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ITERATIVE APPROXIMATION OF SOLUTIONS OF VARIATIONAL INEQUALITY PROBLEMS

ERIC U. OFOEDU

In this paper, we prove strong convergence of implicit and explicit iterative method to solutions of variational inequality problems on fixed point sets of finite family of nonexpansive mappings in q-uniformly smooth real Banach spaces. As an application, we obtain strong convergence of modifications of these iterative methods to solutions of the variational inequality problems on fixed point sets of finite family of strictly pseudo-contractive mappings. Our theorems complement some recently announced results.

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1. Introduction

Let $(E, \|.\|)$ be a real normed space. Let $S := \{x \in E : \|x\| = 1\}$. The space E is said to have a *Gâteaux differentiable* norm if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S$; and the space E is said to have a *uniformly Gâteaux differentiable* norm if for each $y \in S$ the limit is attained uniformly for $x \in S$. The space E is said to be *uniformly smooth* if and only if for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in E$ with $\|x\| = 1$ and $\|y\| \leq \delta$, the inequality

$$\frac{\|x + y\| + \|x - y\|}{2} - 1 < \epsilon \|y\|$$

holds. It is well known that every uniformly smooth real Banach space is *a reflexive real Banach space* and has uniformly Gâteaux

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differentiable norm (see e.g., [13]).

Let E be a real normed linear space with dimension, $dim(E) \geq 2$. The *modulus of smoothness* of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(t) = \sup\Big\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = t\Big\}.$$

In terms of the modulus of smoothness (see e.g. [13]), the space E is called *uniformly smooth* if and only if $\lim_{t \to 0^+} \frac{\rho_E(t)}{t} = 0$. The space E is called *q-uniformly smooth* if and only if there exists a constant $c > 0$ such that $\rho_E(t) \leq ct^q,\ t > 0$. It is easy to see that for $1 < q < +\infty$, every q-uniformly smooth Banach space is uniformly smooth; and thus has a uniformly Gâteaux differentiable norm.

Let E be a real normed linear space with dual E^*. We denote by J_q *the generalized duality mapping* from E to 2^{E^*} defined by

$$J_q x := \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1}\},$$

where $\langle ., . \rangle$ denotes the generalized duality pairing between members of E and members of E^*. For $q = 2$, the mapping $J = J_2$ from E to 2^{E^*} is called *the normalized duality mapping*. It is well known that if E is uniformly smooth or E^* is strictly convex, then the duality mapping is single-valued; and if E has a uniformly Gâteaux differentiable norm then the duality mapping is *norm-to-weak* uniformly continuous* on bounded subsets of E. If $E = H$ is a Hilbert space then the duality mapping becomes the identity map of H (see e.g., [13, 48]). In the sequel, we shall denote the single-valued generalized duality mapping by j_q and the single valued normalized duality mapping by j.

A real normed space E with strictly convex dual is said to have a *weakly sequentially continuous generalized duality mapping* j_q if and only if for each sequence $\{x_n\}_{n\geq1}$ in E such that $\{x_n\}_{n\geq1}$ converges weakly to x^* in E, we have that $\{j_q(x_n)\}_{n\geq1}$ converges in the weak* topology to $j_q(x^*)$. A real Banach space E is said to satisfy *Opial's condition* if for any sequence $\{x_n\}_{n\geq1}$ in E such that $\{x_n\}_{n\geq1}$ converges weakly to x^* in E, we have that

$$\limsup_{n\to\infty} \|x_n - x^*\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in E,\ y \neq x^*$. By Theorem 1 of Gossez and Lami Dozo [15], it is known that if E admits weakly sequentially continuous duality mapping, then E satisfies Opial's condition (see [15] for more details). Aside from Hilbert spaces, it was noted in [5] that the most significant class of Banach spaces having a weakly sequentially continuous generalized duality mapping are the sequence spaces ℓ_q for $1 < q < +\infty$ (see [6] and [7], where it is also shown that $L_q(\mathbb{R})$ has no weakly sequentially continuous duality mapping for $q \neq 2$).

A mapping $f : E \to E$ is said to be a *strict contraction* or simply *a contraction* if and only if there exists $\gamma_0 \in [0, 1)$ such that for all $x, y \in E$,

$$\|f(x) - f(y)\| \leq \gamma_0 \|x - y\|.$$

A mapping $T : E \to E$ is called *nonexpansive* if and only if for all $x, y \in E$,

$$\|Tx - Ty\| \leq \|x - y\|.$$

A point $x \in E$ is called a *fixed point* of an operator T if and only if $Tx = x$. The set of fixed points of an operator T is denoted by $F(T)$, that is, $F(T) := \{x \in E : Tx = x\}$.

A mapping $A : E \to E$ is said to be *accretive* if and if for all $x, y \in E$ there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\Big\langle Ax - Ay, j_q(x - y)\Big\rangle \geq 0.$$

A mapping $A : E \to E$ is called *strongly accretive* if and only if there exists a constant $\eta > 0$ and for all $x, y \in E$ there exist $j_q(x - y) \in J_q(x - y)$ such that

$$\Big\langle Ax - Ay, j_q(x - y)\Big\rangle \geq \eta \|x - y\|^q.$$

When $E = H$ is a Hilbert space, accretive and strongly accretive mappings coincide with monotone and strongly monotone mappings, respectively.

A linear operator $A : H \to H$ is called a $k-$*strongly positive* operator if and only if there exists a constant $k > 0$ such that for all $x \in H$,

$$\Big\langle Ax, x\Big\rangle \geq k\|x\|^2.$$

Thus, every strongly positive bounded linear operator on H is strongly monotone.

Iterative approximation of fixed points and zeros of nonlinear operators has been studied extensively by many authors to solve nonlinear operator equations as well as variational inequality problems (see e.g., [18], [23]-[28], [32]-[38]). In particular, iterative approximation of fixed points of nonexpansive mappings is an important subject in nonlinear operator theory and its applications in image recovery and signal processing is well known (see e.g., [12, 31, 47]). Most published results on nonexpansive mappings centered on the iterative approximation of fixed points of the nonexpansive mappings or approximation of a common fixed point of a given family of nonexpansive mappings. Iterative methods for nonexpansive mappings are now also applicable in solving convex minimization problems (see, for example, [45] and references therein).

Let H be a real Hilbert space with inner product $\langle .,. \rangle$. Let C be a closed convex nonempty subset of H, let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Given $u \in C$ and a real sequence $\{\alpha_n\}_{n\geq1}$ in the interval $(0, 1)$, starting with an arbitrary initial $x_0 \in C$, let a sequence $\{x_n\}_{n\geq1}$ be defined by

$$x_{n+1} = \alpha_{n+1}u + (1 - \alpha_{n+1})Tx_n, n \geq 0. \tag{1.1}$$

Under appropriate conditions on the iterative parameter $\{\alpha_n\}_{n\geq1}$, it has been shown by Halpern [16], Lions [21], Wittmann [41] and Bauschke [2] that $\{x_n\}_{n\geq0}$ converges strongly to $P_{F(T)}u$, the projection of u to the fixed point set, $F(T)$ of T.

H. K. Xu [45] studied the following quadratic minimization problem: find $x^* \in F(T)$ such that

$$\frac{1}{2}\Big\langle Ax^*, x^*\Big\rangle - \Big\langle x^*, u\Big\rangle = \min_{x\in F(T)}\Big(\frac{1}{2}\Big\langle Ax, x\Big\rangle - \Big\langle x, u\Big\rangle\Big), \tag{1.2}$$

where $u \in H$ is fixed and $A : H \to H$ a bounded linear strongly positive operator. Let $C_1, C_2, ..., C_N$ be N closed convex subsets of a real Hilbert space H having a nonempty intersection C. Suppose also that each C_i is a fixed point set of nonexpansive mappings $T_i : H \to H, i = 1, 2, ..., N$, Xu [45] proved strong convergence of the iterative algorithm

$$x_0 \in H,\ x_{n+1} = (I - \alpha_{n+1}A)T_{n+1}x_n + \alpha_{n+1}u,\ n \geq 0 \tag{1.3}$$

(where $T_n = T_{n \bmod N}$ and the mod function takes values in $\{1, 2, ... , N\}$) to a unique solution of the quadratic minimization problem

(1.2).

