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BOUNDEDNESS OF SOLUTIONS OF SOME THIRD ORDER NON-AUTONOMOUS ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. We study the behavior of the solutions of the differential equation

$$\dddot{x}+(a(t,x,\dot{x},\ddot{x})+m(t,x,\dot{x},\ddot{x}))\ddot{x}+b(t)g(x,\dot{x})+c(t,x,\dot{x},\ddot{x})h(x)=p(t,x,\dot{x},\ddot{x})$$

and present sufficient conditions on the functions involved under which the solutions of the differential equation are bounded. Some results on the regularity and asymptotic behavior of the solutions are also obtained.

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1. INTRODUCTION

Here the differential equation

$$\dddot{x}+(a+m)\ddot{x}+b(t)g(x,\dot{x})+ch(x)=p(t,x,\dot{x},\ddot{x}) \qquad (1)$$

is considered, where $a,\ m,\ c$ are continuous functions of $t,x,\dot{x},\ddot{x}$, the functions $b,\ g,\ h,\ and\ p$ depend only on the argument displayed explicitly and the dots denote differentiation with respect to t. It is mainly assumed that the function b is continuous on $\mathbb{R}^+$ ($\mathbb{R}^+ = (0,\infty)$), and the functions $a,\ m,\ c,\ g,\ h\ and\ p$ are continuous for all values of their respective arguments.

Equation (1) for which $(a+m)=f(x,\dot{x},\ddot{x})$, $b(t)=1$ and $c=1$ have been studied extensively by several authors. For example, we refer to the book of Ressig et al [10] as a survey and the papers [2, 3, 7, 9, 11]. The special case for which $(a+m)=f(x,\dot{x},\ddot{x})$, $b(t)\neq 1$ and $c(t)\neq 1$ have received little attention due to the difficulty in constructing suitable scalar function. For example see [4],[5] and [6], and the references cited therein. However, Zarghamee and Mehri [14], Mehri and Shadman [13] and Tunç [12] have studied

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particular cases of equation (1) for the boundedness, regularity and asymptotic behavior of solutions using energy functions.

The motivation for the present work is derived from the papers of the authors mentioned above. Our aim is to further extend their results to the more general equation (1) for the boundedness, regularity and asymptotic behavior of solutions.

2. STATEMENT OF RESULTS

Theorem 1. In our main results we assume the following assumptions:
(i) $b(t) > 0$ and $b'(t) > 0$ for all $t \in \mathbb{R}^+$;
(ii) $|a(t,x,y,z)| \leq \alpha_2(t)$ where $\alpha_2 \in L^1(0,\infty)$ and $t \in \mathbb{R}^+$;
(iii) $m(t,x,y,z) \geq 0$ for all $t \in \mathbb{R}^+$ and $x,y,z \in \mathbb{R}$;
(iv) $yg(x,y) \geq 0$ and $yg_x(x,y) \leq 0$ for all $x,y \in \mathbb{R}$ and $lim_{s\longrightarrow\pm\infty} G(x,s) = +\infty$ where $G(x,s) = \int_0^s g(x,\tau)d\tau$;
(v) $|c(t,x,y,z)| \leq \gamma(t)$ where $\gamma \in L^1(0,\infty)$ and $t \in \mathbb{R}^+$;
(vi) $|h(x)| \leq k|x|$ for all $x \in \mathbb{R}^+$, k a positive constant;
(vii) $|p(t,x,y,z)| \leq e(t)$ for all $t \in \mathbb{R}^+$ and $x,y,z \in \mathbb{R}$;
(viii) There are arbitrary continuous functions α_o, α_1, β and γ on $\mathbb{R}^+$ such that α_0, α_1 and γ are positive and decreasing and β is positive and increasing on $(0,\infty)$;
(ix) $\alpha_2(t)$, $\frac{e(t)}{\sqrt{b(t)}}$, $\left(\frac{\alpha_0(t)}{\alpha_1(t)}\right)^{\frac{1}{2}}$, $\left(\frac{\alpha_1(t)b(t)}{\beta(t)}\right)^{\frac{1}{2}}$, $\gamma(t)\left(\frac{\beta(t)}{\alpha_0(t)b(t)}\right)^{\frac{1}{2}} \in L^1(0,\infty)$, where $L^1(0,\infty)$ is a space of Lebesque integrable functions.
Then, for every solution $x(t)$ of the equation (1),

$$x\sqrt{\frac{\alpha_0(t)}{\beta(t)}},\ \dot{x}\ \sqrt{\frac{\alpha_1(t)}{\beta(t)}}\ \textit{and}\ \ddot{x}\ \frac{1}{\sqrt{b(t)}}\ \textit{are bounded for all}\ t \in \mathbb{R}.$$

