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STRONG CONVERGENCE OF A MODIFIED AVERAGING ITERATIVE ALGORITHM FOR ASYMPTOTICALLY NONEXPANSIVE MAPS

M. O. OSILIKE[1], E. E. CHIMA[2], P. U. NWOKORO AND F. U. OGBUISI

ABSTRACT. Let C be a nonempty closed convex subset of a real Hilbert space and let $T : C \to C$ be an asymptotically nonexpansive mapping with $F(T) = \{x \in C : Tx = x\} \neq \emptyset$. Let $\{\alpha_n\}_{n=1}^{\infty}$, and $\{t_n\}_{n=1}^{\infty}$ be real sequences in $(0,1)$. Let $\{x_n\}_{n=1}^{\infty}$ be the sequence generated from an arbitrary $x_1 \in C$ by

$$\begin{cases} \nu_n := P_C\Big((1-t_n)x_n\Big),\ n \geq 1 \\ x_{n+1} := (1-\alpha_n)\nu_n + \alpha_n T^n \nu_n,\ n \geq 1, \end{cases}$$

where $P_C : H \to C$ is the metric projection. Under some appropriate mild conditions on $\{\alpha_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$, we prove that $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T. No compactness assumption is imposed on T or C and no further requirement is imposed on $F(T)$.

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1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and induced norm $||.||$. Let C be a nonempty closed convex subset of H. A mapping $T : C \to C$ is said to be *L-Lipschitzian* if there exists $L \geq 0$ such that

$$||Tx - Ty|| \leq L||x - y||,\ \forall x, y \in C. \tag{1.1}$$

T is said to be a *contraction* if $L \in [0,1)$ and T is said to be *nonexpansive* if $L = 1$. T is said to be *asymptotically nonexpansive* (see for example [1]) if there exists a sequence $\{k_n\}_{n=1}^{\infty} \subseteq [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$||T^n x - T^n y|| \leq k_n||x - y||,\ \forall x, y \in C. \tag{1.2}$$

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[1]Regular Associate, The Abdus Salam ICTP, Trieste, Italy
[2]Corresponding Author

It is well known (see for example [1]) that the class of nonexpansive mappings is a proper subclass of the class of asymptotically nonexpansive mappings. T is said to be *uniformly L-Lipschitzian* if there exists $L \geq 0$ such that

$$||T^n x - T^n y|| \leq L||x - y||, \ \forall x, y \in C. \tag{1.3}$$

T is said to be *demiclosed at p* if whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in C which converges weakly to $x^* \in C$ and $\{Tx_n\}_{n=1}^{\infty}$ converges strongly to p, then $Tx^* = p$. It is well known that if $T : C \to C$ is asymptotically nonexpansive, then T is uniformly L-Lipschitzian; $(I - T)$ is demiclosed at 0, and $F(T)$ is closed and convex (see for example [2-3]). The modified Mann iteration scheme $\{x_n\}_{n=1}^{\infty}$ generated from an arbitrary $x_1 \in C$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \ n \geq 1, \tag{1.4}$$

where the *control sequence* $\{\alpha_n\}_{n=1}^{\infty}$ is a real sequence in $[0, 1]$ satisfying some appropriate conditions has been used by several authors for the approximation of fixed points of asymptotically nonexpansive maps. (see for example [4-9]). The iteration algorithm (1.4) is a modification of the well known Mann iterative algorithm (see [10]) generated from an arbitrary $x_1 \in C$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \ n \geq 1, \tag{1.5}$$

where the *control sequence* $\{\alpha_n\}_{n=1}^{\infty}$ is a real sequence in $[0, 1]$ satisfying some appropriate conditions.

In real Hilbert spaces, it is known (see for example [8]) that if C is a nonempty closed convex subset of a real Hilbert space H, and $T : C \to C$ is an asymptotically nonexpansive mapping with sequence $\{k_n\}_{n=1}^{\infty} \subseteq [1, \infty)$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, and a nonempty fixed point set $F(T)$, then the modified iteration sequence $\{x_n\}$ generated by (1.4) is an approximate fixed point sequence (i.e., $||x_n - Tx_n|| \to 0$ as $n \to \infty$) if $\alpha_n \in [a, b] \subseteq (0, 1), \ \forall n \geq 1$. This together with the demiclosedness of $(I - T)$ at 0 yield that $\{x_n\}$ converges weakly to a fixed point of T.