Marino and Xu [24], proved that the iteration scheme given by

$$x_0 \in H,\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n,\ n \geq 0 \tag{1.4}$$

converges strongly to a unique solution $x' \in F(T)$ of the variational inequality

$$\Big\langle (\gamma f - A)x', y - x' \Big\rangle \leq 0 \; \forall \; y \in F(T), \tag{1.5}$$

which is the optimality condition for the minimization problem

$$\min_{x \in F(T)} \Big(\frac{1}{2} \Big\langle Ax, x \Big\rangle - h(c) \Big),$$

where h is a potential function for γf (that is, $h'(x) = \gamma f(x)$ for all $x \in H$); provided $f : H \to H$ is a contraction, $T : H \to H$ is nonexpansive and the iterative parameter $\{\alpha_n\}_{n \geq 0}$ satisfies appropriate conditions.

In [46], Yamada introduced the following hybrid iterative method

$$x_0 \in H,\ x_{n+1} = Tx_n - \mu \lambda_n A(Tx_n),\ n \geq 0, \tag{1.6}$$

where T is nonexpansive, A is L-Lipschitzian and strongly monotone operator with constant $\eta > 0$ and $0 < \mu < \frac{2\eta}{L^2}$. He proved that if $\{\lambda_n\}_{n \geq 1}$ satisfied appropriate conditions, then (1.6) converges strongly to a unique solution $x' \in F(T)$ of the varational inequality

$$\Big\langle Ax', y - x' \Big\rangle \geq 0 \; \forall \; y \in F(T).$$

Recently, M. Tian [40] studied the following iterative method:

$$x_0 \in H,\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu \alpha_n A)Tx_n,\ n \geq 0. \tag{1.7}$$

Tian [40] proved that if $f : H \to H$ is a contraction, $A : H \to H$ is an η-strongly monotone mapping, $T : H \to H$ a nonexpansive mapping and the parameter $\{\alpha_n\}_{n \geq 1}$ satisfies appropriate conditions, then the sequence $\{x_n\}_{n \geq 1}$ converges strongly to a unique solution $x' \in F(T)$ of the variational inequality

$$\Big\langle (\gamma f - \mu A)x', y - x' \Big\rangle \leq 0 \; \forall \; y \in F(T).$$

Motivated by the results of the authors mentioned above, it is our aim in this paper to prove strong convergence of implicit and explicit iterative method to solutions of variational inequality problems on fixed point sets of nonexpansive mappings in q-uniformly smooth real Banach spaces. As an application, we obtain strong convergence of modifications of these iterative methods to solutions of the variational inequality problems on fixed point sets of strictly pseudocontractive mappings. Our theorems complement some recently announced results.

2. Preliminary

In what follows, we shall make use of the following Lemmas:

Lemma 2.1. *Let E be a real normed linear space, then for $1 < q < +\infty$, the following inequality holds:*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, j_q(x+y)\rangle \ \forall\, x, y \in E,\ \forall\, j_q(x+y) \in J_q(x+y).$$

Lemma 2.2. *(See e.g., [4, 43, 45]) Let $\{\lambda_n\}_{n\geq 1}$ be a sequence of nonnegative real numbers satisfying the condition*

$$\lambda_{n+1} \leq (1-\alpha_n)\lambda_n + \sigma_n,\ n \geq 0,$$

where $\{\alpha_n\}_{n\geq 0}$ and $\{\sigma_n\}_{n\geq 0}$ are sequences of real numbers such that $\{\alpha_n\}_{n\geq 1} \subset [0,1]$, $\sum_{n=1}^{\infty}\alpha_n = +\infty$. Suppose that $\sigma_n = o(\alpha_n)$, $n \geq 0$ (i.e., $\lim_{n\to\infty}\frac{\sigma_n}{\alpha_n} = 0$) or $\sum_{n=1}^{\infty}|\sigma_n| < +\infty$ or $\limsup_{n\to\infty}\frac{\sigma_n}{\alpha_n} \leq 0$, then $\lambda_n \to 0$ as $n \to \infty$.

Lemma 2.3. *(Compare with Lemma 3 pg. 257 of Bruck [11]) Let C be a nonempty closed and convex subset of a real strictly convex Banach space E. Let $\{T_i\}_{i\geq 1}$ be a sequence of nonself nonexpansive mappings $T_i : C \to E$ such that $F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $\{\sigma_i\} \subset (0,1)$ be such that $\sum_{i=1}^{\infty}\sigma_i = 1$. Then the mapping $T := \sum_{i=1}^{\infty}\sigma_i T_i : C \to E$ is well defined, nonexpansive and $F(T) = \bigcap_{i=1}^{\infty} F(T_i)$.*

Lemma 2.4. *(See [49], p.202 Lemma 3). Let E be a strictly convex Banach space and C be a closed convex subset of E. Let T_1, T_2,...,T_r be nonexpansive mappings of C into itself such that the set of common fixed points of T_1, T_2,...,T_r is nonempty. Let S_1, S_2,...,S_r be mappings of C into itself given by $S_i = (1-\gamma_i)I + \gamma_i T_i$ for any $0 < \gamma_i < 1$, $i = 1, 2, ..., r$, where I denotes the identity mapping on C. Then S_1, S_2,...,S_r satisfies the following: $\bigcap_{i=1}^{r} F(S_i) =$*

$\bigcap_{i=1}^{r} F(T_i)$ *and* $\bigcap_{i=1}^{r} F(S_i) = F(S_r S_{r-1} ... S_1) = F(S_1 S_{r-1} ... S_2) = ... = F(S_{r-1} ... S_1 S_r)$.

Lemma 2.5. *(See e.g. [44]) Let E be a q-uniformly smooth real Banach space for some $q > 1$, then there exists some positive constant d_q such that*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + d_q\|y\|^q \tag{2.1}$$

$\forall\ x, y \in E,\ \forall\ j_q(x) \in J_q(x)$.

If E is L_q (or ℓ_q) space, the constant d_q in (2.1) has been calculated. This is shown in the following lemma.

Lemma 2.6. *(See e.g. [19]) Let E be L_q (or ℓ_q) space ($1 < q < +\infty$) and $x, y \in E$,*

(1) if $1 < q < 2$, then

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + d_q\|y\|^q$$

$\forall\ x, y \in E,\ \forall\ j_q(x) \in J_q(x)$, *where $d_q = \frac{1+b_q^{q-1}}{(1+b_q)^{q-1}}$, and b_q is the unique solution of the equation* $(q-2)b^{q-1} + (q-1)b^{q-2} - 1 = 0,\ 0 < b < 1$.

(2) if $2 \leq q < +\infty$, then

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x)\rangle + (q-1)\|y\|^2$$

$\forall\ x, y \in E,\ \forall\ j(x) \in J(x)$.

Lemma 2.7. *(Lemma 2.2 of [42], p. 1411) Let C be a closed convex nonempty subset of a reflexive Banach space which satisfies Opial's condition and suppose $T : C \to E$ is nonexpansive, then the mapping $I - T$ is demiclosed at zero, that is $\{x_n\}_{n\geq 1}$ is a sequence in C such that $x_n \rightharpoonup x^*$ and $x_n - Tx_n \to 0$ as $n \to \infty$, then $x^* = Tx^*$*

Let μ be a bounded linear functional defined on ℓ_∞ satisfying $\|\mu\| = 1 = \mu(1)$. It is known that μ is a mean on $\mathbb{N}$ if and only if

$$\inf\{a_n : n \in \mathbb{N}\} \leq \mu(a_n) \leq \sup\{a_n : n \in \mathbb{N}\}$$

for every $a = (a_1, a_2, a_3, ...) \in \ell_\infty$. In the sequel, we shall use $\mu_n(a_n)$ instead of $\mu(a)$. A mean μ on $\mathbb{N}$ is called *a Banach limit* if $\mu_n(a_n) = \mu_n(a_{n+1})$ for every $a = (a_1, a_2, a_3, ...) \in \ell_\infty$. It is well known that if μ is a Banach limit, then

$$\liminf_{n\to\infty} a_n \leq \mu_n a_n \leq \limsup_{n\to\infty} a_n$$

for all $a = (a_1, a_2, a_3, ...) \in \ell_\infty$. It is now easy to see that for $a = (a_1, a_2, ...),\ b = (b_1, b_2, ...) \in \ell_\infty$, if $\lim\limits_{n\to\infty} a_n = a^*$, then $\mu_n(a_n) = a^*$ and if $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n$, then $\mu_n(a_n) = \mu_n(b_n)$.