Remark 1. It should be noted that for the special case $p(t,x,y,z) \equiv 0$ in equation (1) the conclusion of Theorem 1 also remains valid.
Remark 2. Consequently, if $a(t,x,y,z) = 0$ and $c(t,x,y,z) \equiv c(t)$ then the equation (1) reduces to the non-linear equation of Cemil Tunç [12] in which the assumptions (i), (iii)-(ix) remain valid.
Proof. Let $\dot{x}= y$, $\dot{y}= z$, and transform Eq. (1) into the system

$$\begin{aligned}
\dot{x} &= y,\\
\dot{y} &= z, \qquad\qquad (2)\\
\dot{z} &= -(a(t,x,y,z)+m(t,x,y,z))z - b(t)g(x,y)\\
&\quad -c(t,x,y,z)h(x) + p(t,x,y,z).
\end{aligned}$$

Now, throughout all the main results established here, our main tool is the continuous differentiable energy function $E = E(t, x, y, z)$ defined by:

$$E := \frac{\alpha_0(t)}{\beta(t)}x^2 + \frac{\alpha_1(t)}{\beta(t)}y^2 + \frac{1}{b(t)}z^2 + 2G(x, y) \qquad (3)$$

where α_0, α_1, β and b are positive continuous functions on $(0, \infty)$ for all $t \in \mathbb{R}^+$. Since the coefficients α_0, α_1, β and b in Eq. (3) are positive and $G(x, y) > 0$, then it is clear that the function E defined by Eq. (3) is positive definite.

Let $(x, y, z) = (x(t), y(t), z(t))$ be an arbitrary solution of the system Eq. (2). Differentiating the function $E = E(t, x, y, z)$ given by Eq. (3) along the solution (x, y, z) of the system of the Eq. (2), it can easily be followed that

$$\dot{E} \equiv \frac{dE}{dt} = \frac{\partial E}{\partial x}y + \frac{\partial E}{\partial y}z + \frac{\partial E}{\partial z}\dot{z} + \frac{\partial E}{\partial t} \qquad (4)$$

$$= \Big(\frac{\alpha_0'(t)}{\beta(t)} - \frac{\alpha_0(t)\beta'(t)}{\beta^2(t)}\Big)x^2 + \Big(\frac{\alpha_1'(t)}{\beta(t)} - \frac{\alpha_1(t)\beta'(t)}{\beta^2(t)}\Big)y^2$$

$$-\frac{b'(t)}{b^2(t)}z^2 - \frac{2a(t, x, y, z)}{b(t)}z^2 - \frac{2m(t, x, y, z)}{b(t)}z^2 + 2y\int_0^y g_x(x, \tau)d\tau$$

$$+\frac{2\alpha_0(t)}{\beta(t)}xy + \frac{2\alpha_1(t)}{\beta(t)}yz - \frac{2c(t, x, y, z)}{b(t)}h(x)z + \frac{2p(t, x, y, z)}{b(t)}z. \qquad (5)$$

Clearly, using the assumptions imposed on the functions α_0, α_1, β, b and the assumptions (iii)-(iv), show that

$$\Big(\frac{\alpha_0'(t)}{\beta(t)} - \frac{\alpha_0(t)\beta'(t)}{\beta^2(t)}\Big) < 0, \ \Big(\frac{\alpha_1'(t)}{\beta(t)} - \frac{\alpha_1(t)\beta'(t)}{\beta^2(t)}\Big) < 0, -\frac{b'(t)}{b^2(t)} < 0,$$

$$-\frac{2m(t, x, y, z)}{b(t)} \leq 0 \textit{ and } 2y\int_0^y g_x(x, \tau)d\tau \leq 0.$$

