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AN APPLICATION OF THE NEW DISSIPATIVE CRITERION FOR SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS.

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Abstract. Using the new dissipative criterion towards Yacubovich Oscillations, we give some extensions of earlier result on dissipativity of some nonlinear differential equations of the second order. The frequency domain methods are used on Rayleigh and Liénard equations.

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1. Introduction

Over the years, the study of qualitative properties of non-linear differential equations of the form

$$X^{'} = F(t, X) \tag{1.1}$$

have been linked to obtaining a Lyapunov function $V = V(t, X)$ where properties are linked to $F(t, X)$. For example,[9, 10, 17], it is known that if $V(t, X)$ is positive definite and $\dot{V}|_{(1.1)}$ is negative definite, we are guaranteed the qualitative properties as boundedness, stability and many other properties for (1.1). However, it is very difficult to always construct such a function $V(t, X)$. In attempt to overcome this difficulty, Yacubovich [18], Kalman [12], Popov [15] and later Yacubovich [19, 20], came out with what we now call the Kalman-Yacubovich -Popov (KYP) lemma,(cf:[1, 2, 3, 4], [5, 7, 11, 21]) which helped to discuss the dissipative properties of systems of the form

$$X^{'} = AX - B\varphi(\sigma) + P(t, X), \ \sigma = C^{*}X \tag{1.2}$$

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with A an $n \times n$ stable matrix, B and C $n \times m$-matrices, $\varphi(\sigma) = col(\varphi_j(\sigma_j)), (j = 1, 2, \ldots, m)$, ($\varphi_j(\sigma_j)$ being real-valued functions), $P(t, X)$ is a perturbation term, a vector function of t and X, and C^* is the complex conjugate transpose of matrix C.

The development came from the consideration of the non - perturbed system

$$X' = AX - B\varphi(\sigma),\ \sigma = C^*X \tag{1.3}$$

such that

$$\underline{\varphi}_j \leq \frac{\varphi_j(\sigma_j)}{\sigma_j} \leq \bar{\varphi}_j,\ (j = 1, 2, \ldots, m) \tag{1.4}$$

that is $\varphi_j \in \aleph(\underline{\varphi}_j, \bar{\varphi}_j)$, with $\underline{\varphi}_j,\ \bar{\varphi}_j \in \mathbb{R}$. (cf [16]).

The absolute stability of (1.3), i.e. a global asymptotic stability of the equilibrium $X \equiv 0$ for all φ, with $\varphi_j \in \aleph(\underline{\varphi}_j, \bar{\varphi}_j)$ was studied using the special Lyapunov function

$$V(X) = X^*HX + \beta \int_0^{\sigma = C^*X} \varphi(\lambda) d\lambda,$$

" a quadratic form plus integral of the nonlinear function".

The KYP lemma introduced the same problem to the frequency domain inequality

$$\frac{1}{\bar{\varphi}} + Re(1 + i\omega\beta)G(i\omega) > 0 \tag{1.5}$$

ensuring absolute stability for $\varphi \in \aleph(0, \bar{\varphi})$, here $G(s) = C^*(sI - A)^{-1}B$ is the transfer function.

Our main objective in this paper is focused on the application of the new criterion for dissipative systems of the second order nonlinear differential equations. This will allow us to generalize and expand the scope for the applicability of such to equations with two nonlinear functions.

Let us first give the basic definitions and theorem that will be used in the paper.

Definition 1.1. [10, 13, 14] *A system of equations, $\dot{X} = F(t, X)$, satisfying the conditions of uniqueness and continuity with respect to initial conditions, will be said to be* dissipative *if there exist a constant $\rho > 0$ such that* $\limsup_{t \to \infty} \|X(t; t_0, X_0)\| < \rho$ *for every solution.*

Definition 1.2. *Moreover, this will be said to be* uniformly dissipative *if there exists $\rho > 0$ such that for all $\alpha > 0$ and $t_0 \geq 0$,* *there exists* $T(\alpha)$ *such that for all* X_0 *with* $\|X_0\| \leq \alpha$ *we have* $\|X(t;t_0,X_0)\| \leq \rho$ *for* $t \geq t_0 + T(\alpha)$.