3. Main results.

We start as follows.

Lemma 3.1. *Let E be a real normed linear space. Let $A : E \to E$ be a $\eta-$strongly accretive mapping. Let $T : E \to E$ be a nonexpansive mapping such that $F(T) \neq \emptyset$, $\lambda > 0$ and $u \in E$ be a fixed vector. Suppose that the solution $x' \in F(T)$ of the variational inequality $\Big\langle u - \lambda Ax', j_q(p - x')\Big\rangle \leq 0\ \forall\ p \in F(T)$ exists, then x' is unique.*

Proof. Suppose for contradiction that the variational inequality has two solutions in $F(T)$, say $x' \neq y'$, then we have, in particular, that

$$\Big\langle u - \lambda Ax', j_q(y' - x')\Big\rangle \leq 0$$

and

$$-\Big\langle u - \lambda Ay', j_q(y' - x')\Big\rangle \leq 0.$$

Thus, adding these two inequalities and using strong accretivity of A, we obtain that

$$\lambda\eta\|y' - x'\|^q \leq \Big\langle \lambda Ay' - \lambda Ax', j_q(y' - x')\Big\rangle \leq 0,$$

a contradiction. Hence, the variational inequality has a unique solution, provided the solution exists. This completes the proof. □

Remark 3.2. If E is a q-uniformly smooth real Banach space; a mapping $A : E \to E$ an L-Lipschitzian $\eta-$strongly accretive and $T : E \to E$ a nonexpansive mapping, then for the mapping $(I - t\lambda A)T : E \to E$ (where I is the identity map of E, $\lambda > 0$ and

$t \in (0,1)$), we obtain using Lemma 2.5 that

$$\begin{aligned}\|(I-t\lambda A)Tx-(I-t\lambda A)Ty\|^q &= \|Tx-Ty-t\lambda(ATx-ATy)\|^q\\ &\leq \|Tx-Ty\|^q - qt\lambda\big\langle ATx-ATy,\\ &\quad j_q(Tx-Ty)\big\rangle + d_q(t\lambda)^q\\ &\quad \times\|ATx-ATy\|^q\\ &\leq \|Tx-Ty\|^q - qt\eta\lambda\|Tx-Ty\|^q\\ &\quad +d_q(tL\lambda)^q\|Tx-Ty\|^q\\ &\leq \Big(1-t\lambda(q\eta-d_qL^q\lambda^{q-1})\Big)\\ &\quad \times\|Tx-Ty\|^q\end{aligned}$$

So that if λ is such that $0<\lambda<\left(\frac{q\eta}{L^q d_q}\right)^{\frac{1}{q-1}}$, we have that

$$0<1-t\lambda(q\eta-d_qL^q\lambda^{q-1})<1$$

and since T is a nonexpansive mapping of E into E and $1<q<+\infty$, we obtain

$$\begin{aligned}\|(I-t\lambda A)Tx-(I-t\lambda A)Ty\| &\leq \Big(1-t\lambda(q\eta-d_qL^q\lambda^{q-1})\Big)^{\frac{1}{q}}\\ &\quad \times\|Tx-Ty\|\\ &\leq \Big(1-t\lambda(q\eta-d_qL^q\lambda^{q-1})\Big)^{\frac{1}{q}}\\ &\quad \times\|x-y\|. \qquad (3.1)\end{aligned}$$

Lemma 3.3. *Let E be a q-uniformly smooth real Banach space. Let $A:E\to E$ be an L-Lipschitzian $\eta-$strongly accretive mapping. Let $T:E\to E$ be a nonexpansive mapping such that $F(T)\neq\emptyset$. Let $u\in E$ be fixed, $0<\lambda<\left(\frac{q\eta}{L^qd_q}\right)^{\frac{1}{q-1}}$ and $t\in\Big(0,\min\Big\{1,\frac{1}{q\eta\lambda-d_q(L\lambda)^q}\Big\}\Big)$, then there exists a unique $z_t\in E$ such that*

$$z_t = tu+(I-t\lambda A)Tz_t. \qquad (3.2)$$

Proof. For each $t\in\Big(0,\min\Big\{1,\frac{1}{q\eta\lambda-d_q(L\lambda)^q}\Big\}\Big)$, define $S_t:E\to E$ by $S_tx = tu+(I-t\lambda A)Tx$ for all $x\in E$. Then,

$$\begin{aligned}\|S_tx-S_ty\| &= \|(I-t\lambda A)Tx-(I-t\lambda A)Ty\|\\ &\leq \Big(1-t\lambda(q\eta-d_q(L)^q\lambda^{q-1})\Big)^{\frac{1}{q}}\|x-y\|.\end{aligned}$$

Thus, for all $t \in \Big(0, \min\Big\{1, \frac{1}{q\eta\lambda - d_q(L\lambda)^q}\Big\}\Big)$, we have that S_t is a strict contraction on E. Hence, there exists a unique $z_t \in E$ which satisfies (3.2). This completes the proof. □

Lemma 3.4. *Let E, A, $u \in E$ and T be as in Lemma 3.3. Let $\{z_t\}$ satisfy (3.2), then*

(a) $\{z_t\}$ is bounded;
(b) The mapping $\Psi : \Big(0, \min\Big\{1, \frac{1}{q\eta\lambda - d_q(L\lambda)^q}\Big\}\Big) \to E$ given by $\Psi(t) = z_t$ is continuous;
(c) $\lim_{t\to 0} \|z_t - Tz_t\| = 0$.

Proof.
(a) Let $p \in F(T)$. If $z_t = p$ for all $t \in \Big(0, \min\Big\{1, \frac{1}{q\eta\lambda - d_q(L\lambda)^q}\Big\}\Big)$, then we are done. Otherwise, let $t^* \in \Big(0, \min\Big\{1, \frac{1}{q\eta\lambda - d_q(L\lambda)^q}\Big\}\Big)$ be such that $z_t \neq p \ \forall\ t \in (0, t^*]$; then using (3.1), (3.2) and Lemma 2.1, we have that for all $t \in (0, t^*]$,

$$\begin{aligned} \|z_t - p\|^q &= \|(I - t\lambda A)Tz_t - (I - t\lambda A)p + t(u - \lambda Ap)\|^q \\ &\leq \Big(1 - t(q\eta\lambda - d_q(L\lambda)^q)\Big)\|z_t - p\|^q \\ &\quad + qt\Big\langle u - \lambda Ap, j_q(z_t - p)\Big\rangle \\ &\leq \Big(1 - t(q\eta\lambda - d_q(L\lambda)^q)\Big)\|z_t - p\|^q \\ &\quad + qt\|u - \lambda Ap\|.\|z_t - p\|^{q-1} \end{aligned} \tag{3.3}$$

Thus, we obtain from (3.3) that

$$\|z_t - p\| \leq \Big(1 - t(q\eta\lambda - d_q(L\lambda)^q)\Big)\|z_t - p\| + qt\|u - \lambda Ap\|. \tag{3.4}$$

Inequality (3.4) therefore gives

$$\|z_t - p\| \leq \frac{q\|u - \mu Ap\|}{q\eta\lambda - d_q(L\lambda)^q}.$$

Hence, the path $\{z_t\}$ is bounded; and so is $\{ATz_t\}$.
(b) Let $t_0 \in \Big(0, \min\Big\{1, \frac{1}{q\eta\lambda - d_q(L\lambda)^q}\Big\}\Big)$ be arbitrary. It is enough to show that the mapping $\Psi : \Big(0, \min\Big\{1, \frac{1}{q\eta\lambda - d_q(L\lambda)^q}\Big\}\Big) \to E$ given by $\Psi(t) = z_t$ is continuous at t_0. Now, for some constant $M_1 > 0$ and

using Lemma 2.1, we have that

$$\begin{aligned}\|z_t - z_{t_0}\|^q &= \|(t-t_0)u - (t-t_0)\lambda ATz_t + (I - t_0\lambda A)Tz_t \\ &\quad -(I-t_0\lambda A)Tz_{t_0}\|^q \\ &\leq \|(I-t_0\lambda A)Tz_t - (I-t_0\lambda A)Tz_{t_0}\|^q \\ &\quad +q\Big\langle (t-t_0)u - (t-t_0)\lambda ATz_t, j_q(z_t - z_{t_0})\Big\rangle \\ &\leq \Big(1 - t_0(q\eta\lambda - d_q(L\lambda)^q)\Big)\|z_t - z_{t_0}\|^q \\ &\quad +|t-t_0|M_1\|z_t - z_{t_0}\|^{q-1}. \qquad (3.5)\end{aligned}$$