Hence, we now obtain from Eq. (5) that

$$\dot{E} \leq -\frac{2a(t, x, y, z)}{b(t)}z^2 + \frac{2\alpha_0(t)}{\beta(t)}xy + \frac{2\alpha_1(t)}{\beta(t)}yz - \frac{2c(t, x, y, z)}{b(t)}h(x)z$$

$$+\frac{2p(t, x, y, z)}{b(t)}z.$$

Using the assumptions (ii), (v) - (vii) and in view of the inequality achieved above, we obtain

$$\dot{E} \leq \frac{2\alpha_2(t)}{b(t)}z^2+\frac{2\alpha_0(t)}{\beta(t)}|x||y|+\frac{2\alpha_1(t)}{\beta(t)}|y||z|+\frac{2k\gamma(t)}{b(t)}|x||z|+\frac{2|e(t)|}{b(t)}|z|. \tag{6}$$

Considering $yg(x,y) \geq 0$ and with the properties of the functions α_0, α_1, β, b of the theorem, it is clear from Eq. (3) that

$$|x| \leq \Big(\frac{\beta(t)}{\alpha_0(t)}\Big)^{\frac{1}{2}}E^{\frac{1}{2}},$$

$$|y| \leq \Big(\frac{\beta(t)}{\alpha_1(t)}\Big)^{\frac{1}{2}}E^{\frac{1}{2}}$$

and

$$|z| \leq b^{\frac{1}{2}}(t)E^{\frac{1}{2}} \leq b^{\frac{1}{2}}(t)\Big(\frac{1}{2}+\frac{E}{2}\Big).$$

Therefore, each term of inequality (6) becomes

$$\begin{aligned} \frac{2\alpha_2(t)}{b(t)}z^2 &\leq 2\alpha_2(t)E, \\ \frac{2\alpha_0(t)}{\beta(t)}|x||y| &\leq 2\Big(\frac{\alpha_0(t)}{\alpha_1(t)}\Big)^{\frac{1}{2}}E, \\ \frac{2\alpha_1(t)}{\beta(t)}|y||z| &\leq 2\Big(\frac{\alpha_1(t)b(t)}{\beta(t)}\Big)^{\frac{1}{2}}E, \\ \frac{2|e(t)|}{b(t)}|z| &\leq \frac{|e(t)|}{b^{\frac{1}{2}}(t)}+\frac{|e(t)|}{b^{\frac{1}{2}}(t)}E, \\ \frac{2k\gamma(t)}{b(t)}|z||x| &\leq 2k\gamma(t)\Big(\frac{\beta(t)}{\alpha_0(t)b(t)}\Big)^{\frac{1}{2}}E. \end{aligned} \tag{7}$$

Hence, as a result of what is obtained in inequalities (7), inequality (6) now implies

$$\begin{aligned} \dot{E} \leq\ & 2\alpha_2(t)E+2\Big(\frac{\alpha_0(t)}{\alpha_1(t)}\Big)^{\frac{1}{2}}E+2\Big(\frac{\alpha_1(t)b(t)}{\beta(t)}\Big)^{\frac{1}{2}}E+\frac{|e(t)|}{b^{\frac{1}{2}}(t)}+\frac{|e(t)|}{b^{\frac{1}{2}}(t)}E \\ & +2k\gamma(t)\Big(\frac{\beta(t)}{\alpha_0(t)b(t)}\Big)^{\frac{1}{2}}E. \end{aligned} \tag{8}$$

We re-write inequality (8) as

$$\dot{E} \leq \frac{|e(t)|}{b^{\frac{1}{2}}(t)}+\Phi(t)E(t), \tag{9}$$

where

$$\Phi(t) = 2\Big[\alpha_2(t)+\Big(\frac{\alpha_0(t)}{\alpha_1(t)}\Big)^{\frac{1}{2}}+\Big(\frac{\alpha_1(t)b(t)}{\beta(t)}\Big)^{\frac{1}{2}}+\frac{|e(t)|}{2b^{\frac{1}{2}}(t)}+k\gamma(t)\Big(\frac{\beta(t)}{\alpha_0(t)b(t)}\Big)^{\frac{1}{2}}\Big]. \tag{10}$$

Integrating inequality (9) from 0 to t, we have

$$E(t)-E(0) \le \int_0^t \frac{|e(\tau)|}{b^{\frac{1}{2}}(\tau)}d\tau + \int_0^t \Phi(\tau)E(\tau)d\tau.$$

Having assumption (viii) in mind, and using Gronwall's inequality, we finally obtain

$$E(t) \le A\exp\Big(\int_0^t \Phi(\tau)d\tau\Big)$$

for some constant A, where $A = E(0)+\int_0^t \frac{|e(\tau)|}{b^{\frac{1}{2}}(\tau)}d\tau$.