The recent effort by Vl. Rasvan ([16]) has motivated our thinking back to this work.

The generalized theorem for discussing uniform dissipativity of system (1.2) is

Theorem 1.1. *Consider*

$$X' = AX - B\varphi(\sigma) + P(t,X),\ \sigma = C^*X \tag{1.6}$$

Suppose for $|\alpha| \geq \lambda_0, 0 \leq \alpha\varphi_j(\alpha) \leq \mu_o^j\alpha^2; -\alpha_1^j \leq \varphi_j'(\alpha) \leq \alpha_2^j; \mu_o^j \leq \alpha_1^j; |P(t,X)| \leq \rho$

If there exist diagonal matrices $D_1 > 0, D_2, D_3 \geq 0$ *such that*

$$\begin{aligned}\pi(\omega) \ \equiv\ & D_1 + Re[D_1(diag(\mu_o^j) + i\omega D_2)G(i\omega)] \\ & + \omega^2[D_3(I + Re(diag(\alpha_2^j - \alpha_1^j)G(i\omega)))] \\ & - G^*(-i\omega)D_3 diag(\alpha_1^j\alpha_2^j)G(i\omega) > 0 \end{aligned} \tag{1.7}$$

for all real ω. *Furthermore, assume that*

$$\liminf_{|\lambda|\to\infty} D_2|\lambda|^{-2} diag[\int_0^\lambda \varphi_j(\alpha)d\alpha - \frac{\lambda}{2}\varphi_j(\lambda)] \geq 0,$$

then system (1.6) is uniformly dissipative.

Remark 1.1. *We note that unlike the earlier results, that had only the sector conditions, this is having conditions on the derivatives of the nonlinearities (* $\varphi(\sigma)$*.)*

Remark 1.2. *We also note that, when* $D_3 \equiv 0$ *in (1.7), the inequality reduces to*

$$\pi(\omega) \equiv D_1 + Re[D_1(diag(\mu_o^j) + i\omega D_2)G(i\omega)] > 0 \tag{1.8}$$

for all $\omega \in I\!R$.

2. Application to equation with one non-linearity

Let us first consider the second order equation with one nonlinearity, just to see the simple application of theorem 1.1.

Consider

$$\ddot{x} + 2p\omega_n\dot{x} + \omega_n^2 x + \omega_n^2\gamma(x) = q(t,x,\dot{x}) \tag{2.1}$$

which can be re-written as

$$\ddot{x} + 2p\omega_n\dot{x} + \omega_n^2\varepsilon x + \omega_n^2[(1-\varepsilon)x + \gamma(x)] = q(t,x,\dot{x})$$

or

$$\ddot{x} + 2p\omega_n\dot{x} + \omega_n^2\varepsilon x + \omega_n^2\varphi(x) = q(t, x, \dot{x})$$

where $\varphi(x) = (1-\varepsilon)x + \gamma(x)$.

Let us put this in the form of (1.2) by setting

$$\begin{cases} \dot{x} = y \\ \dot{y} = -2p\omega_n y - \omega_n^2\varepsilon x - \omega_n^2\varphi(x) + q(t, x, y). \end{cases} \tag{2.2}$$

Thus by choosing

$$\begin{aligned} A &= \begin{pmatrix} 0 & 1 \\ -\omega_n^2\varepsilon & -2p\omega_n \end{pmatrix}; X = \begin{pmatrix} x \\ y \end{pmatrix}; C = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \\ B &= \begin{pmatrix} 0 \\ \omega_n^2 \end{pmatrix}; P(t, X) = \begin{pmatrix} 0 \\ q(t, x, y) \end{pmatrix}, \end{aligned} \tag{2.3}$$

we have system (1.2).