Thus, we obtain from (3.5) that

$$\|\Psi(t) - \Psi(t_0)\| = \|z_t - z_{t_0}\| \leq \frac{M_1}{t_0(q\eta\lambda - d_q(L\lambda)^q)}\Big|t - t_0\Big|$$

and the result follows.
(c) Using (3.2), we have that for some constant $M_0 > 0$,

$$\|z_t - Tz_t\| \leq t\|u - \lambda ATz_t\| \leq tM_0 \to 0 \text{ as } t \to 0.$$

This completes the proof. □

Theorem 3.5. *Let E, A, $u \in E$ and T be as in Lemma 3.3. Let $\{z_t\}$ satisfy* (3.2), *then $\{z_t\}$ converges strongly to some $x' \in F(T)$ which is a solution of the variational inequality*

$$\Big\langle u - \lambda Ax', j(p - x')\Big\rangle \leq 0 \;\forall\, p \in F(T). \qquad (3.6)$$

Proof. By Lemma 3.1, if solution of (3.6) exists in $F(T)$, then it is unique. Let $p \in F(T)$, then using (3.2), we have that

$$z_t - p = t\big(u - \lambda Ap\big) + (I - t\lambda A)Tz_t - (I - t\lambda A)p. \qquad (3.7)$$

Thus, using (3.1), (3.7) and Lemma 2.1, we obtain that

$$\begin{aligned}\|z_t - p\|^q &\leq \|(I - t\lambda A)Tz_t - (I - t\lambda A)p\|^q \\ &\quad +qt\Big\langle u - \lambda Ap, j_q(z_t - p)\Big\rangle \\ &\leq \Big(1 - t(q\eta\lambda - d_q(\lambda L)^q\Big)\|z_t - p\|^q \\ &\quad +qt\Big\langle u - \lambda Ap, j_q(z_t - p)\Big\rangle. \qquad (3.8)\end{aligned}$$

So, (3.8) implies that

$$\|z_t - p\|^q \leq \frac{1}{q\eta\lambda - d_q(\lambda L)^q}\Big\langle u - \lambda Ap, j_q(z_t - p)\Big\rangle. \qquad (3.9)$$

From Lemma 3.4 we know that $\{z_t\}$ is bounded. Let $\{t_n\}_{n\geq 1}$ in $\Big(0, \min\Big\{1, \frac{1}{q\eta\lambda - d_q(L\lambda)^q}\Big\}\Big)$ be such that $\lim\limits_{n\to\infty} t_n = 0$ and set $z_n = z_{t_n}$. Then, defining $\Phi : E \to \mathbb{R} \cup \{+\infty\}$ by $\Phi(x) = \mu_n\|z_n - x\|^q$ for all $x \in E$ (for some Banach limit, μ, on ℓ_∞), we have that Φ is continuous, convex and $\lim\limits_{\|x\|\to+\infty} \Phi(x) = +\infty$. Thus, setting

$$K = \Big\{y \in E : \Phi(y) = \min_{x\in E} \Phi(x)\Big\},$$

then K is bounded closed convex and nonempty subset of E and since by Lemma 3.4, $\lim\limits_{n\to\infty} \|z_n - Tz_n\| = 0$, we obtain using Lemma 2.1 that for some constant $M_0 > 0$ and for all $y \in K$,

$$\begin{aligned}
\Phi(Ty) &= \mu_n\|z_n - Ty\|^q = \mu_n\|Tz_n - Ty + z_n - Tz_n\|^q \\
&\leq \mu_n\Big(\|Tz_n - Ty\|^q + q\Big\langle z_n - Tz_n, j_q(z_n - Ty)\Big\rangle\Big) \\
&\leq \mu_n\Big(\|Tz_n - Ty\|^q + \|z_n - Tz_n\|M_0\Big) \\
&\leq \mu_n\|z_n - y\|^q + \mu_n\|z_n - Tz_n\|M_0 \\
&\quad = \mu_n\|z_n - y\|^q = \Phi(y).
\end{aligned} \tag{3.10}$$

Thus, $T(K) \subset K$; that is, K is invariant under T. Since E is uniformly smooth real Banach space and thus every bounded closed convex nonempty subset of E has the fixed point property for nonexpansive mappings, then there exists $x' \in K$ such that $Tx' = x'$. Since x' is also a minimizer of Φ over E, it follows that for arbitrary $x \in E$ and for all $\xi \in (0,1)$, $\Phi(x') \leq \Phi\big(x' + \xi(x - x')\big)$. Lemma 2.1 gives

$$\begin{aligned}
\|z_n - x' - \xi(x - x')\|^q &\leq \|z_n - x'\|^q \\
&- q\xi\Big\langle x - x', j_q\big(z_n - x' - \xi(x - x')\big)\Big\rangle.
\end{aligned} \tag{3.11}$$

Inequality (3.11) implies that

$$\mu_n\Big\langle x - x', j_q\big(z_n - x' - \xi(x - x')\big)\Big\rangle \leq 0. \tag{3.12}$$

Furthermore, since E is uniformly smooth, the duality mapping j_q is norm-to-weak* uniformly continuous on bounded subsets of E; and hence

$$\lim_{\xi\to 0}\Big(\Big\langle x - x', j_q(z_n - x')\Big\rangle - \Big\langle x - x', j_q\big(z_n - x' - \xi(x - x')\big)\Big\rangle\Big) = 0.$$

So, for all $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that for all $\xi \in (0, \delta_\epsilon)$ and for all $n \in \mathbb{N}$, we have that

$$\Big\langle x - x', j_q(z_n - x')\Big\rangle - \Big\langle x - x', j_q\big(z_n - x' - \xi(x - x')\big)\Big\rangle < \epsilon.$$

This implies that for all $\xi \in (0, \delta_\epsilon)$,

$$\mu_n\Big\langle x - x', j_q(z_n - x')\Big\rangle \le \epsilon + \mu_n\Big\langle x - x', j_q\big(z_n - x' - \xi(x - x')\big)\Big\rangle \le \epsilon$$

and since $\epsilon > 0$ is arbitrary, we obtain that

$$\mu_n\Big\langle x - x', j_q(z_n - x')\Big\rangle \le 0 \;\forall\; x \in E.$$

Hence, we have in particular, that for the vectors $u \in E$ and $2x' - \lambda Ax' \in E$

$$\mu_n\Big\langle u - x', j_q(z_n - x')\Big\rangle \le 0 \tag{3.13}$$

and

$$\begin{aligned}\mu_n\Big\langle (2x' - \lambda Ax') - x', j_q(z_n - x')\Big\rangle &= \mu_n\Big\langle x' - \lambda Ax', j_q(z_n - x')\Big\rangle \\ &\le 0. \end{aligned} \tag{3.14}$$

Moreover, since $\{t_n\}_{n\ge1}$ is in $\Big(0, \min\Big\{1, \frac{1}{q\eta\lambda - d_q(L\lambda)^q}\Big\}\Big)$ and $x' \in F(T)$, we obtain from (3.9) that

$$\begin{aligned}\|z_n - x'\|^q &\le \frac{1}{q\eta\lambda - d_q(\lambda L)^q}\Big\langle u - \lambda Ax', j_q(z_n - x')\Big\rangle \\ &= \frac{1}{\omega}\Big(\Big\langle u - x', j_q(z_n - x')\Big\rangle \\ &\quad + \Big\langle x' - \lambda Ax', j_q(z_n - x')\Big\rangle\Big), \end{aligned} \tag{3.15}$$

where $\omega = q\eta\lambda - d_q(\lambda L)^q$. So, from (3.15) (using (3.13), (3.14) and the linearity of Banach limit) we obtain that

$$\begin{aligned}\mu_n\|z_n - x'\|^q &\le \frac{1}{\omega}\mu_n\Big\langle u - x', j_q(z_n - x')\Big\rangle + \\ &\quad + \frac{1}{\omega}\mu_n\Big\langle x' - \lambda Ax', j_q(z_n - x')\Big\rangle \le 0. \end{aligned} \tag{3.16}$$

So, $\mu_n\|z_n - x'\|^q = 0$ and this implies that there exists a subsequence $\{z_{n_i}\}_{i\ge1}$ of $\{z_n\}_{n\ge1}$ such that $z_{n_i} \to x'$ as $i \to \infty$.