Thus, by assumption (ix) of Theorem 1, we have $\Phi \in L^1(0,\infty)$, which implies the boundedness of the function E. Hence, we can conclude that $\frac{\alpha_0(t)}{\beta(t)}x^2$, $\frac{\alpha_1(t)}{\beta(t)}y^2$ and $\frac{1}{b(t)}z^2$ are bounded, so this result guarantees the boundedness of

$x\sqrt{\frac{\alpha_0(t)}{\beta(t)}}$, $\dot{x}\ \sqrt{\frac{\alpha_1(t)}{\beta(t)}}$, $\ddot{x}\ \frac{1}{\sqrt{b(t)}}$. This completes our proof.

Theorem 2. The assumptions of Theorem 1 remains valid. Thus:
(x) there exist a positive constant M such that $m(t,x,y,z) \ge M$ for all $t \in \mathbb{R}^+$ and $x, y\ and\ z \ne 0\ \in \mathbb{R}$ and $b'(t)+2Mb(t) > 0$ for all $t \in \mathbb{R}$;
(xi) there are arbitrary continuous functions α_o, α_1, β, and γ on $\mathbb{R}^+ = (0,\infty)$ such that α_0, α_1 and γ are positive and decreasing, β is positive and increasing for all $t \in \mathbb{R}^+$, and

$$\frac{e^2(t)}{b'(t)+2Mb(t)},\ \frac{e(t)}{\sqrt{b(t)}},\ \alpha_2(t),\ \Big(\frac{\alpha_0(t)}{\alpha_1(t)}\Big)^{\frac{1}{2}},\ \Big(\frac{\alpha_1(t)b(t)}{\beta(t)}\Big)^{\frac{1}{2}},$$

$$\gamma(t)\Big(\frac{\beta(t)}{\alpha_1(t)b(t)}\Big)^{\frac{1}{2}} \in L^1(0,\infty).$$

Then, the conclusion of Theorem 1 holds.

Proof. As known, the function E defined in Eq. (3) is positive definite. Now, subject to the assumptions of Theorem 2, an easy calculation from Eq. (3) and Eq. (2) show that

$$\dot{E} \equiv \frac{dE}{dt} = \Big(\frac{\alpha_0'(t)}{\beta(t)} - \frac{\alpha_0(t)\beta'(t)}{\beta^2(t)}\Big)x^2 + \Big(\frac{\alpha_1'(t)}{\beta(t)} - \frac{\alpha_1(t)\beta'(t)}{\beta^2(t)}\Big)y^2$$
$$-\frac{b'(t)}{b^2(t)}z^2 - \frac{2}{b(t)}(a(t,x,y,z)+m(t,x,y,z))z^2 + 2y\int_0^y g_x(x,\tau)d\tau$$

