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HYPONORMALITY-PRESERVING FINITE RANK PERTURBATIONS OF TERRACED MATRICES

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ABSTRACT. Suppose M is a terraced matrix that is a hyponormal bounded linear operator on ℓ^2. Here we determine conditions under which there exists a finite rank terraced matrix $F \neq 0$ such that $M + F$ is also hyponormal. Two different approaches are employed. One approach uses Sylvester's criterion, and the other uses the recently defined concept of supraposinormality. Examples include generalized Cesàro operators of order one and terraced matrices associated with some logistic sequences.

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1. INTRODUCTION

A lower triangular matrix M, acting as a bounded operator on ℓ^2, is *terraced* if its row segments are constant (see [**2**], [**3**]). Recall that M is *hyponormal* if it satisfies

$$[M^*, M] :\equiv M^*M - MM^* \geq 0.$$

The best-known hyponormal terraced matrix is the Cesàro matrix (see [**1**]), whose row segments are given by the sequence

$$\{\tfrac{1}{n+1} : n \geq 0\}.$$

Let $M(a)$ denote the terraced matrix whose row segments are given by the positive-termed sequence $a :\equiv \{a_n\}_{n=0}^{\infty}$. The following example is what motivated this paper.

Example 1.1: (Modified Cesàro matrix). Consider the terraced matrix $M(b)$ with $b :\equiv \{b_n\}$ given by $b_n = \frac{1}{n+1}$ for all $n \geq 1$ and $b_0 > 0$. In [**5**] it was demonstrated that $M(b)$ is not hyponormal for $b_0 \neq 1$.

It seems natural to ask whether this fragile behavior is typical of hyponormal terraced matrices, and that has led to the following question. As usual, $B(H)$ denotes the set of all bounded linear operators on the Hilbert space H.

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Question 1.2: Given a hyponormal terraced matrix $M \in B(\ell^2)$, when is it possible to identify a finite rank operator $F \neq 0$ such that $M + F$ is also a hyponormal terraced matrix?

In answering this question, two distinct approaches will be employed. The approach in Section 2 involves determinants, while that of Section 3 avoids them.

2. USING SYLVESTER'S CRITERION

Suppose that $a :\equiv \{a_n\}$ is a sequence of positive numbers such that the terraced matrix $M :\equiv M(a) \in B(\ell^2)$ is hyponormal, so $[M^*, M]$ is a positive operator; by Sylvester's criterion, all the leading principal minors of $[M^*, M]$ must be positive. The entries of $[M^*, M] :\equiv [\Delta_{mn}]$ are given by

$$\Delta_{mn} = \begin{cases} \sum_{k=m}^{\infty} a_k^2 - (n+1)a_m a_n & if \quad m > n; \\ \sum_{k=n}^{\infty} a_k^2 - (n+1)a_n^2 & if \quad m = n; \\ \sum_{k=n}^{\infty} a_k^2 - (m+1)a_m a_n & if \quad m < n. \end{cases}$$

If U denotes the unilateral shift and n is a fixed positive integer, then since M is hyponormal, $(U^*)^n[M^*, M]U^n$ is also a positive operator, and again by Sylvester's criterion, all the leading principal minors of $(U^*)^n[M^*, M]U^n$ must be positive.

Proposition 2.1: Assume that the entries a_n of the terraced matrix $M :\equiv M(a) \in B(\ell^2)$ satisfy $2a_1 \neq 3a_2$. If M is hyponormal and $M(b)$ denotes the terraced matrix associated with the sequence $\{b_n\}$ satisfying $b_0 = 3a_2$ and $b_1 = \frac{3}{2}a_2$, while $b_n = a_n$ for all $n \geq 2$, then $M(b)$ is hyponormal.