To obtain strong convergence of the modified Mann algorithm to a fixed point of an asymptotically nonexpansive mapping, additional conditions are usually required on T and or the subset C (see for example [6-9]). Even for nonexpansive maps, additional conditions are required on T or C to obtain strong convergence using the Mann algorithm (1.5). In [11], Genel and Lindenstraus provided an example of a nonexpansive mapping defined on a bounded closed convex subset of a Hilbert space for which the Mann iteration does not converge to a fixed point of T.

Recently Yao, Zhou and Liou [12] (see also [13-14]) studied a modified Mann iteration algorithm $\{x_n\}$ generated from an arbitrary $x_1 \in H$ by

$$\begin{cases} \nu_n = (1-t_n)x_n \\ x_{n+1} = (1-\alpha_n)\nu_n + \alpha_n T\nu_n. \end{cases} \tag{1.6}$$

where $\{t_n\}$ and $\{\alpha_n\}$ are real sequences in $(0,1)$ satisfying some appropriate conditions. They proved strong convergence of the modified algorithm to a fixed point of a nonexpansive mapping $T : H \to H$ when $F(T) \neq \emptyset$.

Clearly, the modified Mann iteration algorithm reduces to the normal Mann iteration algorithm when $t_n \equiv 0$.

It is our purpose in this paper to modify the algorithm (1.6) and prove that the modified algorithm converges strongly to a fixed point of asymptotically nonexpansive mapping $T : C \to C$, where C is a nonempty closed convex subset of a real Hilbert space H and $F(T) \neq \emptyset$.

2. PRELIMINARY

In what follows, we shall need the following results.

Lemma 2.1 **[15]** Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of non negative real numbers such that

$$a_{n+1} \leq (1-\lambda_n)a_n + \lambda_n\gamma_n + \sigma_n,\ n \geq 1,$$

where $\{\lambda_n\} \subseteq (0,1), \{\gamma_n\} \subseteq \Re$, and $\{\sigma_n\}$ is a sequence of nonnegative real numbers such that

(i) $\lambda_n \in [0,1], \Sigma_{n=0}^{\infty}\lambda_n = \infty,$ or equivalently $\Pi_{n=0}^{\infty}(1-\lambda_n) = 0,$
(ii) $\limsup\limits_{n\to\infty} \gamma_n \leq 0,$ and
(iii) $\Sigma_{n=0}^{\infty}\sigma_n < \infty.$

Then $\lim_{n\to\infty} a_n = 0$.

It is also well known that in real Hilbert spaces H, we have the following (see for example [16]).

(i) $\|x+y\|^2 \leq \|y\|^2 + 2\langle x, x+y\rangle, \ \forall x, y \in H, \qquad (2.2)$

(ii) $$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2, \qquad (2.3)$$

$\forall x, y \in H$ and $\alpha \in [0,1]$,

(iii) if $\{x_n\}_{n=1}^{\infty}$ is a sequence in H which converges weakly to z, then,

$$\limsup_{n\to\infty} ||x_n - z||^2 = \limsup_{n\to\infty} ||x_n - y||^2 + ||z-y||^2, \ \forall y \in H. \quad (2.4)$$

Let C be a closed convex subset of a real Hilbert space H. Let $P_C : H \to C$ denote the metric projection (the proximity map) which assigns to each point $x \in H$ the unique nearest point in C, denoted by $P_C(x)$. It is well known that

$$z = P_C(x) \text{ if and only if } \langle x - z, z - y\rangle \geq 0, \ \forall y \in C, \qquad (2.5)$$

and that P_C is nonexpansive.

3. MAIN RESULTS

We now introduce the following iterative algorithm analogous to the one studied in [12]:

Modified Averaging Mann Algorithm. Let C be a nonempty closed convex subset of a real Hilbert space and let $T : C \to C$ be a giving mapping. For arbitrary $x_1 \in C$ our iteration sequence $\{x_n\}$ is given by

$$\begin{cases} \nu_n = P_C\Big((1-t_n)x_n\Big) \\ x_{n+1} = (1-\alpha_n)\nu_n + \alpha_n T^n \nu_n. \end{cases} \qquad (3.1)$$

where $\{t_n\}$ and $\{\alpha_n\}$ are real sequences in $(0,1)$ satisfying some appropriate conditions that will be made precise in our strong convergence theorem.