$$
\begin{aligned}
&+\frac{2\alpha_0(t)}{\beta(t)}xy+\frac{2\alpha_1(t)}{\beta(t)}yz-\frac{2c(t,x,y,z)}{b(t)}h(x)z+\frac{2p(t,x,y,z)}{b(t)}z\\
&\le -\frac{2}{b(t)}m(t,x,y,z)z^2-\frac{b'(t)}{b^2(t)}z^2+\frac{2p(t,x,y,z)}{b(t)}z-\frac{2}{b(t)}a(t,x,y,z)z^2\\
&\quad+\frac{2\alpha_0(t)}{\beta(t)}xy+\frac{2\alpha_1(t)}{\beta(t)}yz-\frac{2c(t,x,y,z)}{b(t)}h(x)z\\
&\le -2M\frac{1}{b(t)}z^2-\frac{b'(t)}{b^2(t)}z^2+2\frac{|e(t)|}{b(t)}|z|+\frac{2\alpha_2(t)}{b(t)}z^2+2\frac{\alpha_0(t)}{\beta(t)}|x||y|\\
&\quad+2\frac{\alpha_1(t)}{\beta(t)}|y||z|+2k\frac{\gamma(t)}{b(t)}|z||x|\\
&= -(b'(t)+2Mb(t))\frac{z^2}{b^2(t)}+2\frac{|e(t)|}{b(t)}|z|+\frac{2\alpha_2(t)}{b(t)}z^2+2\frac{\alpha_0(t)}{\beta(t)}|x||y|\\
&\quad+2\frac{\alpha_1(t)}{\beta(t)}|y||z|+2k\frac{\gamma(t)}{b(t)}|z||x|\\
&\le -(b'(t)+2Mb(t))\Big(\frac{z^2}{b^2(t)}-2\frac{|e(t)||z|}{b(t)(b'(t)+2Mb(t))}+\frac{e^2(t)}{(b'(t)+2Mb(t))^2}\\
&\quad-\frac{e^2(t)}{(b'(t)+2Mb(t))^2}\Big)+\frac{2\alpha_2(t)}{b(t)}z^2+2\frac{\alpha_0(t)}{\beta(t)}|x||y|+2\frac{\alpha_1(t)}{\beta(t)}|y||z|\\
&\quad+2k\frac{\gamma(t)}{b(t)}|z||x|\\
&= -(b'(t)+2Mb(t))\Big(\frac{|z|}{b(t)}-\frac{|e(t)|}{(b'(t)+2Mb(t))}\Big)^2+\frac{e^2(t)}{b'(t)+2Mb(t)}\\
&\quad+2\Big[\alpha_2(t)+\Big(\frac{\alpha_0(t)}{\alpha_1(t)}\Big)^{\frac{1}{2}}+\Big(\frac{\alpha_1(t)b(t)}{\beta(t)}\Big)^{\frac{1}{2}}+k\gamma(t)\Big(\frac{\beta(t)}{\alpha_0(t)b(t)}\Big)^{\frac{1}{2}}\Big]E.
\end{aligned}
$$

This implies that

$$
\dot{E}\le \frac{e^2(t)}{b'(t)+2Mb(t)}+\Big[\Phi(t)-\frac{|e(t)|}{2b^{\frac{1}{2}}(t)}\Big]E, \tag{11}
$$

where $\Phi(t)$ is the same as in Eq. (10). Similarly, as in the proof of Theorem 1, integrating inequality (11) from 0 to t, and considering assumption (xi) of Theorem 2 and the application of Gronwall's inequality, one can easily deduce the boundedness of the function E. The proof is now complete.

Remark 3. Theorem 2 improves the second theorem of Mehri and Shadman [13]. The assumptions

$$
b'(t)+2Mb(t)>0 \quad \textit{and} \quad \frac{e^2(t)}{b'(t)+2Mb(t)}\in L^1(0,\infty)
$$

of Theorem 2 are less restrictive than that

$$b'(t)+2Ma(t)b(t)>0 \quad and \quad \frac{e^2(t)}{b'(t)+2Ma(t)b(t)}\in L^1(0,\infty),$$

where $a(t)\geq 0$ established in Mehri and Shadman (Theorem 2, [13]), and the equation (1) includes the equation considered therein.

The next theorem is concerned with the regularity of solutions of differential equation (1).

Theorem 3. Let all the conditions of Theorem 2 hold. Then, every solution of the equation (1) satisfies

$$\Big(\frac{|\alpha_0'(t)|}{\beta(t)}\Big)^{\frac{1}{2}}x\in L^2(0,\infty) \ and \ \Big(\frac{|\alpha_1'(t)|}{\beta(t)}\Big)^{\frac{1}{2}}\dot{x}\in L^2(0,\infty).$$

If in addition, we assume

$$l.u.b\ \frac{b^2(t)}{b'(t)+2Mb(t)}=\mu<\infty,\ t\geq 0,$$

then

$$\ddot{x}\in L^2(0,\infty).$$

Proof. However, following the procedure as indicated in Theorem 1 and Theorem 2 above, except for some minor modification, we obtain as follows:

$$\begin{aligned}\dot{E}\leq &\Big(\frac{\alpha_0'(t)}{\beta(t)}-\frac{\alpha_0(t)\beta'(t)}{\beta^2(t)}\Big)x^2+\Big(\frac{\alpha_1'(t)}{\beta(t)}-\frac{\alpha_1(t)\beta'(t)}{\beta^2(t)}\Big)y^2-\frac{b'(t)}{b^2(t)}z^2-\frac{2M}{b(t)}z^2\\ &+\frac{|e(t)|}{\sqrt{b(t)}}+2\Big[\alpha_2(t)+\Big(\frac{\alpha_0(t)}{\alpha_1(t)}\Big)^{\frac{1}{2}}+\Big(\frac{\alpha_1(t)b(t)}{\beta(t)}\Big)^{\frac{1}{2}}+\frac{|e(t)|}{2b^{\frac{1}{2}}(t)}\\ &+k\gamma(t)\Big(\frac{\beta(t)}{\alpha_0(t)b(t)}\Big)^{\frac{1}{2}}\Big]E\\ =&\Big(\frac{\alpha_0'(t)}{\beta(t)}-\frac{\alpha_0(t)\beta'(t)}{\beta^2(t)}\Big)x^2+\Big(\frac{\alpha_1'(t)}{\beta(t)}-\frac{\alpha_1(t)\beta'(t)}{\beta^2(t)}\Big)y^2-\frac{b'(t)}{b^2(t)}z^2\\ &-\frac{2M}{b(t)}z^2+\frac{|e(t)|}{\sqrt{b(t)}}+\Phi(t)E(t),\end{aligned}$$

where $\Phi(t)$ is given by the Eq. (10).

Hence, it follows that

$$\Big(\frac{\alpha_0(t)\beta'(t)}{\beta^2(t)}-\frac{\alpha_0'(t)}{\beta(t)}\Big)x^2+\Big(\frac{\alpha_1(t)\beta'(t)}{\beta^2(t)}-\frac{\alpha_1'(t)}{\beta(t)}\Big)y^2+\frac{b'(t)+2Mb(t)}{b^2(t)}z^2$$

$$\leq -\dot{E} + \frac{|e(t)|}{\sqrt{b(t)}} + \Phi(t)E(t). \tag{12}$$

Integrating both sides of inequality (12) from 0 to t, we obtain

$$\int_0^t \Big[\Big(\frac{\alpha_0(\tau)\beta'(\tau)}{\beta^2(\tau)} - \frac{\alpha_0'(\tau)}{\beta(\tau)}\Big)x^2 + \Big(\frac{\alpha_1(\tau)\beta'(\tau)}{\beta^2(\tau)} - \frac{\alpha_1'(\tau)}{\beta(\tau)}\Big)y^2$$
$$+\frac{b'(\tau)+2Mb(\tau)}{b^2(\tau)}z^2\Big]d\tau$$
$$\leq E(0) - E(t) + \int_0^t \frac{|e(\tau)|}{\sqrt{b(\tau)}}d\tau + k_1\int_0^t \Phi(\tau)d\tau,$$

where it is assumed that $E(t) \leq k_1$, $t > 0$. In view of the assumptions of Theorem 3, and the boundedness of the function $E(t)$, we can easily conclude that

$$\int_0^t \frac{|\alpha_0'(\tau)|}{\beta(\tau)}x^2 d\tau < \infty,\ \int_0^t \frac{|\alpha_1'(\tau)|}{\beta(\tau)}y^2 d\tau < \infty\ and\ \int_0^t z^2 d\tau < \infty,\ t \geq 0.$$

The proof of Theorem 3 is now complete.

Theorem 4. Let all the hypothesis of Theorem 3 be satisfied except
$m(t,x,y,z) \geq M$, and besides, we assume that $b(t)$, $a(t,x,y,z)$, $c(t,x,y,z)$, $p(t,x,y,z)$ are bounded for all $t \geq 0$ and $m(t,x,y,z) \leq M_2$ for all $t \in \mathbb{R}^+$ and x, y *and* $z \neq 0 \in \mathbb{R}$, where M_2 is a positive constant. Then the solutions of the equation (1) for which $\ddot{x}$ is bounded, satisfy

$$\lim_{t\longrightarrow\infty} \ddot{x}(t) = 0.$$

Proof. It can easily be written from Eq. (2) that

$$|z\,\dot{z}\,| \leq z^2|a(t,x,y,z) + m(t,x,y,z)| + |z||b(t)||g(x,y)|$$
$$+|z||c(t,x,y,z)||h(x)| + |z||p(t,x,y,z)|.$$