Then the transfer matrix $G(s) = C^*(sI - A)^{-1}B = \frac{1}{\Delta(s)}\omega_n^2$ with $\Delta(s) = (s^2 + 2p\omega_n s + \omega_n^2\varepsilon)$. Clearly, $G(0) = \varepsilon^{-1} > 0$.

Taking $\underline{\varphi} = 0$ and choosing in (1.7), $D_1 = \frac{1}{\bar{\varphi}},\ D_2 = \theta,\ D_3 = 0$, we have the frequency domain inequality as

$$\pi(\omega) \equiv \frac{1}{\bar{\varphi}} + \frac{(\omega_n^2\varepsilon - \omega^2 - 2p\omega_n\theta)\omega_n^2}{(\omega_n^2\varepsilon - \omega^2)^2 = 4p^2\omega_n^2\omega^2} > 0$$

for all $\omega \geq 0$.

Now, this will be true if and only if

$$(\omega_n^2\varepsilon - \omega^2)^2 + 4p^2\omega_n^2\omega^2 + \bar{\varphi}(\omega_n^2\varepsilon - \omega^2 - 2p\omega_n\theta)\omega_n^2 > 0. \tag{2.4}$$

After some simplifications, this reduces to

$$\omega^4 + \omega^2\{\omega_n^2[4p^2 - 2\varepsilon - \bar{\varphi}]\} + \omega_n^4[\varepsilon^2 + \bar{\varphi}\varepsilon - \beta\bar{\varphi}] > 0 \tag{2.5}$$

with $\beta = \frac{2p\theta}{\omega_n}$.

We note the fact that a quadratic inequality $v^2 + kv + l > 0$ for all $v \geq 0$ is true if and only if $k^2 - 4l < 0$.

Using this fact, we have that (2.5) will be true for all $\omega \geq 0$, if

$$(4p^2 - 2\varepsilon - \bar{\varphi})^2 - 4(\varepsilon^2 + \bar{\varphi}\varepsilon - \beta\bar{\varphi}) < 0, \quad for\ \bar{\varphi} > 0.$$

That is,

$$\bar{\varphi}^2 + 4(\beta - 2p^2)\bar{\varphi} + 4(\varepsilon - 2p^2)^2 - 4\varepsilon^2 < 0.$$

Again, using a quadratic property, this will hold if

$$(\beta - 2p^2)^2 - [(\varepsilon - 2p^2)^2 - \varepsilon^2] > 0, \quad for\ \bar{\varphi} > 0.$$

That is

$$\beta^2 - 4p^2\beta + 4p^2\varepsilon > 0, \quad for\ \bar{\varphi} > 0. \tag{2.6}$$

Now, choosing β between two positive roots of

$$\beta^2 - 4p^2\beta + 4p^2\varepsilon = 0$$

then, (2.6) will be true for all $\bar{\varphi} > 0$, which may be arbitrarily large.

This will then give us the sector and slope conditions.

$$\frac{\varphi(\sigma)}{\sigma} > -G(0)^{-1} = -\varepsilon,$$

for $|\sigma| \geq \lambda_o$.

This will again give us condition on our original nonlinear function $\gamma(x)$

$$\frac{\gamma(x)}{x} > -1 - \varepsilon,\ -1 < \gamma'(x) < -1 + \bar{\varphi},\ \forall\ |x| \geq \lambda_o. \tag{2.7}$$

Theorem 2.1. *Subject to conditions (2.7), equation (2.1) is uniformly dissipative.*

Remark 2.1. *This only shows the type of difficulties that occur for one nonlinear function in the considered equation.*

3. Further Applications to equations with two non-linearities

The main contribution will be to consider equations of two nonlinear functions of the second order, namely the Liénard equation

$$\ddot{y} + f(y)\dot{y} + g(y) = q(t, y, \dot{y}) \tag{3.1}$$

and the Rayleigh equation

$$\ddot{x} + F(\dot{x}) + g(x) = q(t, x, \dot{x}) \tag{3.2}$$

where $F(z) = \int_0^z f(s)ds$.