We now show that x' is a solution of the variational inequality (3.6). From (3.2), we obtain that

$$z_{n_i} = t_{n_i}u + (I - t_{n_i}\lambda A)Tz_{n_i}.$$

This gives

$$\lambda A z_{n_i} - u = -\frac{1}{t_{n_i}}(z_{n_i} - Tz_{n_i}) + \lambda(Az_{n_i} - ATx_{n_i}). \qquad (3.17)$$

Thus, for all $p \in F(T)$, we obtain from (3.17) that

$$\Big\langle \lambda A z_{n_i} - u, j_q(z_{n_i} - p)\Big\rangle = -\frac{1}{t_{n_i}}\Big\langle z_{n_i} - Tz_{n_i}, j_q(z_{n_i} - p)\Big\rangle + \lambda\Big\langle Az_{n_i} - ATx_{n_i}, j_q(z_{n_i} - p)\Big\rangle \quad (3.18)$$

Since T is nonexpansive, we have that $(I - T)$ is accretive. Thus, $\Big\langle z_{n_i} - Tz_{n_i}, j_q(z_{n_i} - p)\Big\rangle = \Big\langle (I-T)z_{n_i} - (I-T)p, j_q(z_{n_i} - p)\Big\rangle \geq 0.$ So, (3.18) gives

$$\Big\langle \lambda A z_{n_i} - u, j_q(z_{n_i} - p)\Big\rangle \leq \lambda\Big\langle Az_{n_i} - ATx_{n_i}, j_q(z_{n_i} - p)\Big\rangle \quad (3.19)$$

Using (3.19), we obtain (for some constant $M_2 > 0$) that

$$\begin{aligned}
\Big\langle \lambda Ax' - u, j_q(x' - p)\Big\rangle &= \Big\langle \lambda A z_{n_i} - u, j_q(z_{n_i} - p)\Big\rangle \\
&\quad -\Big\langle \lambda A z_{n_i} - \lambda A x', j_q(z_{n_i} - p)\Big\rangle \\
&\quad +\Big\langle u - \lambda Ax', j_q(z_{n_i} - p) - j_q(x' - p)\Big\rangle \\
&\leq M_2\Big(\|Az_{n_i} - ATz_{n_i}\| + \|\lambda Az_{n_i} - \lambda Ax'\|\Big) \\
&\quad +\Big\langle u - \lambda Ax', j_q(z_{n_i} - p) - j_q(x' - p)\Big\rangle \\
&. \qquad (3.20)
\end{aligned}$$

Hence, since A, T are continuous, $z_{n_i} \to x' \in F(T)$ as $i \to \infty$ and the duality mapping is norm-to-weak* uniformly continuous on bounded subsets of E, we obtain from (3.20) that (as $i \to \infty$)

$$\Big\langle \lambda Ax' - u, j_q(x' - p)\Big\rangle \leq 0 \;\forall\, p \in F(T).$$

So, $x' \in F(T)$ is a solution (3.6).

Finally, we show that $\{z_n\}_{n\geq 1}$ converges to x'. Suppose that there is another subsequence $\{z_{n_l}\}_{l\geq 1}$ of $\{z_n\}_{n\geq 1}$ such that $z_{n_l} \to z' \in E$ as $l \to \infty$. Then by (c) of Lemma 3.4, we have that $z' \in F(T)$.

Similar argument (from (3.17) to (3.20) with z_{n_i} replaced by z_{n_l}) shows that

$$\Big\langle \lambda A z' - u, j_q(z' - p)\Big\rangle \leq 0 \; \forall\, p \in F(T).$$

Uniqueness of solution of (3.6) shows that $x' = z'$. Hence, $z_n = z_{t_n} \to x'$ and $n \to \infty$. Consequently, we obtain that the path $\{z_t\}$ given by (3.2) converges strongly (as $t \to 0$) to unique $x' \in F(T)$ which solves the variational inequality (3.6). This completes the proof. □

The following corollaries follow from our discussion so far.

Corollary 3.6. *Let E be a real L_p or (ℓ_p) space. Let A and T be as in Lemma 3.3 and $\{z_t\}$ satisfy* (3.2)*, then we obtain the same conclusion as in Theorem 3.5.*

Remark 3.7. Since every Hilbert space is a 2-uniformly smooth Banach space, it follows from Lemma 2.6 that if $E = H$ is a Hilbert space, then $d_q = d_2 = 1$. Furthermore, we recall that in a Hilbert space H, the duality mapping coincide with the identity mapping on H. Thus, we have the following corollary.

Corollary 3.8. *Let H be a real Hilbert space, $T : H \to H$ a nonexpansive mapping such that $F(T) \neq \emptyset$ and $A : H \to H$ be an L-Lipschitzian strongly accretive mapping with a constant $\eta > 0$. Let $u \in H$ be fixed. Suppose that $0 < \lambda < \frac{2\eta}{L^2}$ and $t \in \Big(0, \min\Big\{1, \frac{1}{2\eta\lambda - (\lambda L)^2}\Big\}\Big)$, then there exists a unique $z_t \in H$ satisfying* (3.2)*. Moreover, $\{z_t\}$ converges strongly (as t $\to 0$) to a unique solution $x' \in F(T)$ of the variational inequality $\Big\langle u - \lambda A x', p - x'\Big\rangle \leq 0 \; \forall\, p \in F(T)$.*

Corollary 3.9. *Let E be a strictly convex q-uniformly smooth real Banach space, let $A : E \to E$ be an L-Lipschitzian strongly accretive mapping with a constant $\eta > 0$. Let $T_i : E \to E, i = 1, 2, ...$ be a countable family of nonexpansive mappings such that $F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $T := \sum_{i=1}^{\infty} \sigma_i T_i$, where $\{\sigma_i\}_{i \geq 1} \subset (0, 1)$ is such that $\sum_{i=1}^{\infty} \sigma_i = 1$. Let $u \in E$ be fixed. Suppose that the conditions of Lemma 3.3 are satisfied, then there exist a unique $z_t \in E$ satisfying $z_t = tu + (I - t\lambda A)T z_t$. Moreover, $\{z_t\}$ converges strongly (as*

$t \to 0$) to a unique solution $x' \in F(T)$ of the variational inequality $\Big\langle (u - \lambda A x', j(p - x')\Big\rangle \leq 0 \; \forall \; p \in F$.

Proof. By Lemma 2.3, $T := \sum_{i=1}^{\infty} \sigma_i T_i$ is well defined, nonexpansive and $F(T) = F$. The rest follows as in the proof of Theorem 3.5. $\square$

Corollary 3.10. *Let E be a strictly convex q-uniformly smooth real Banach space, let $A : E \to E$ be an L-Lipschitzian strongly accretive mapping with a constant $\eta > 0$. Let $T_i : E \to E, i = 1, 2, ..., m$ be a finite family of nonexpansive mappings such that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Let $T := \sum_{i=1}^{m} \sigma_i T_i$, where $\{\sigma_i\}_{i=1}^{m} \subset (0, 1)$ is such that $\sum_{i=1}^{m} \sigma_i = 1$. Let $u \in E$ be fixed. Suppose that the conditions of Lemma 3.3 are satisfied, then there exist a unique $z_t \in E$ satisfying $z_t = tu + (I - t\lambda A)T z_t$. Moreover, $\{z_t\}$ converges strongly to a unique solution $x' \in F(T)$ of the variational inequality $\Big\langle u - \lambda A x', j(p - x')\Big\rangle \leq 0 \; \forall \; p \in F$.*

4. Strong convergence of explicit iteration scheme for finite family of nonexpansive mappings.

In the sequel, we shall assume that $0 < \lambda < \left(\frac{q\eta}{L^q d_q}\right)^{\frac{1}{q-1}}$, $\alpha_n \in \left(0, \min\Big\{1, \frac{1}{q\eta\lambda - d_q(L\lambda)^q}\Big\}\right) \; \forall \; n \in \mathbb{N}$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \frac{\alpha_{n+r} - \alpha_n}{\alpha_{n+r}} = 0 \Longleftrightarrow \lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+r}} = 1$.