We now prove the following convergence theorem:

Theorem 3.1 Let C be a nonempty closed convex subset of a real Hilbert space and let $T : C \to C$ be an asymptotically nonexpansive mapping with sequence $\{k_n\}_{n=1}^{\infty} \subseteq [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $F(T) \neq \emptyset$ and let $\{t_n\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$ be

sequences in $(0,1)$ satisfying the conditions:
(c1) $\lim_{n\to\infty} t_n = 0$;

(c2) $\sum_{n=1}^{\infty} t_n = \infty$;

(c3) $\lim_{n\to\infty} \frac{1}{t_n}(k_n - 1) = 0$;

(c4) $\alpha_n \in [a,b]\ \forall n \geq 1$ and for some $a, b \in (0,1)$.
Then the modified averaging iteration sequence $\{x_n\}_{n=1}^{\infty}$ generated from an arbitrary $x_1 \in C$ by (3.1) converges strongly to a fixed point of T.

Proof. Let $p \in F(T)$ be arbitrary. Then using (2.3) we obtain

$$\begin{aligned}
||x_{n+1} - p||^2 &= ||(1-\alpha_n)(\nu_n - p) + \alpha_n(T^n\nu_n - p)||^2 \\
&= (1-\alpha_n)||\nu_n - p||^2 + \alpha_n||T^n\nu_n - p||^2 \\
&\quad -\alpha_n(1-\alpha_n)||\nu_n - T^n\nu_n||^2 \\
&\leq \Big[1 + \alpha_n(k_n^2 - 1)\Big]||\nu_n - p||^2 \\
&\quad -\alpha_n(1-\alpha_n)||\nu_n - T^n\nu_n)||^2 \qquad (3.2) \\
&\leq \Big[1 + \alpha_n(k_n^2 - 1)\Big]||\nu_n - p||^2. \qquad (3.3)
\end{aligned}$$

It follows from (3.3) that

$$\begin{aligned}
||x_{n+1} - p|| &\leq \Big[1 + \alpha_n(k_n^2 - 1)\Big]||\nu_n - p|| \\
&\leq \Big[1 + \alpha_n(k_n^2 - 1)\Big]||(1-t_n)x_n - p|| \\
&\leq \Big[1 + \alpha_n(k_n^2 - 1)\Big]\Big[(1-t_n)||x_n - p|| + t_n||p||\Big] \\
&\leq \Big[1 + \alpha_n(k_n^2 - 1)\Big]\max\Big\{||x_n - p||, ||p||\Big\} \\
&\ \vdots \\
&\leq \prod_{j=1}^{n}\Big[1 + \alpha_j(k_j^2 - 1)\Big]\max\Big\{||x_1 - p||, ||p||\Big\}. \qquad (3.4)
\end{aligned}$$

Since $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ if and only if $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$, it follows from (3.4) that $\{x_n\}_{n=1}^{\infty}$ is bounded. Hence $\{\nu_n\}_{n=1}^{\infty}$ is also bounded. Furthermore, it follows from (2.2) that

$$\begin{aligned} ||x_n - x_{n+1}||^2 &= ||x_n - \nu_n + \nu_n - x_{n+1}||^2 \\ &\leq ||\nu_n - x_{n+1}||^2 + 2\langle x_n - \nu_n, x_n - x_{n+1}\rangle \\ &\leq ||\nu_n - x_{n+1}||^2 + 2||x_n - \nu_n||||x_n - x_{n+1}|| \\ &\leq ||\nu_n - x_{n+1}||^2 + 2t_n||x_n||||x_n - x_{n+1}||. \end{aligned} \tag{3.5}$$

From (3.1) and (3.5) we obtain

$$\begin{aligned} ||\nu_n - T^n\nu_n||^2 &= \frac{1}{\alpha_n^2}||\nu_n - x_{n+1}||^2 \\ &\geq \frac{1}{\alpha_n^2}\Big[||x_n - x_{n+1}||^2 - 2t_n||x_n||||x_n - x_{n+1}||\Big]. \end{aligned} \tag{3.6}$$