Proof: First we note that

$$\sum_{k=j}^{\infty} a_k^2 - (j+1)a_j^2 \geq 0$$

for each j, since these are the diagonal elements of the self-commutator of the hyponormal operator $M(a)$. Let $S_n(b)$ denote the n^{th} finite section of $[M^*(b), M(b)]$, where n = 0, 1, 2, ... It must be shown that $det(S_n(b)) \geq 0$ for all $n \geq 0$. It is clear that

$$det(S_0(b)) = \sum_{k=0}^{\infty} b_k^2 - b_0^2 = \sum_{k=1}^{\infty} b_k^2 > 0.$$

For $S_1(b)$, we subtract the second column from the first and obtain an upper triangular matrix with the same determinant as $S_1(b)$, so

$$det(S_1(b)) = b_0 b_1 (\sum_{k=1}^{\infty} b_k^2 - 2b_1^2) = \frac{9}{2}a_2^2(\sum_{k=2}^{\infty} a_k^2 - \frac{9}{4}a_2^2) > \frac{9}{2}a_2^2(\sum_{k=2}^{\infty} a_k^2 - 3a_2^2) \geq 0.$$

For $n \geq 2$, we subtract the second column of $S_n(b)$ from the first column, then the third column from the second, and we obtain $S_n^{'}(b)$. Observe that $S_2^{'}(b)$ is an upper triangular matrix, so

$$det(S_2(b)) = det(S_2^{'}(b)) = b_0 b_1 (b_1)(2b_2 - b_1)(\sum_{k=2}^{\infty} b_k^2 - 3b_2^2) =$$

$$\frac{27}{8}a_2^4(\sum_{k=2}^{\infty} a_k^2 - 3a_2^2) \geq 0.$$

Before proceeding, we illustrate the situation for $n = 3$ with

$$S_3(b) = \begin{pmatrix} \sum_{k=0}^{\infty} b_k^2 - b_0^2 & \sum_{k=1}^{\infty} b_k^2 - b_0b_1 & \sum_{k=2}^{\infty} b_k^2 - b_0b_2 & \sum_{k=3}^{\infty} b_k^2 - b_0b_3 \\ \sum_{k=1}^{\infty} b_k^2 - b_0b_1 & \sum_{k=1}^{\infty} b_k^2 - 2b_1^2 & \sum_{k=2}^{\infty} b_k^2 - 2b_1b_2 & \sum_{k=3}^{\infty} b_k^2 - 2b_1b_3 \\ \sum_{k=2}^{\infty} b_k^2 - b_0b_2 & \sum_{k=2}^{\infty} b_k^2 - 2b_1b_2 & \sum_{k=2}^{\infty} b_k^2 - 3b_2^2 & \sum_{k=3}^{\infty} b_k^2 - 3b_2b_3 \\ \sum_{k=3}^{\infty} b_k^2 - b_0b_3 & \sum_{k=3}^{\infty} b_k^2 - 2b_1b_3 & \sum_{k=3}^{\infty} b_k^2 - 3b_2b_3 & \sum_{k=3}^{\infty} b_k^2 - 4b_3^2 \end{pmatrix}$$

and the corresponding transformed matrix with the same determinant,

$$S_3^{'}(b) = \begin{pmatrix} b_0b_1 & * & * & * \\ 0 & b_1(2b_2 - b_1) & * & * \\ 0 & 0 & \sum_{k=2}^{\infty} b_k^2 - 3b_2^2 & \sum_{k=3}^{\infty} b_k^2 - 3b_2b_3 \\ 0 & 0 & \sum_{k=3}^{\infty} b_k^2 - 3b_2b_3 & \sum_{k=3}^{\infty} b_k^2 - 4b_3^2 \end{pmatrix}.$$

For $n \geq 3$, the entries below the main diagonal in the first two columns of $S_n^{'}(b)$ will all be 0, so

$$det(S_n(b)) = det(S_n^{'}(b)) = b_0b_1[b_1(2b_2 - b_1)]det(T_{n-2}(b)) = \frac{27}{8}a_2^4 det(T_{n-2}(b))$$

where $T_n(b)$ is the $n^{th}(n = 0, 1, 2, ...)$ finite section of

$$(U^*)^2[M(b)^*, M(b)]U^2 = (U^*)^2[M(a)^*, M(a)]U^2.$$

Since $det(T_n(b)) \geq 0$ for all n, it follows that $det(S_n(b)) \geq 0$ for all $n \geq 3$.