Since the function g is continuous and x and y are bounded on $\mathbb{R}$, the function g is bounded on $\mathbb{R}$, and by using the assumptions (vii), $m(t,x,y,z) \leq M_2$ of Theorem 4 and the boundedness of the function g, say $|g(x,y)| \leq k_2$ where k_2 is a positive constant which we now assume, we have

$$|z\,\dot{z}\,| \leq (\alpha_2(t) + M_2)b(t)\frac{z^2}{b(t)} + k_2 b^{\frac{3}{2}}(t)\frac{|z|}{\sqrt{b(t)}}$$
$$+k\gamma(t)\sqrt{b(t)}\sqrt{\frac{\beta(t)}{\alpha_0(t)}}\frac{|z|}{\sqrt{b(t)}}\frac{|x|}{\sqrt{\beta(t)/\alpha_0(t)}} + |e(t)|\sqrt{b(t)}\frac{|z|}{\sqrt{b(t)}}.$$

Hence, the expression on the right is bounded. Thus

$$\lim_{t \longrightarrow \infty} z(t) = 0,$$

(see Bihari [1]).

Similar results could be obtained for x and y by imposing more restrictive conditions on the differential equation (1). For example, if we take *l.u.b.* $\frac{\beta(t)}{\sqrt{\alpha_1'(t)}} \leq c_1$, then, by Theorem 3, we have $y \in L^2(0, \infty)$.

Since $\dot{y} = z$, then $\dot{y}$ is also bounded. Hence

$$\lim_{t \longrightarrow \infty} y(t) = 0,$$

(see argument in Lefschetz [8], p.289 - 291, Mehri and Shadman [13]).

Example: Here we consider suitable application for the first theorem in the form

$$\dddot{x} + \Big(-(1 + t^2 + x^2 + (\dot{x})^2 + (\ddot{x})^2)^{-1} + (2 + t^2 + x^2 + (\dot{x})^2 + (\ddot{x})^2) \Big) \ddot{x}$$

$$+ (t^2+1)^3 \dot{x} + \frac{1}{(t^2+1)^7} x = e(t).$$

The equivalent system form become

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= \Big((1 + t^2 + x^2 + y^2 + z^2)^{-1} - (2 + t^2 + x^2 + y^2 + z^2) \Big) z \\ &\quad -(t^2+1)^3 y - \frac{1}{(t^2+1)^7} x + e(t). \end{aligned}$$

Now, let

$$\alpha_2(t) = \frac{1}{t^2+1}, \ \alpha_0(t) = \frac{1}{(t^2+1)^5}, \ \alpha_1(t) = \frac{1}{(t^2+1)^3},$$

$$\beta(t) = (t^2+1)^6, \ \gamma(t) = \frac{1}{(t^2+1)^7}.$$

Clearly, $\alpha_0(t)$, $\alpha_1(t)$ and γ are positive and decreasing functions, $\beta(t)$ is positive and increasing functions on $(0, \infty)$, and

$$\Big(\frac{\alpha_0(t)}{\alpha_1(t)} \Big)^{\frac{1}{2}} = \frac{1}{t^2+1} \in L^1(0, \infty),$$

$$\Big(\frac{\alpha_1(t)b(t)}{\beta(t)}\Big)^{\frac{1}{2}} = \frac{1}{(t^2+1)^3} \in L^1(0,\infty),$$
$$\gamma(t)\Big(\frac{\beta(t)}{\alpha_0(t)b(t)}\Big)^{\frac{1}{2}} = \frac{1}{(t^2+1)^3} \in L^1(0,\infty).$$

Now, if we choose $e(t)$ such that $\frac{e(t)}{\sqrt{b(t)}} = \frac{e(t)}{(t^2+1)^{\frac{3}{2}}} \in L^1(0,\infty)$, then for every solution $x(t)$ of equation (1), one can reach the following conclusion:

$$\frac{x(t)}{(t^2+1)^{\frac{11}{2}}},\ \frac{\dot{x}\ (t)}{(t^2+1)^{\frac{9}{2}}}\ and\ \frac{\ddot{x}\ (t)}{(t^2+1)^{\frac{3}{2}}}\ are\ bounded\ for\ all\ t \geq 0.$$

In view of the above choice, we have that

$$\begin{aligned} E(t,x,y,z) &= \frac{\alpha_0(t)}{\beta(t)}x^2 + \frac{\alpha_1(t)}{\beta(t)}y^2 + \frac{1}{b(t)}z^2 + 2\int_0^y \phi(\tau)d\tau \\ &= \frac{1}{(t^2+1)^{11}}x^2 + \frac{1}{(t^2+1)^9}y^2 + \frac{1}{(t^2+1)^3}z^2 + 2\int_0^y \tau d\tau \\ &= \frac{1}{(t^2+1)^{11}}x^2 + \frac{1}{(t^2+1)^9}y^2 + \frac{1}{(t^2+1)^3}z^2 + y^2. \end{aligned}$$

It is clear that the function $E = E(t,x,y,z)$ is a positive definite function.