Theorem 4.1. *Let E be a q-uniformly smooth real Banach space which admits weakly sequentially continuous generalized duality mapping, let $A : E \to E$ be an L-Lipschitzian strongly accretive mapping with a constant $\eta > 0$. Let $T_i : E \to E, i = 1, 2, ..., r$ be a finite family of nonexpansive mappings such that $\Omega := \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$. Let $u \in E$ be fixed and $\{x_n\}_{n\geq 1}$ be a sequence in E generated iteratively by*

$$x_0 \in E, x_{n+1} = \alpha_{n+1} u + (I - \alpha_{n+1}\lambda A)T_{n+1} x_n, \; n \geq 0, \qquad (4.1)$$

where $T_n = T_{n \bmod r}$ and mod function takes values in $\{1, 2, ..., r\}$. Suppose that $\Omega = F(T_r T_{r-1} ... T_1) = F(T_1 T_{r-1} ... T_2) = ...$

$= F(T_{r-1}...T_1T_r)$, *then,* $\{x_n\}_{n\geq 0}$ *converges strongly to a solution of the variational inequality*

$$\Big\langle u - \lambda Ax', j_q(p - x')\Big\rangle \leq 0 \; \forall \, p \in \Omega. \tag{4.2}$$

.

Proof. Since the mapping $G = T_1T_2...T_r : E \to E$ is nonexpansive, then following the method of proof of Theorem 3.5, we have that for all $t \in \Big(0, \min\Big\{1, \frac{1}{q\eta\lambda - d_q(L\lambda)^q}\Big\}\Big)$, there exists unique $z_t \in E$ which satisfies $z_t = tu + (I - t\lambda A)Gz_t$, moreover, $\{z_t\}$ converges to a unique solution $x' \in F(G) = \Omega$ of (4.2). We now show that the explicit scheme (4.1) converges strongly to x'. We start by showing first that $\{x_n\}_{n\geq 0}$ is bounded. Now, let $p \in \Omega$ and set

$$r := \max\Big\{\|x_0 - p\|, \frac{q\|u - \lambda Ap\|}{q\eta\lambda - d_q(L\lambda)^q}\Big\}.$$

We show by induction that

$$\|x_n - p\| \leq r \; \forall \, n \geq 0. \tag{4.3}$$

Observe that for $n = 0$ (4.3) clearly holds. Assume for $n \geq 0$ that (4.3) is true. We show that (4.3) is also true for $n+1$. Suppose for contradiction that this does not hold, then

$$\|x_{n+1} - p\| > r \geq \|x_n - p\|.$$

Thus, using (3.1), (4.1) and Lemma 2.1, we obtain that

$$\begin{aligned}
\|x_{n+1} - p\|^q &= \|\alpha_{n+1}u + (I - \alpha_{n+1}\lambda A)T_{n+1}x_n - p\|^q \\
&= \|(I - \alpha_{n+1}\lambda A)T_{n+1}x_n \\
&\quad -(I - \alpha_{n+1}\lambda A)p + \alpha_{n+1}(u - \lambda Ap)\|^q \\
&\leq \|(I - \alpha_{n+1}\lambda A)T_{n+1}x_n - (I - \alpha_{n+1}\lambda A)p\|^q \\
&\quad +q\alpha_{n+1}\Big\langle u - \lambda Ap, j_q(x_{n+1} - p)\Big\rangle \\
&\leq \Big(1 - \alpha_{n+1}(q\eta\lambda - d_q(L\lambda)^q)\Big)\|x_n - p\|^q \\
&\quad +q\alpha_{n+1}\|u - \lambda Ap\|.\|x_{n+1} - p\|^{q-1} \\
&< \Big(1 - \alpha_{n+1}(q\eta\lambda - d_q(L\lambda)^q)\Big)\|x_{n+1} - p\|^q \\
&\quad +q\alpha_{n+1}\|u - \lambda Ap\|.\|x_{n+1} - p\|^{q-1}.
\end{aligned} \tag{4.4}$$

Inequality (4.4) implies that

$$\|x_{n+1} - p\| < \frac{q\|u - \lambda Ap\|}{q\eta\lambda - d_q(L\lambda)^q},$$

a contradiction. Hence, the sequence $\{x_n\}_{n\geq 1}$ is bounded. Consequently,
$\{T_{n+1}x_n\}_{n\geq 1}$ and $\{AT_{n+1}x_n\}_{n\geq 1}$ are also both bounded.

Furthermore, using (4.1),

$$\begin{aligned} x_{n+r} - x_n &= (\alpha_{n+r} - \alpha_n)u + (I - \alpha_{n+r}\lambda A)T_{n+r}x_{n+r-1} \\ &\quad -(I - \alpha_{n+r}\lambda A)T_n x_{n-1} - (\alpha_{n+r} - \alpha_n)\lambda AT_n x_{n-1}. \end{aligned} \tag{4.5}$$

Thus, using the fact that $T_{n+r} = T_n$, we obtain from (4.5) using Lemma 2.1 that for some constant $M_5 > 0$,

$$\begin{aligned} \|x_{n+r} - x_n\|^q &\leq \Big(1 - \alpha_{n+r}(q\eta\lambda - d_q(L\lambda)^q)\Big)\|x_{n+r-1} - x_{n-1}\|^q \\ &\quad +|\alpha_{n+r} - \alpha_n|M_5 \\ &= \Big(1 - \alpha_{n+r}(q\eta\lambda - d_q(L\lambda)^q)\Big)\|x_{n+r-1} - x_{n-1}\|^q \\ &\quad +M_5\alpha_{n+r}\frac{|\alpha_{n+r} - \alpha_n|}{\alpha_{n+r}} \end{aligned} \tag{4.6}$$

So, using (4.6), we obtain from Lemma 2.2 that

$$\lim_{n\to\infty} \|x_{n+r} - x_n\| = 0 \tag{4.7}$$

Furthermore, from the recursion formula (4.1) and for some constant $M_6 > 0$, we obtain

$$\|x_{n+1} - T_{n+1}x_n\| = \alpha_{n+1}\|u - \lambda AT_{n+1}x_n\| \leq \alpha_{n+1}M_6.$$

Thus,

$$\lim_{n\to\infty} \|x_{n+1} - T_{n+1}x_n\| = 0.$$

Similar argument shows that

$$x_{n+r} - T_{n+r}x_{n+r-1} \to 0 \text{ as } n \to \infty. \tag{4.8}$$

Thus, we obtain using (4.8) and the fact that T_n is nonexpansive that

$$\begin{aligned} &x_{n+r} - T_{n+r}x_{n+r-1} \to 0 \text{ as } n \to \infty \\ &T_{n+r}x_{n+r-1} - T_{n+r}T_{n+r-1}x_{n+r-2} \to 0 \text{ as } n \to \infty \\ &T_{n+r}T_{n+r-1}x_{n+r-2} - T_{n+r}T_{n+r-1}T_{n+r-2}x_{n+r-3} \to 0 \text{ as } n \to \infty \\ &\vdots \\ &T_{n+r}...T_{n+2}x_{n+1} - T_{n+r}...T_{n+2}T_{n+1}x_n \to 0 \text{ as } n \to \infty. \end{aligned}$$

Adding up these yields

$$x_{n+r} - T_{n+r}...T_{n+2}T_{n+1}x_n \to 0 \text{ as } n \to \infty. \tag{4.9}$$

But,

$$\|x_n - T_{n+r}...T_{n+2}T_{n+1}x_n\| \leq \|x_n - x_{n+r}\| + \|x_{n+r} - T_{n+r}...T_{n+2}T_{n+1}x_n\| \tag{4.10}$$

Thus, using (4.7), (4.9) and (4.10), we get that

$$\lim_{n\to\infty} \|x_n - T_{n+r}...T_{n+2}T_{n+1}x_n\| = 0. \tag{4.11}$$

Next, we show that

$$\limsup_{n\to\infty} \Big\langle u - \lambda Ax', j_q(x_n - x') \Big\rangle \leq 0.$$