Lemma 2.2: Assume that the entries a_n of the terraced matrix $M :\equiv M(a) \in B(\ell^2)$ satisfy $(N+1)a_N \neq (N+2)a_{N+1}$ for some N. If M is hyponormal and $M(b)$ denotes the terraced matrix associated with the sequence $\{b_n\}$ satisfying $b_n = \frac{N+2}{n+1}a_{N+1}$ for $n = 0, 1, ..., N$ and $b_n = a_n$ for all $n \geq N + 1$, then

$$\sum_{k=j}^{\infty} b_k^2 - (j + 1)b_j^2 \geq 0 \text{ for all } j.$$

Proof: For $j \geq N + 1$, we have

$$\sum_{k=j}^{\infty} b_k^2 - (j + 1)b_j^2 = \sum_{k=j}^{\infty} a_k^2 - (j + 1)a_j^2 \geq 0$$

since $M(a)$ is hyponormal.

For $j = 0$ the result is clear. In the case $0 < j \leq N$, we have

$$\sum_{k=j}^{\infty} b_k^2 - (j + 1)b_j^2 =$$
$$\sum_{k=j}^{N}(\frac{N+2}{k+1}a_{N+1})^2 + \sum_{k=N+1}^{\infty} a_k^2 - (j + 1)(\frac{N+2}{j+1}a_{N+1})^2 =$$
$$\sum_{k=N+1}^{\infty} a_k^2 + \{(\frac{1}{j+1})^2 + (\frac{1}{j+2})^2 + ... + (\frac{1}{N+1})^2 - \frac{1}{j+1}\}(N + 2)^2a_{N+1}^2$$
$$\geq \sum_{k=N+1}^{\infty} a_k^2 - (N + 2)a_{N+1}^2 \geq 0,$$

again because of the hyponormality of $M(a)$.

Theorem 2.3: Assume that the entries a_n of the terraced matrix $M :\equiv M(a) \in B(\ell^2)$ satisfy $(N+1)a_N \neq (N+2)a_{N+1}$ for some N. If M is hyponormal and $M(b)$ denotes the terraced matrix associated with the sequence $\{b_n\}$ satisfying $b_n = \frac{N+2}{n+1}a_{N+1}$ for $n = 0, 1, ..., N$ and $b_n = a_n$ for all $n \geq N + 1$, then $M(b)$ is hyponormal.

Proof: The proof is modeled on that of Proposition 2.1, which handled the case $N = 1$. First we note that Lemma 2.2 guarantees that the diagonal elements of $[M(b)^*, M(b)]$ are all positive. For $1 \leq n \leq N$, $S'_n(b)$ is obtained from $S_n(b)$ by successively subtracting the $(k+1)^{st}$ column from the k^{th} column for each $k \in \{1, ..., n\}$. For $n \geq N+1$, $S'_n(b)$ is obtained from $S_n(b)$ by successively subtracting the $(k+1)^{st}$ column from the k^{th} column for each $k \in \{1, ..., N+1\}$. Note that when $1 \leq n \leq N+1$, $S'_n(b)$ is upper triangular; when $n \geq N+2$, the entries below the main diagonal in the first $N+1$ columns of $S'_n(b)$ will all be 0. Note also that $det(S_n(b)) = det(S'_n(b))$ for all $n \geq 1$. Now we are ready to show that $det(S_n(b)) \geq 0$ for all n. Once again, the case $n = 0$ is clear. For $1 \leq n \leq N$, we have

$$det(S_n(b)) = \{\prod_{j=0}^{n-1} \frac{1}{(j+1)^2(j+2)}\}[(N+2)a_{N+1}]^{2n}[\sum_{j=n}^{\infty} b_j^2 - (n+1)b_n^2] \geq 0.$$

For $n \geq N+1$, we have

$$det(S_n(b)) = \{\prod_{j=0}^{N} \frac{1}{(j+1)^2(j+2)}\}[(N+2)a_{N+1}]^{2(N+1)} det(T_{n-(N+1)}(b)) \geq 0,$$

where $T_n(b)$ is the $n^{th}(n = 0, 1, 2, ...)$ finite section of the positive operator

$$(U^*)^{N+1}[M(b)^*, M(b)]U^{N+1} = (U^*)^{N+1}[M(a)^*, M(a)]U^{N+1}.$$

Note that $F :\equiv M(b) - M(a) \neq 0$ is clearly a finite rank terraced matrix. The following corollary provides an answer to the question posed in the Introduction.