Now, differentiating the function E along the above system, we obtain (following Eq. (4)),

$$\begin{aligned} \dot{E} &= -\frac{22t}{(t^2+1)^{12}}x^2 + \frac{2}{(t^2+1)^{11}}x\ \dot{x} - \frac{18t}{(t^2+1)^{10}}y^2 + \frac{2}{(t^2+1)^9}y\ \dot{y} \\ &\quad -\frac{6t}{(t^2+1)^4}z^2 + \frac{2}{(t^2+1)^3}z\ \dot{z} + 2y\ \dot{y} \\ &= -\frac{22t}{(t^2+1)^{12}}x^2 + \frac{2}{(t^2+1)^{11}}xy - \frac{18t}{(t^2+1)^{10}}y^2 + \frac{2}{(t^2+1)^9}yz - \frac{6t}{(t^2+1)^4}z^2 \\ &\quad + \frac{2(1+t^2+x^2+y^2+z^2)^{-1}}{(t^2+1)^3}z^2 - \frac{2(2+t^2+x^2+y^2+z^2)}{(t^2+1)^3}z^2 \\ &\quad - \frac{2}{(t^2+1)^{10}}xz + \frac{2e(t)}{(t^2+1)^3}z. \end{aligned}$$

Based on assumptions imposed on $\alpha_2(t)$, $\alpha_0(t)$, $\alpha_1(t)$, $\beta(t)$, in Theorem 1 and the assumptions (i) and (iii), we now obtain

$$\begin{aligned} \dot{E} \leq\ & \frac{2(1+t^2+x^2+y^2+z^2)^{-1}}{(t^2+1)^3}z^2 + \frac{2}{(t^2+1)^{11}}xy + \frac{2}{(t^2+1)^9}yz \\ & - \frac{2}{(t^2+1)^{10}}xz + \frac{2e(t)}{(t^2+1)^3}z \end{aligned}$$

$$\leq \frac{2}{(t^2+1)^4}z^2 + \frac{2}{(t^2+1)^{11}}|x||y| + \frac{2}{(t^2+1)^9}|y||z|$$
$$+\frac{2}{(t^2+1)^{10}}|x||z| + \frac{2e(t)}{(t^2+1)^3}|z|.$$

Now, from the scalar function employed for the equation in example 1 and the results obtained in Eq. (6), we can deduce that

$$\begin{aligned}\dot{E} \leq\ & 2\Big[\frac{2}{t^2+1} + \frac{e(t)}{2(t^2+1)^{\frac{3}{2}}} + \frac{2}{(t^2+1)^3}\Big]E(t) + \frac{e(t)}{(t^2+1)^{\frac{3}{2}}} \\ =\ & \Phi(t)E(t) + \frac{e(t)}{(t^2+1)^{\frac{3}{2}}},\end{aligned}$$

where

$$\Phi(t) = 2\Big[\frac{2}{t^2+1} + \frac{e(t)}{2(t^2+1)^{\frac{3}{2}}} + \frac{2}{(t^2+1)^3}\Big].$$

Integrating the above from 0 to t, we obtain

$$E(t) - E(0) \leq \int_0^t E(\tau)\Phi(\tau)d\tau + \int_0^t \frac{e(\tau)}{(\tau^2+1)^{\frac{3}{2}}}d\tau.$$

Applying the Gronwall's inequality, we finally find

$$E(t) \leq B\exp\Big(\int_0^t \Phi(\tau)d\tau\Big)$$

where $B = E(0) + \int_0^t \frac{e(\tau)}{(\tau^2+1)^{\frac{3}{2}}}d\tau$. Thus, $\Phi \in L^1(0,\infty)$ implies the boundedness of E, and hence the boundedness of

$$\frac{x}{\sqrt{\beta(t)/\alpha_0(t)}}, \frac{\dot{x}}{\sqrt{\beta(t)/\alpha_1(t)}}\ and\ \frac{\ddot{x}}{\sqrt{b(t)}}.$$

This shows the useful application of Theorem 1.

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