Since $\{\nu_n\}_{n=1}^{\infty}$ is bounded, then

$$||\nu_n - p||^2 \leq D, \ \forall n \geq 1 \text{ and for some } D > 0,$$

and hence using condition (c4) and (3.6) in (3.2) we obtain

$$\begin{aligned} ||x_{n+1} - p||^2 &\leq \Big[1 + \alpha_n(k_n^2 - 1)\Big]||\nu_n - p||^2 \\ &\quad -\alpha_n(1-\alpha_n)||\nu_n - T^n\nu_n||^2 \\ &\leq ||\nu_n - p||^2 - \frac{(1-\alpha_n)}{\alpha_n}\Big[||x_n - x_{n+1}||^2 \\ &\quad -2t_n||x_n||||x_n - x_{n+1}||\Big] + \alpha_n(k_n^2 - 1)D \\ &\leq ||(1-t_n)x_n - p||^2 - \frac{(1-b)}{b}\Big[||x_n - x_{n+1}||^2 \\ &\quad -2t_n||x_n||||x_n - x_{n+1}||\Big] + \alpha_n(k_n^2 - 1)D \\ &= ||x_n - p||^2 - 2t_n\langle x_n, x_n - p\rangle + t_n^2||x_n||^2 \\ &\quad -k\Big[||x_n - x_{n+1}||^2 - 2t_n||x_n||||x_n - x_{n+1}||\Big] \\ &\quad +\alpha_n(k_n^2 - 1)D, (\text{ where } k := \frac{(1-b)}{b} > 0) \\ &= ||x_n - p||^2 - k||x_n - x_{n+1}||^2 + t_n\Big[t_n||x_n||^2 \\ &\quad +2k||x_n||||x_n - x_{n+1}|| - 2\langle x_n, x_n - p\rangle\Big] \\ &\quad +\alpha_n(k_n^2 - 1)D. \end{aligned} \tag{3.7}$$

Since $\{x_n\}_{n=1}^{\infty}$ is bounded, we have that there exists $M > 0$ such that

$$t_n||x_n||^2 + 2k||x_n||||x_n - x_{n+1}|| - 2\langle x_n, x_n - p\rangle \leq M, \ \forall n \geq 1. \tag{3.8}$$

From (3.7) and (3.8) we obtain

$$||x_{n+1}-p||^2-||x_n-p||^2+k||x_n-x_{n+1}||^2 \leq Mt_n+\alpha_n(k_n^2-1)D. \quad (3.9)$$

To complete the proof, we now consider the following two cases:
Case 1. Suppose $\{||x_n - p||\}_{n=1}^{\infty}$ is a monotone sequence, then we may assume that $\{||x_n - p||\}$ is monotone decreasing. Then $\lim_{n\to\infty} ||x_n - p||$ exists and it follows from (3.9), conditions (c1) and $\lim_{n\to\infty} k_n = 1$ that

$$\lim_{n\to\infty} ||x_n - x_{n+1}|| = 0. \quad (3.10)$$

Furthermore,

$$||\nu_n - x_n|| \leq t_n||x_n|| \to 0 \text{ as } n \to \infty, \quad \text{and}$$

$$||\nu_n - x_{n+1}|| \leq ||\nu_n - x_n|| + ||x_n - x_{n+1}|| \to 0 \text{ as } n \to \infty.$$

Hence

$$||\nu_n - T^n\nu_n|| \leq \frac{1}{\alpha_n}||\nu_n - x_{n+1}|| \leq \frac{1}{a}||\nu_n - x_{n+1}|| \to 0 \text{ as } n \to \infty,$$

and

$$\begin{aligned} ||x_n - T^n x_n|| &\leq ||x_n - \nu_n|| + ||\nu_n - T^n\nu_n|| \\ &\quad +||T^n\nu_n - T^n x_n|| \\ &\leq (1 + k_n)||x_n - \nu_n|| \\ &\quad +||\nu_n - T^n\nu_n|| \to 0 \text{ as } n \to \infty. \end{aligned}$$

Observe also that since T is uniformly L-Lipschitzian we obtain

$$\begin{aligned} ||\nu_n - T\nu_n|| &\leq ||\nu_n - T^n\nu_n|| + ||T^n\nu_n - T\nu_n|| \\ &\leq ||\nu_n - T^n\nu_n|| + L||T^{n-1}\nu_n - \nu_n|| \\ &\leq ||\nu_n - T^n\nu_n|| + L||T^{n-1}\nu_n - T^{n-1}\nu_{n-1}|| \\ &\quad +L||T^{n-1}\nu_{n-1} - \nu_{n-1}|| + L||\nu_{n-1} - \nu_n|| \\ &\leq ||\nu_n - T^n\nu_n|| + L||T^{n-1}\nu_{n-1} - \nu_{n-1}|| \\ &\quad +L(1 + L)||\nu_n - \nu_{n-1}|| \\ &\leq ||\nu_n - T^n\nu_n|| + L||T^{n-1}\nu_{n-1} - \nu_{n-1}|| \\ &\quad +L(1 + L)\Big[||\nu_n - x_n|| + ||x_n - x_{n-1}|| \\ &\quad +||x_{n-1} - \nu_{n-1}||\Big] \to 0 \text{ as } n \to \infty. \end{aligned} \quad (3.11)$$