Let $\{x_{n_k}\}_{k\geq 1}$ be a subsequence of $\{x_n\}_{n\geq 1}$ such that

$$\limsup_{n\to\infty} \Big\langle u - \lambda Ax', j_q(x_n - x') \Big\rangle = \lim_{k\to\infty} \Big\langle u - \lambda Ax', j_q(x_{n_k} - x') \Big\rangle.$$

Since $\{x_n\}_{n\geq 1}$ is bounded and since E is a reflexive real Banach space, there exists a subsequence $\{x_{n_{k_m}}\}_{m\geq 1}$ of $\{x_{n_k}\}_{k\geq 1}$ such that $\{x_{n_{k_m}}\}_{m\geq 1}$ converges weakly to some $p^* \in E$. Without loss of generality, we may assume that n_{k_m} is such that $T_{n_{k_m}} = T_i$ for some $i \in \{1, 2, ..., r\}$, for all $m \geq 1$. It therefore follows from (4.11) that

$$\lim_{m\to\infty} \|x_{n_{k_m}} - T_{i+r}...T_{i+2}T_{i+1}x_{n_{k_m}}\| = 0.$$

So, by Lemma 2.7, $p^* \in F(T_{i+r}...T_{i+2}T_{i+1}) = \Omega$. Thus, since E has weakly sequential continuous generalized duality mapping, we have that

$$\begin{aligned} \limsup_{n\to\infty} \Big\langle u - \lambda Ax', j_q(x_n - x') \Big\rangle &= \lim_{k\to\infty} \Big\langle u - \lambda Ax', j_q(x_{n_k} - x') \Big\rangle \\ &= \lim_{m\to\infty} \Big\langle u - \lambda Ax', j_q(x_{n_{k_m}} - x') \Big\rangle \\ &= \Big\langle u - \lambda Ax', j_q(p^* - x') \Big\rangle \leq 0 \end{aligned} \tag{4.12}$$

Thus, setting

$$\theta_n = \max\Big\{0, \Big\langle u - \lambda Ax', j_q(x_n - x') \Big\rangle\Big\},$$

then it is easy to see that $\lim_{n\to\infty}\theta_n = 0$. Furthermore, we obtain from the recursion formula (4.1) using Lemma 2.1 that

$$\begin{aligned}\|x_{n+1}-x'\|^q &\le \Big(1-\alpha_{n+1}(q\eta\lambda - d_q(\lambda L)^q)\Big)\|x_n - x'\|^q \\ &\quad + q\alpha_{n+1}\Big\langle u-\lambda Ax', j_q(x_{n+1}-x')\Big\rangle \\ &\le \Big(1-\alpha_{n+1}(q\eta\lambda - d_q(\lambda L)^q)\Big)\|x_n - x'\|^q + \delta_n\end{aligned}$$

$\forall\ n\ge 0$, where $\delta_n = \alpha_{n+1}q(q\eta\lambda - d_q(\lambda L)^q)\Big[\frac{\theta_{n+1}}{q\eta\lambda - d_q(\lambda L)^q}\Big]$, which is clear $o(\alpha_{n+1})$. Hence, by Lemma 2.2, $\{x_n\}_{n\ge 1}$ converges strongly to $x'\in\Omega$ which is a unique solution of (4.2). This completes the Proof. $\square$

Remark 4.2. We note that some authors (see e.g., Xu [45]) had made the assumption that

$$\Omega = F(T_rT_{r-1}...T_1) = F(T_1T_{r-1}...T_2) = ... = F(T_{r-1}...T_1T_r).$$

We now consider a situation where this condition is dispensed with.

Example 4.3. Let E be a strictly convex q-uniformly smooth real Banach space which admits weakly sequentially continuous generalized duality mapping, let $A: E\to E$ be an L-Lipschitzian strongly accretive mapping with a constant $\eta>0$. Let $T_i : E\to E, i = 1,2,...,r$ be a finite family of nonexpansive mappings such that $\Omega := \bigcap_{i=1}^{r} F(T_i)\neq\emptyset$ and $S_i = (1-\omega_i)I + \omega_i T_i, i = 1,2,...,r$. Let $u\in E$ be fixed and let $\{x_n\}_{n\ge 1}$ be a sequence in E generated iteratively by

$$x_0\in E, x_{n+1} = \alpha_{n+1}u + (I-\alpha_{n+1}\lambda A)S_{n+1}x_n,\ n\ge 0, \quad (4.13)$$

where $S_n = S_{n \bmod r}$ and mod function takes values in $\{1,2,...,r\}$, then $\{x_n\}_{n\ge 0}$ converges strongly to a solution of the variational inequality (4.2)

Proof. By Lemma 2.4, $\bigcap_{i=1}^{r} F(S_i) = \bigcap_{i=1}^{r} F(T_i)$ and $\bigcap_{i=1}^{r} F(S_i)$ $= F(S_rS_{r-1}...S_1) = F(S_1S_{r-1}...S_2) = ... = F(S_{r-1}...S_1S_r)$. The rest follows as in the proof of Theorem 4.1. $\square$

Corollary 4.4. *Let E be a q-uniformly smooth real Banach space which admits weakly sequentially continuous generalized duality mapping, let $A: E\to E$ be an L-Lipschitzian strongly accretive mapping with a constant $\eta>0$. Let $T: E\to E$, be a nonexpansive* *mapping such that* $F(T) \neq \emptyset$*. Let* $u \in E$ *be fixed and let* $\{x_n\}_{n\geq 1}$ *be a sequence in* E *generated iteratively by*

$$x_0 \in E, x_{n+1} = \alpha_{n+1}u + (I - \alpha_{n+1}\lambda A)Tx_n,\ n \geq 0, \tag{4.14}$$

then $\{x_n\}_{n\geq 0}$ *converges strongly to a unique* $x' \in F(T)$ *which is a solution of the variational inequality* (3.6)

Proof. Follows as in the proof of Theorem 4.1 using Theorem 3.5. □

Remark 4.5. We remark that corollaries synonymous to Corollaries 3.6, 3.8, 3.9 and 3.10 are obtainable in this section. But we must note, however, that though Theorem 4.1, Corollary 4.3 and Corollary 4.4 hold in the real sequence space ℓ^q, they do not hold in $L^q(\mathbb{R})$ for $1 < q < +\infty,\ q \neq 2$ since $L^q(\mathbb{R}),\ q \neq 2$ do not possess weakly sequentially continuous duality mapping. All our theorems thus hold in real Hilbert space.

5. Applications

Convergence Theorem for families of strictly pseudocontractive mappings.

Let E be a normed space. A mapping $T : E \to E$ is called *k-strictly pseudocontractive* if and only if there exists a real constant $k > 0$ such that for all $x, y \in D(T)$ there exists $j(x-y) \in J(x-y)$ such that

$$\big\langle Tx - Ty, j(x-y)\big\rangle \leq \|x-y\|^2 - k\|x - y - (Tx - Ty)\|^2. \tag{5.1}$$

Without loss of generality we may assume that $k \in (0,1)$. If I denotes the identity operator, then (5.1) can be re-written as

$$\big\langle (I-T)x - (I-T)y, j(x-y)\big\rangle \geq k\|(I-T)x - (I-T)y)\|^2. \tag{5.2}$$

In Hilbert spaces, (5.1) (or equivalently (5.2)) is equivalent to the inequality

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + \beta\|(I-T)x - (I-T)y\|^2,$$

where $\beta = (1-k) < 1$. It was shown in [30] that if T is k-strictly pseudocontractive, then the following inequality holds

$$\begin{aligned}&\big\langle (I-T)x - (I-T)y, j_q(x-y)\big\rangle \\ &\qquad\qquad \geq k^{q-1}\|(I-T)x - (I-T)y)\|^q.\end{aligned} \tag{5.3}$$

Thus, if E is a q-uniformly smooth real Banach space; and $T : E \to E$ is a k-strictly pseudocontractive mapping, then for the map $T_a := (1-a)I + aT) : E \to E$ (where I is the identity map of E and $a > 0$), we obtain by Lemma 2.5 using (5.3) that:

$$\begin{aligned}\|T_ax - T_ay\|^q &= \|x - y - a\Big((I-T)x - (I-T)y\Big)\|^q \\ &\leq \|x-y\|^q - qa\big\langle (I-T)x - (I-T)y, j_q(x-y)\big\rangle \\ &\quad + d_q a^q \|(I-T)x - (I-T)y\|^q \\ &\leq \|x-y\|^q - a(k^{q-1}q - d_q a^{q-1})\|Ax - Ay\|^q,\end{aligned}$$

where $A = (I-T)$. If a is such that $0 < a < \left(\frac{qk^{q-1}}{d_q}\right)^{\frac{1}{q-1}}$, we have that the mapping T_a is nonexpansive. It is also easy to see that the fixed point set of T_a and that of T coincide.