Corollary 2.4: If $a :\equiv \{a_n\}$ is a strictly decreasing positive sequence and $M(a)$ is a hyponormal terraced matrix that is not a scalar multiple of the Cesàro matrix, then there exists a finite rank terraced matrix $F \neq 0$ such that $M(a) + F$ is also hyponormal.

For $0 \leq t \leq 1$, let $M(t)$ denote the terraced matrix that arises from $M(a)$ when a_0 is replaced by $ta_0 + (1-t)(2a_1)$; the rest of the entries of $M(a)$ are left unchanged.

Theorem 2.5: If $M(a)$ is hyponormal for $a :\equiv \{a_n\}$ strictly decreasing and $a_0 \neq 2a_1$, then $M(t)$ is hyponormal for all $t \in [0, 1]$.

Proof: First we note that $M(t)$ is hyponormal when $t = 0$ by Theorem 2.3 and when $t = 1$ by hypothesis, so we focus our attention on the case $0 < t < 1$. Replace a_0 in $[M(a)^*, M(a)]$ by $ta_0 + (1-t)(2a_1)$ to obtain $[M(t)^*, M(t)]$. Let $S_n(t)$ denote the n^{th} finite section of $[M(t)^*, M(t)]$. Clearly,

$$det(S_0(t)) = \sum_{k=1}^{\infty} a_k^2 > 0.$$

For $n \geq 1$, we subtract the second column of $S_n(t)$ from the first to obtain $S_n^{'}(t)$, and then subtract the second row of $S_n^{'}(t)$ from the first to obtain $S_n''(t)$. To illustrate what happens next, we consider

$$S_2^{''}(t) = \begin{pmatrix} 2ta_1(a_0 - 2a_1) + 2a_1^2 & ta_1(2a_1 - a_0) & ta_2(2a_1 - a_0) \\ ta_1(2a_1 - a_0) & \sum_{k=1}^{\infty} a_k^2 - 2a_1^2 & \sum_{k=2}^{\infty} a_k^2 - 2a_1a_2 \\ ta_2(2a_1 - a_0) & \sum_{k=2}^{\infty} a_k^2 - 2a_1a_2 & \sum_{k=2}^{\infty} a_k^2 - 3a_2^2 \end{pmatrix}.$$

It follows that $det(S_2(t)) = det(S_2''(t)) = t^2(detX_2(t) + detY_2(t))$ where

$$X_2(t) :\equiv \begin{pmatrix} 2a_1(a_0 - a_1) & a_1(2a_1 - a_0) & a_2(2a_1 - a_0) \\ a_1(2a_1 - a_0) & \sum_{k=1}^{\infty} a_k^2 - 2a_1^2 & \sum_{k=2}^{\infty} a_k^2 - 2a_1a_2 \\ a_2(2a_1 - a_0) & \sum_{k=2}^{\infty} a_k^2 - 2a_1a_2 & \sum_{k=2}^{\infty} a_k^2 - 3a_2^2 \end{pmatrix}$$

and

$$Y_2(t) :\equiv \begin{pmatrix} 2a_1[(\frac{1}{t} - 1)a_0 + (1 - \frac{1}{t})^2 a_1] & a_1(2a_1 - a_0) & a_2(2a_1 - a_0) \\ 0 & \sum_{k=1}^{\infty} a_k^2 - 2a_1^2 & \sum_{k=2}^{\infty} a_k^2 - 2a_1a_2 \\ 0 & \sum_{k=2}^{\infty} a_k^2 - 2a_1a_2 & \sum_{k=2}^{\infty} a_k^2 - 3a_2^2 \end{pmatrix}.$$