Furthermore,

$$\begin{aligned}
||x_n - Tx_n|| &\leq ||x_n - T^n x_n|| + ||T^n x_n - Tx_n|| \\
&\leq ||x_n - T^n x_n|| + L||T^{n-1}x_n - x_n|| \\
&\leq ||x_n - T^n x_n|| + L||T^{n-1}x_n - T^{n-1}x_{n-1}|| \\
&\quad +L||T^{n-1}x_{n-1} - x_{n-1}|| + L||x_{n-1} - x_n|| \\
&\leq ||x_n - T^n x_n|| + L||T^{n-1}x_{n-1} - x_{n-1}|| \\
&\quad +L(1+L)||x_n - x_{n-1}|| \to 0 \text{ as } n \to \infty. \qquad (3.12)
\end{aligned}$$

Since $\lim\limits_{n\to\infty} ||x_n - Tx_n|| = \lim\limits_{n\to\infty} ||\nu_n - T\nu_n|| = \lim\limits_{n\to\infty} ||\nu_n - x_n|| = 0$, then the demiclosedness property of $(I - T)$, (2.4) and usual standard argument yield that $\{x_n\}_{n=1}^\infty$ and $\{\nu_n\}_{n=1}^\infty$ converge weakly to some $x^* \in F(T)$.

Since $||\nu_n - x^*||^2 \leq D_2, \ \forall n \geq 1,$ and for some $D_2 > 0$, then using (3.3) we obtain

$$\begin{aligned}
||x_{n+1} - x^*||^2 &\leq ||\nu_n - x^*||^2 + \alpha_n(k_n^2 - 1)D_2 \\
&\leq ||(1-t_n)(x_n - x^*) - t_n x^*||^2 + \alpha_n(k_n^2 - 1)D_2 \\
&\leq (1-t_n)^2||x_n - x^*||^2 - 2t_n(1-t_n)\langle x_n - x^*, x^*\rangle \\
&\quad +t_n^2||x^*||^2 + \alpha_n(k_n^2 - 1)D_2 \\
&\leq (1-t_n)||x_n - x^*||^2 - 2t_n(1-t_n)\langle x_n - x^*, x^*\rangle \\
&\quad +t_n^2||x^*||^2 + \alpha_n(k_n^2 - 1)D_2. \qquad (3.13)
\end{aligned}$$

Thus

$$||x_{n+1} - x^*||^2 \leq (1-t_n)||x_n - x^*||^2 + t_n\gamma_n + \sigma_n, \ \forall n \geq 1,$$

where $\gamma_n := -2(1-t_n)\langle x_n - x^*, x^*\rangle + t_n||x^*||^2 \to 0$ as $n \to \infty,$ and $\sigma_n = \alpha_n(k_n^2 - 1)D_2$, with $\sum_{n=1}^\infty \sigma_n < \infty$. It now follows from Lemma 2.1 that $\{x_n\}_{n=1}^\infty$ converges strongly to x^*. Consequently, $\{\nu_n\}_{n=1}^\infty$ converges strongly to x^*.

Case 2. Suppose $\{||x_n - p||\}_{n=1}^\infty$ is not a monotone decreasing sequence, then set $\Gamma_n := ||x_n - p||^2$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined for all $n \geq N_0$ for some sufficiently large N_0 by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \ \Gamma_k \leq \Gamma_{k+1}\}.$$

Then τ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$, for $n \geq N_0$. Using (c1) and (c2) in (3.9) we obtain

$$||x_{\tau(n)+1} - x_{\tau(n)}||^2 \leq \frac{1}{k}\Big[Mt_{\tau(n)} + \alpha_{\tau(n)}(k_{\tau(n)}^2 - 1)D\Big] \to 0 \text{ as } n \to \infty. \qquad (3.14)$$

Following the same argument as in Case 1, we obtain

$||\nu_{\tau(n)} - T\nu_{\tau(n)}|| \to 0$ as $n \to \infty$ and $||x_{\tau(n)} - Tx_{\tau(n)}|| \to 0$ as $n \to \infty$.