Thus, we have the following theorem

Theorem 5.1. *Let E be a q-uniformly smooth real Banach space which admits weakly sequential continuous generalized duality mapping, let $A : E \to E$ be an L-Lipschitzian strongly accretive mapping with a constant $\eta > 0$. Let $T_i : E \to E, i = 1, 2, ..., r$ be a finite family of k-striclty pseudocontractive mappings such that $\Omega^* := \bigcap\limits_{i=1}^{r} F(T_i) \neq \emptyset$. Let $\{a_i\}_{i=1}^r$ be such that $0 < a_i < \left(\frac{qk^{q-1}}{d_q}\right)^{\frac{1}{q-1}}, i = 1, 2, ..., r$ and define $T_{a_i} = (1-a_i)I + a_iT_i$. Let $u \in E$ be fixed and $\{x_n\}_{n\geq 1}$ be a sequence in E generated iteratively by*

$$x_0 \in E, x_{n+1} = \alpha_{n+1}u + (I - \alpha_{n+1}\lambda A)T_{a_{n+1}}x_n, \ n \geq 0, \tag{5.4}$$

where $T_{a_n} = T_{a_{n \bmod r}}$ and mod function takes values in $\{1, 2, ..., r\}$. Suppose that $\Omega^* = F(T_{a_r}T_{a_{r-1}}...T_{a_1}) = F(T_{a_1}T_{a_{r-1}}...T_{a_2}) = ... = F(T_{a_{r-1}}...T_{a_1}T_{a_r})$, then, $\{x_n\}_{n\geq 0}$ converges strongly to a solution of the variational inequality

$$\Big\langle u - \lambda Ax', j_q(p - x')\Big\rangle \leq 0 \ \forall \ p \in \Omega^*. \tag{5.5}$$

.

Corollary 5.2. *Let E be a strictly convex q-uniformly smooth real Banach space which admits weakly sequentially continuous generalized duality mapping, let $A : E \to E$ be an L-Lipschitzian strongly accretive mapping with a constant $\eta > 0$. Let $T_i : E \to E, i = 1, 2, ..., r$ be a finite family of k-striclty pseudocontractive mappings such that $\Omega := \bigcap\limits_{i=1}^{r} F(T_i) \neq \emptyset$. Let $\{a_i\}_{i=1}^r$ be such that $0 < a_i < \left(\frac{qk^{q-1}}{d_q}\right)^{\frac{1}{q-1}}, i = 1, 2, ..., r$ and define $T_{a_i} = (1-a_i)I + a_iT_i$ and*

$S_{a_i} = (1-\omega_i)I + \omega_i T_{a_i}, i = 1, 2, ..., r$. Let $u \in E$ be fixed and let $\{x_n\}_{n\geq 1}$ be a sequence in E generated iteratively by

$$x_0 \in E, x_{n+1} = \alpha_{n+1}u + (I - \alpha_{n+1}\lambda A)S_{a_{n+1}}x_n, \; n \geq 0, \qquad (5.6)$$

where $S_{a_n} = S_{n \bmod r}$ and mod function takes values in $\{1, 2, ..., r\}$. Then, $\{x_n\}_{n\geq 0}$ converges strongly to a solution of the variational inequality (5.5).

Corollary 5.3. *Let E be a q-uniformly smooth real Banach space which admits weakly sequentially continuous generalized duality mapping, let $A : E \to E$ be an L-Lipschitzian strongly accretive mapping with a constant $\eta > 0$. Let $T : E \to E$, be a k-striclty pseudocontractive mapping such that $F(T) \neq \emptyset$. For some $0 < a < \left(\frac{qk^{q-1}}{d_q}\right)^{\frac{1}{q-1}}$ and define $T_a = (1-a)I + aT$. Let $u \in E$ be fixed and let $\{x_n\}_{n\geq 1}$ be a sequence in E generated iteratively by*

$$x_0 \in E, x_{n+1} = \alpha_{n+1}u + (I - \alpha_{n+1}\lambda A)T_a x_n, \; n \geq 0, \qquad (5.7)$$

then, $\{x_n\}_{n\geq 0}$ converges strongly to a unique $x' \in F(T)$ which is a solution of the variational inequality (3.6)

Remark 5.4. Prototype for our iteration parameter $\{\alpha_n\}_{n\geq 1}$ (see e.g. [45]) is given by

$$\alpha_n = \begin{cases} \frac{1}{\sqrt{n}} \; if \; n \text{ is odd,} \\ \\ \frac{1}{\sqrt{n}-1} \; if \; n \text{ is even.} \end{cases}$$

If we assume that r is odd, since the case r being even is similar, it is not difficult to see that

$$\frac{\alpha_n}{\alpha_{n+r}} = \begin{cases} \frac{\sqrt{n+r}-1}{\sqrt{n}} \; if \; n \text{ is odd,} \\ \\ \frac{\sqrt{n+r}}{\sqrt{n}-1} \; if \; n \text{ is even.} \end{cases}$$

Remark 5.5. It is easy to see that Corollary 4.3 is an obvious improvement on the corresponding results of [45] in the sense that the condition

$$\Omega = F(T_rT_{r-1}...T_1) = F(T_1T_{r-1}...T_2) = ... = F(T_{r-1}...T_1T_r)$$

imposed in [45] is dispensed with.

Remark 5.6. When $E = H$, a Hilbert space, $\lambda = 1$ and A is a bounded linear strongly positive operator, our iteration process (4.1) reduces to the iteration scheme studied by Xu [45]. If the fixed vector $u \in E$ is identically equal to the zero vector of H; if the mapping A a strongly monotone operator; and we consider a single nonexpansive mapping, (4.1) reduces to the scheme studied by Yamada [46]. We recall that the results of Xu [45] and Yamada [46] remain in Hilbert spaces. Our theorems, therefore, complement the results of these authors.

Remark 5.7. It is well known that in a uniformly convex and uniformly smooth real Banach space, $x' \in F(T)$ is a solution of variational inequality $\big\langle (\gamma f - \mu A)x', y - x' \big\rangle \leq 0 \;\forall\; y \in F(T)$ if and only if $P_{F(T)}\big(I - (\gamma f - \mu A)\big)x' = x'$, where $P_{F(T)}$ is the metric projection of E onto $F(T)$. Furthermore, if $E = H$ is a Hilbert space, we can easily show that if $0 < \mu < \frac{2\eta}{L^2}$ (where L is the Lipschitz constant of the strongly accretive operator A), then the mapping $P_{F(T)}\big(I - (\gamma f - \mu A)$ is a strict contraction on H (using the fact that the metric projection $P_{F(T)}$ is nonexpansive in this case). Thus, *Banach contraction mapping principle* gives existence of a unique $x^* \in H$ such that $P_{F(T)}\big(I - (\gamma f - \mu A)x^* = x^*$. One may therefore wonder why the Picard's iterative method given by

$$(**) \quad x_0 \in H,\; x_{n+1} = P_{F(T)}\big(I - (\gamma f - \mu A)x_n,\; n \geq 0$$

was not employed for this problem in Hilbert space. It happens that theoretically, the Picard's iteration method $(**)$ works, but in application, it seems difficult to be used since the projection operator $P_{F(T)}$ is not readily handy. Besides, we must not that the projection operator is not necessarily nonexpansive in spaces more general than Hilbert space. Hence, construction of iterative methods which do not involve metric projections become a necessity.

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DEPARTMENT OF MATHEMATICS, NNAMDI AZIKIWE UNIVERSITY, P. M. B. 5025, AWKA, ANAMBRA STATE, NIGERIA.

E-mail address: `euofoedu@yahoo.com`