We have the following definition:

Definition 3.1. *Under the frequency-domain criteria (1.7) we shall say that the system*

$$X' = AX - B\varphi(\sigma) + P(t, X),\ \sigma = C^*X \tag{3.3}$$

has the dual system

$$X' = A^*X - C\tilde{\varphi}(\tilde{\sigma}) + \tilde{P}(t, X),\ \tilde{\sigma} = B^*X \tag{3.4}$$

where A^*, B^* are complex conjugates transpose of matrices A and B and C as given above.

Remark 3.1. *The frequency domain conditions, (1.7), for dual systems are equivalent. (cf.*[4]*)*

Lemma 3.1. *Under the frequency domain criteria the Rayleigh equation (3.2) is a dual to the Liénard equation (3.1).*

Proof: Let $F(y) = 2py + \Phi(y), g(x) = \nu^2 x + \gamma(x),\ (p > 0,\ \nu > 0)$. This is clear if we re-write (3.2) as a system form:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\nu^2 x - 2py - \gamma(x) - \Phi(y) + q(t,x,y) \end{aligned} \tag{3.5}$$

for which

$$\begin{aligned} A &= \begin{pmatrix} 0 & 1 \\ -\nu^2 & -2p \end{pmatrix}; X = \begin{pmatrix} x \\ y \end{pmatrix}; B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}; \\ C &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; P(t,X) = \begin{pmatrix} 0 \\ q(t,x,y) \end{pmatrix}; \\ \sigma &= C^*X = \begin{pmatrix} x \\ y \end{pmatrix}; \varphi(\sigma) = \begin{pmatrix} \gamma(x) \\ \Phi(y) \end{pmatrix}. \end{aligned} \tag{3.6}$$

The Liénard equation is equivalent to the system

$$\begin{aligned} \dot{x} &= -\nu^2 y - \gamma(y) + q(t,x,y) \\ \dot{y} &= x - 2py - \Phi(y) \end{aligned} \tag{3.7}$$

which indeed is

$$\begin{aligned} \ddot{y} &= \dot{x} - 2p\dot{y} - \Phi'(y)\dot{y} \\ &= -\nu^2 y - \gamma(y) - F'(y)\dot{y} + q(t,x,y) \\ &= -g(y) - f(y)\dot{y} + q(t,x,y). \end{aligned}$$

Thus

$$\ddot{y} + f(y)\dot{y} + g(y) = q(t,y,\dot{y}).$$

Hence system (3.7) is a dual for system (3.5) with

$$\begin{aligned} \tilde{A} &= \begin{pmatrix} 0 & -\nu^2 \\ 1 & -2p \end{pmatrix} = A^*; X = \begin{pmatrix} x \\ y \end{pmatrix}; \tilde{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = C; \\ \tilde{C} &= \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = B; \tilde{P}(t,X) = \begin{pmatrix} q(t,x,y) \\ 0 \end{pmatrix}; \\ \tilde{\sigma} &= \tilde{C}^*X = \begin{pmatrix} y \\ y \end{pmatrix}; \tilde{\varphi}(\tilde{\sigma}) = \begin{pmatrix} \gamma(y) \\ \Phi(y) \end{pmatrix}. \end{aligned}$$

This completes the proof of the lemma.

Now, let us consider the main results of this paper on the systems (3.5) and (3.7).

Consider the Rayleigh equation

$$\ddot{x} + F(\dot{x}) + g(x) = q(t,x,\dot{x})$$

or equivalently,

$$\ddot{x} + 2p\dot{x} + \Phi(\dot{x}) + \nu^2 x + \gamma(x) = q(t, x, \dot{x})$$

with $F(y) = 2py + \Phi(y), g(x) = \nu^2 x + \gamma(x),\ (p > 0,\ \nu > 0)$, such that the nonlinearities $\Phi(y),\ \gamma(x)$ satisfy

$$\begin{aligned} 0 \le y\Phi(y) \le \mu_2 y^2, \qquad & 0 \le x\gamma(x) \le \mu_1 x^2, \\ 0 \le \Phi'(y) \le \mu_2, \qquad & 0 \le \gamma'(x) \le \mu_1 \end{aligned} \tag{3.8}$$

for $|y| \ge \lambda_o$, $|x| \ge \lambda_o$ and $|q(t, x, \dot{x})| \le \rho_o$.