Note that $det(X_2(t)) = det(S_2''(1)) \geq 0$ since $M(a)$ is hyponormal, and observe that

$$\begin{aligned} det(Y_2(t)) &= 2a_1[(\tfrac{1}{t} - 1)a_0 + (1 - \tfrac{1}{t})^2 a_1] \begin{vmatrix} \sum_{k=1}^{\infty} a_k^2 - 2a_1^2 & \sum_{k=2}^{\infty} a_k^2 - 2a_1a_2 \\ \sum_{k=2}^{\infty} a_k^2 - 2a_1a_2 & \sum_{k=2}^{\infty} a_k^2 - 3a_2^2 \end{vmatrix} \\ &\geq 0 \end{aligned}$$

since the leading principal minors of $U^*[M(a)^*, M(a)]U$ are all positive. A similar argument holds for each $n \geq 1$, so $det(S_n(t)) \geq 0$, and it follows that $M(t)$ is hyponormal for $0 < t < 1$. This completes the proof.

Corollary 2.6: Suppose $M(a)$ and $M(b)$ are hyponormal terraced matrices associated with positive decreasing sequences $a :\equiv \{a_n\}$ and $b :\equiv \{b_n\}$ satisfying $a_0 \neq b_0$ and $a_n = b_n$ for all $n \geq 1$. If $c :\equiv \{c_n\}$ satisfies $c_n = a_n$ for all $n \geq 1$ and c_0 is between a_0 and b_0, then $M(c)$ is also a hyponormal terraced matrix.

Example 2.7: (Generalized Cesàro matrix of order one). Recall that for fixed $k > 0$, the generalized Cesàro matrices of order one are the terraced matrices $C_k :\equiv M(a)$ that occur when $a :\equiv \{a_n\}$ satisfies $a_n = \frac{1}{k+n}$ for all n. C_k is hyponormal for $k \geq 1$; see [**8**].

(a) Suppose $k > 1$. If $M(b)$ is the terraced matrix associated with the sequence $b :\equiv \{b_n\}$ defined by $b_0 = \frac{2}{k+1}$ and $b_n = a_n$ for all $n \geq 1$, then Theorem 2.3 tells us that $M(b)$ is hyponormal.

(b) For $k > 1$ and $0 \leq t \leq 1$, take $b_0 = t(\frac{1}{k}) + (1 - t)(\frac{2}{k+1})$ and $b_n = a_n$ for all $n \geq 1$. By Theorem 2.5, $M(b)$ is hyponormal.

(c) For $k > 1$, define $b_0 = \frac{3}{k+2}$, $b_1 = \frac{\frac{3}{2}}{k+2}$, and $b_n = a_n$ for all $n \geq 2$. By Theorem 2.3, $M(b)$ is hyponormal.

3. USING SUPRAPOSINORMALITY

In this section we will make use of a recently-defined concept (see [**7**]) that does not require determinants.

Definition 3.1: If H is a Hilbert space and $A \in B(H)$, then A is *supraposinormal* with interrupter pair (Q, P) if there exist positive operators P, $Q \in B(H)$ such that $AQA^* = A^*PA$, where at least one of P, Q has dense range.

The following proposition will aid in the proof of our next theorem.

Proposition 3.2: Assume that the terraced matrix $M = M(\{a_n\})$ is a bounded operator on ℓ^2 and $\{a_n\}$ is strictly decreasing to 0. If

$$P :\equiv diag\{\frac{a_n - a_{n+1}}{a_n^2} : n = 0, 1, 2, ...\} \in B(\ell^2)$$

and

$$Q :\equiv diag\{\frac{1}{a_0}, \frac{a_n - a_{n+1}}{a_n a_{n+1}} : n = 0, 1, 2, ...\} \in B(\ell^2),$$

then M is supraposinormal with interrupter pair (Q, P).

Proof: Once the hypothesis is assumed, it is straightforward to verify that $MQM^* = M^*PM$ (reverse L-shaped). Clearly the positive operators P and Q are one-to-one, so they both have dense range; thus M is supraposinormal with interrupter pair (Q, P).