As in Case 1 we also obtain that $\{x_{\tau(n)}\}$ and $\{\nu_{\tau(n)}\}$ converge weakly to some x^* in $F(T)$. Furthermore, for all $n \geq N_0$, we obtain from (3.13) that

$$\begin{aligned} 0 &\leq ||x_{\tau(n)+1} - x^*||^2 - ||x_{\tau(n)} - x^*||^2 \\ &\leq t_{\tau(n)}\Big[-2(1 - t_{\tau(n)})\langle x_{\tau(n)} - x^*, x^*\rangle + t_{\tau(n)}||x^*||^2 \\ &\quad + D_2\alpha_{\tau(n)}\frac{(k^2_{\tau(n)} - 1)}{t_{\tau(n)}} - ||x_{\tau(n)} - x^*||^2\Big] \end{aligned} \tag{3.15}$$

It follows from (3.15) that

$$\begin{aligned} ||x_{\tau(n)} - x^*||^2 &\leq 2(1 - t_{\tau(n)})\langle x^* - x_{\tau(n)}, x^*\rangle + t_{\tau(n)}||x^*||^2 \\ &\quad + D_2\alpha_{\tau(n)}\frac{(k^2_{\tau(n)} - 1)}{t_{\tau(n)}} \to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus

$$\lim_{n\to\infty} \Gamma_{\tau(n)} = \lim_{n\to\infty} \Gamma_{\tau(n)+1}.$$

Furthermore, for $n \geq N_0$, we have $\Gamma_n \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (i.e $\tau(n) < n$), because $\Gamma_j > \Gamma_{j+1}$ for $\tau(n) + 1 \leq j \leq n$. It then follows that for all $n \geq N_0$ we have

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

This implies $\lim_{n\to\infty} \Gamma_n = 0$, and hence $\{x_n\}_{n=1}^\infty$ converges strongly to $x^* \in F(T)$. $\square$

Corollary 3.1 Let C be a nonempty closed convex subset of a real Hilbert space H with $0 \in C$. Let $T : C \to C$ be an asymptotically nonexpansive mapping with sequence $\{k_n\}_{n=1}^\infty \subseteq [1, \infty)$ such that $\sum_{n=1}^\infty (k_n - 1) < \infty$. Let $F(T) \neq \emptyset$ and Let $\{t_n\}_{n=1}^\infty$ and $\{\alpha_n\}_{n=1}^\infty$ be sequences in $(0, 1)$ satisfying the conditions:

(c1) $\lim_{n\to\infty} t_n = 0$;

(c2) $\sum_{n=1}^\infty t_n = \infty$;

(c3) $\lim_{n\to\infty} \frac{1}{t_n}(k_n - 1) = 0$;

(c4) $\alpha_n \in [a, b]$ $\forall n \geq 1$ and for some $a, b \in (0, 1)$.

Then the modified averaging iteration sequence $\{x_n\}_{n=1}^\infty$ generated from an arbitrary $x_1 \in C$ by

$$\begin{cases} \nu_n = (1 - t_n)x_n \\ x_{n+1} = (1 - \alpha_n)\nu_n + \alpha_n T^n \nu_n. \end{cases}$$

converges strongly to a fixed point of T.

Remark 3.1 Prototype for our real sequences $\{t_n\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$ are:

$$t_n := \sqrt{k_n - 1 + \frac{1}{n+1}},\ n \geq 1;\ \alpha_n := \frac{n}{2(n+1)} \subseteq \left[\frac{1}{4}, \frac{1}{2}\right],\ n \geq 1.$$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NIGERIA, NSUKKA, NIGERIA

E-mail addresses: `osilike@yahoo.com; micah.osilike@unn.edu.ng`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NIGERIA, NSUKKA, NIGERIA

E-mail address: `epukeezinneuk@yahoo.com`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NIGERIA, NSUKKA, NIGERIA

E-mail address: `petaun28@yahoo.co.uk`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NIGERIA, NSUKKA, NIGERIA

E-mail address: `fudochukwu@yahoo.com`