Theorem 3.1. *The Rayleigh equation (3.2) is uniformly dissipative if there exist* $\varepsilon > 0,\ \eta > 0,\ \mu_1 > 0,$ *and* $\mu_2 > 0$ *such that for* $|y| \ge \lambda_o$ *and* $|x| \ge \lambda_o$ *and* $|q(t, x, \dot{x})| \le \rho_o$

$$\varepsilon \le \frac{F(y)}{y} \le \mu_2,\ \eta \le \frac{g(x)}{x} \le \mu_1.$$

In the same vein, the Liénard equation is uniformly dissipative as a DUAL of Rayleigh equation.

Theorem 3.2. *The Liénard equation (3.1) is uniformly dissipative if there exist* $\varepsilon > 0,\ \eta > 0,\ \mu_1 > 0,$ *and* $\mu_2 > 0$ *such that for* $|z| \ge \lambda_o$ *and* $|y| \ge \lambda_o$ *and* $|q(t, y, \dot{y})| \le \rho_o$

$$\varepsilon \le \frac{\int_0^y f(s)ds}{y} \le \mu_2,\ \eta \le \frac{g(y)}{y} \le \mu_1.$$

4. Sketch of proof of Theorem 3.1

First, we note that, with the choice of

$$\begin{aligned} A &= \begin{pmatrix} 0 & 1 \\ -\nu^2 & -2p \end{pmatrix}; X = \begin{pmatrix} x \\ y \end{pmatrix}; B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}; \\ C &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; P(t, X) = \begin{pmatrix} 0 \\ q(t, x, y) \end{pmatrix}; \\ \sigma &= C^* X = \begin{pmatrix} x \\ y \end{pmatrix}; \varphi(\sigma) = \begin{pmatrix} \gamma(x) \\ \Phi(y) \end{pmatrix}; \end{aligned}$$

we have

$$(sI - A) = \begin{pmatrix} s & -1 \\ \nu^2 & s + 2p \end{pmatrix}; det(sI - A) = s^2 + 2ps + \nu^2.$$

Hence A is stable.

If we denote $\Delta(i\omega) = det(i\omega - A) = \nu^2 - \omega^2 + 2pi\omega$, then,

$$(i\omega I - A)^{-1} = \frac{1}{\Delta(i\omega)} \begin{pmatrix} 2p + i\omega & 1 \\ -\nu^2 & i\omega \end{pmatrix}$$

and the transfer matrix is

$$G(i\omega) = C^*(i\omega I - A)^{-1}B = \frac{1}{\Delta(i\omega)}\begin{pmatrix} 1 & 1 \\ i\omega & i\omega \end{pmatrix}.$$

If we choose $\tau_1 > 0, \tau_2 > 0, D_1 = diag(\tau_1, \tau_2), D_2 = diag(\lambda_1, \lambda_2)$ and $D_3 = 0$, the frequency domain inequality (1.7) becomes

$$\pi(\omega) \equiv \begin{pmatrix} 2\tau_1 + 2Re[\frac{\tau_1\mu_1 + i\omega\lambda_1}{\Delta(i\omega)}] & [\frac{\tau_1\mu_1 + i\omega\lambda_1}{\Delta(i\omega)} - \frac{\lambda_2\omega^2 + i\omega\tau_2\mu_2}{\Delta(-i\omega)}] \\ \frac{\tau_1\mu_1 - i\omega\lambda_1}{\Delta(i\omega)} + \frac{-\lambda_2\omega^2 + i\omega\tau_2\mu_2}{\Delta(-i\omega)} & 2\tau_2 + 2Re[\frac{-\lambda_2\omega^2 + i\omega\tau_2\mu_2}{\Delta(i\omega)}] \end{pmatrix} > 0. \tag{4.1}$$