We are now ready to present restrictions on the sequence $a :\equiv \{a_n\}$ that are sufficient to guarantee that the terraced matrix $M :\equiv M(a) \in B(\ell^2)$ is hyponormal.

Theorem 3.3: Assume that the terraced matrix M is a bounded operator on ℓ^2, $\{a_n\}$ is strictly decreasing to 0, and $\{\frac{a_n}{a_{n+1}}\}$ is bounded. If $0 < a_0 < 1$ and

$$a_n(1 - a_n) \leq a_{n+1} \leq \frac{a_n}{1 + a_n}$$

for each nonnegative integer n, then M is hyponormal.

Proof: By Proposition 3.2, we have $MQM^* = M^*PM$. Note that $I - P \geq 0$ since $a_n(1 - a_n) \leq a_{n+1}$ for all n, and $Q - I \geq 0$ since $0 < a_0 < 1$ and $a_{n+1} \leq \frac{a_n}{1+a_n}$ for all n. It follows that

$$\langle [M^*, M]f, f\rangle = \langle (I - P)Mf, Mf\rangle + \langle (Q - I)M^*f, M^*f\rangle \geq 0$$

for all f, so M is hyponormal.

We note that Proposition 3.2 and Theorem 3.3 are special cases of results presented in [**7**]. A result similar to Theorem 3.3 appears in [**4**], obtained using a different approach.

Example 3.4: (a) (A logistic sequence). Consider the terraced matrix $M(a)$ associated with the sequence $a :\equiv \{a_n\}$ determined as follows: Choose $a_0 = \frac{1}{2}$ and $a_{n+1} = a_n(1 - a_n)$ for each $n \geq 1$. Note that $a_1 = \frac{1}{4}$ and $a_2 = \frac{3}{16}$. We know that $M(a)$ is hyponormal by Theorem 3.3.

(b) Let $M(b)$ denote the terraced matrix associated with a sequence $\{b_n\}$ such that $\frac{1}{3} \leq b_0 < 1$ and $b_n = a_n$ for all $n \geq 1$. Then $M(b)$ will also be hyponormal by Theorem 3.3.

(c) Let $M(b)$ denote the terraced matrix associated with a sequence $\{b_n\}$ satisfying $b_n = a_n$ for all $n \geq 2$ but $b_0 \neq a_0$ and $b_1 \neq a_1$. If we take $b_0 = \frac{3}{10}$ and $b_1 = \frac{3}{13}$, then $M(b)$ is hyponormal by Theorem 3.3.

The next example involves both Theorem 2.3 and Theorem 3.3.

Example 3.5: Let $M(a)$ denote the terraced matrix with $a_0 = \frac{1}{10}$ and $a_{n+1} = a_n(1 - a_n)$ for all $n > 0$. By Theorem 3.3, $M(a)$ is hyponormal. Note that $a_1 = \frac{9}{100}$, so $a_0 \neq 2a_1$. If we define $\{b_n\}$ so that $b_0 = 2a_1 = \frac{9}{50}$ and $b_n = a_n$ for all $n \geq 1$, then $M(b)$ is hyponormal by Theorem 2.3. It is interesting to note, however, that $M(b)$ does not satisfy the hypothesis of Theorem 3.3 since $b_1 < b_0(1 - b_0)$.

4. REMARKS

If C_1 denotes the Cesàro matrix and $Q :\equiv diag\{q, 1, 1, 1, 1, 1, ...\}$ where $\frac{1}{2} \leq q < 1$, it follows from [**6**, Proposition 1] that $\sqrt{Q}C_1\sqrt{Q}$ is a hyponormal operator, although it is not a terraced matrix. However, $\sqrt{Q}C_1\sqrt{Q}$ is factorable, as is C_1; recall that a lower triangular matrix $M :\equiv M(\{a_i\}, \{c_j\})$ is *factorable* if its nonzero entries m_{ij} satisfy $m_{ij} = a_i c_j$ where a_i depends only on i and c_j depends only on j. Also, since $\sqrt{Q}C_1\sqrt{Q} - C_1$ has nonzero entries only in the first column, it is not hard to verify that this difference is a rank one operator and is therefore compact.

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