Augmented Cubic line search would be a very useful tool in any conjugate gradient method. This is true since the method successfully led to the optimization of high dimensional objective functions, given in problems 1 to 4, through different conjugate gradient methods.

4. CONCLUDING REMARKS

Augmented Cubic line search method is highly effective and accurate. It complements the conventional cubic line search procedure when applied in any conjugate gradient algorithm. Numerical examples on standard optimization problems reveal that the method led to the successful optimization of objective functions with 5000 and 10,000 independent variables. Thus, any large scale conjugate gradient scheme could utilize the fast convergent property of this Augmented Cubic line search method.

In Fig. 1, the critical value of the temperature is plotted against Newtonian parameter δ.

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