After some calculations, this becomes

$$\pi(\omega) \equiv \begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix} > 0 \tag{4.2}$$

where

$$\begin{aligned} \pi_{11} &= 2\tau_1 + \frac{2}{|\Delta|^2}[\mu_1\tau_1(\nu^2 - \omega^2) + 2p\omega^2\lambda_1], \\ \pi_{12} &= \frac{2}{|\Delta|^2}\{[(\nu^2 - \omega^2)(\mu_1\tau_1 - \omega^2\lambda_1) + 2p\omega^2(\lambda_1 + \mu_2\tau_2)] \\ &\quad -2pi\omega(\mu_1\tau_1 + \omega^2\lambda_2)\} \\ &= \bar{\tau}_{21}, \\ \pi_{22} &= 2\tau_2 + \frac{2\omega^2}{|\Delta|^2}[2p\mu_2\tau_2 - \lambda_2(\nu^2 - \omega^2)], \end{aligned}$$

with $|\Delta|^2 = \Delta(i\omega)\Delta(-\omega) = (\nu^2 - \omega^2)^2 + 4p^2\omega^2$.

On setting $\lambda_2 = 0, \gamma_1 = \frac{\tau_1\mu_1}{\tau_2\mu_2}, \gamma_2 = \frac{\lambda_1}{\tau_1}$, the frequency domain condition can be verified using Sylvester's criterion on the positivity of the determinant of the principal minors of $\pi(\omega)$.

Hence, for this to be true, it is sufficient to have the determinant of $\pi(\omega) > 0$.

This simplifies to

$$\begin{aligned} \omega^2 &+ \frac{1}{\omega^2}(\nu^4 + \mu_1\nu^2 - \frac{\gamma_1\mu_1\mu_2}{4}) \\ &> 2\nu^2 - 2p(2p + \mu_2) + \mu_1 - (2p + \frac{\mu_2}{2})\gamma_2 + \frac{\mu_1\mu_2}{4\gamma_1} + \frac{\gamma_2^2}{4}.\frac{\gamma_1\mu_2}{\mu_1} \end{aligned} \tag{4.3}$$

We note that a function $H(v) = v + \frac{k}{v}$ for positive values of v attains it minimum at $v_1 = \sqrt{k}$. Moreover, it has its minimum value as $H_{min}(v_1) = 2\sqrt{k}$.

Using this fact, we have that the minimum of the left hand side of the inequality (4.3) for $\omega \in \mathbb{R}$ is

$$\sqrt{4\nu^2(\nu^2+\mu_1)-\mu_1\mu_2\gamma_1},$$

whenever $0<\gamma_1<4\nu^2(\nu^2+\mu_1)(\mu_1\mu_2)^{-1}$.

Hence the required conditions can be obtained.

Let,

$$C(\mu_1,\mu_2,\gamma_1)=\sqrt{4\nu^2(\nu^2+\mu_1)-\mu_1\mu_2\gamma_1}+2p(2p+\mu_2)-2\nu^2-\frac{\mu_1\mu_2}{4\gamma_1}-\mu_1$$

the frequency domain condition will be true if we have γ_1 and γ_2 such that

$$(\frac{\gamma_1\mu_2}{4\mu_1})\gamma_2^2-(2p+\frac{\mu_2}{2})\gamma_2-C(\mu_1,\mu_2,\gamma_1)<0.$$

This is possible if

$$(2p+\frac{\mu_2}{2})^2+(\frac{\gamma_1\mu_2}{\mu_1})C(\mu_1,\mu_2,\gamma_1)<0.$$

Using the expression for $C(\mu_1,\mu_2,\gamma_1)$, as

$$\frac{\gamma_1\mu_2}{\mu_1}C(\mu_1,\mu_2,\gamma_1)=-\frac{\mu_2^2}{4}+O(\gamma_1)$$

then we have (4.3) will be true for small values if γ_1.

This concludes the proof of Theorem 3.1, using theorem 1.1.

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