Journal of the Nigerian Mathematical Society
Vol. 32, pp. 289-301, 2013

On n-WEAK AMENABILITY OF SEMIGROUP ALGEBRAS

O. T. MEWOMO[1] AND O. J. OGUNSOLA

ABSTRACT. In this work, we examine the n-weak amenability of some semigroup algebras and the second dual A'' of a Banach algebra A. Let S be a semigroup, for the Banach semigroup algebra $\ell^1(S)$ on S, we show that, in the case where S is right zero semigroup, $\ell^1(S)$ is n-weakly amenable if and only if n is odd.

Keywords and phrases: Banach algebra, n-weak amenability, bounded derivation, dual space, semigroup.

2010 Mathematical Subject Classification: Primary: 46H20; Secondary: 46H10, 46H25

1. INTRODUCTION

In [4], Dales, Ghahramani, and Gronbaek introduced the concept of n-weak amenability for Banach algebras for $n \in \mathbb{N}$. They determined some relations between m- and n-weak amenability for general Banach algebras and for Banach algebras in various classes, and proved that, for each $n \in \mathbb{N}$, $(n+2)$- weak amenability always implies n-weak amenability. Let A be a weakly amenable Banach algebra. Then it is also proved in [4] that in the case where A is an ideal in its second dual $(A'', \Box), A$ is necessarily $(2m-1)$-weakly amenable for each $m \in \mathbb{N}$. The authors of [4] asked the following questions: (i) Is a weakly amenable Banach algebra necessarily 3-weakly amenable? (ii) Is a 2-weakly amenable Banach algebra necessarily 4-weakly amenable? A counter-example resolving question (i) was given by Zhang in [14], but it seems that question (ii) is still open.

It is also shown in [4, Corollary 5.4] that for certain Banach space E the Banach algebra $\mathcal{N}(E)$ of nuclear operators on E is n-weakly amenable if and only if n is odd.

A class of Banach algebra that was not considered in [4] is the Banach algebra on Semigroup. Mewomo in [12], considered the

Received by the editors January 16, 2012; Revised: August 28, 2012; Accepted: October 2, 2012

[1]Corresponding author

Rees matrix semigroup S, and show that $l^1(S)$ is $(2k+1)$-weakly amenable for a Rees matrix semigroup S.
In this work, we examine the n-weak amenability of some semigroup algebras, and give an easier example of a Banach algebra which is n-weakly amenable if and only if n is odd. We also examine the n-weak amenability of the second dual A''.
Let $L^1(G)$ be the group algebra of a locally compact group G [3, §3.3]. Then Johnson has proved that $L^1(G)$ is amenable if and only if G is amenable ([9], [3, Theorem 5.6.42]) and that $L^1(G)$ is always weakly amenable ([10], [3, Theorem 5.6.48]). It is proved in [4, Theorem 4.1] that each group algebra is n-weakly amenable whenever n is odd, and it is conjectured that $L^1(G)$ is n-weakly amenable for each $n \in \mathbb{N}$; this is true whenever G is amenable, and it is true when G is a free group [11].

2. PRELIMINARY

First, we recall some standard notions; for further details, see [3]. Let A be an algebra. The character space of A is denoted by Φ_A. Let X be an A-bimodule. A *derivation* from A to X is a linear map $D : A \to X$ such that

$$D(ab) = Da \cdot b + a \cdot Db \quad (a, b \in A)\,.$$

For example, for each $x \in X$, the map $\delta_x : A \to X$ defined by $\delta_x(a) = a \cdot x - x \cdot a, \quad (a \in A)$ is a derivation; derivations of this form are the *inner derivations*.
Let X be a Banach space. Then the spaces $X^{(n)}$ for $n \in \mathbb{Z}^+$ are the iterated duals of X, where we take $X^{(0)} = X$. Let $\lambda \in X'$. We denote by $\lambda^{(2n)} \in X^{(2n+1)}$ the $2n^{\text{th.}}$ dual of λ for $n \in \mathbb{Z}^+$, where $\lambda^{(0)} = \lambda$. Clearly $\lambda^{(2n)} \mid X = \lambda$, where we regard X as a closed subspace of $X^{(2n)}$. The second dual of X is X'', and the canonical embedding of X in X'' is denoted by i or $\widehat{\cdot}$. The weak-* topology on X' is denoted by $\sigma(X', X)$. We shall use the Goldstine's theorem: for each $\Lambda \in X''$, there is a net x_α in X such that $\|x_\alpha\| \leq \|\Lambda\|$ and $\hat{x}_\alpha \to \Lambda$ in $(X'', \sigma(X'', X'))$.
Let A be a Banach algebra, and let X be an A-bimodule. Then X is a Banach A-bimodule if X is a Banach space and if there is a constant $k > 0$ such that

$$\|a \cdot x\| \leq k\,\|a\,\|\,\|x\|\,, \quad \|x \cdot a\| \leq k\,\|a\,\|\,\|x\| \quad (a \in A,\, x \in X)\,.$$

By renorming X, we can suppose that $k = 1$. For example, A itself is a Banach A-bimodule, and X', the dual space of a Banach A-bimodule X, is a Banach A-bimodule with respect to the module operations specified for by

$$\langle x,\, a \cdot \lambda\rangle = \langle x \cdot a,\, \lambda\rangle, \quad \langle x,\, \lambda \cdot a\rangle = \langle a \cdot x,\, \lambda\rangle \quad (x \in X)$$

for $a \in A$ and $\lambda \in X'$, where $\langle .,. \rangle$ is the duality pairing of X and X'; we say that X' is the *dual module* of X. Successively, the duals $X^{(n)}$ are Banach A-bimodules; in particular $A^{(n)}$ is a Banach A-bimodule for each $n \in \mathbb{N}$. We take $X^{(0)} = X$.

Let A be a Banach algebra, and let X be a Banach A-bimodule. Then $\mathcal{Z}^1(A, X)$ is the space of all continuous derivations from A into X, $\mathcal{N}^1(A, X)$ is the space of all inner derivations from A into X, and the first cohomology group of A with coefficients in X is the quotient space

$$\mathcal{H}^1(A, X) = \mathcal{Z}^1(A, X)/\mathcal{N}^1(A, X)\,.$$

The Banach algebra A is *amenable* if $\mathcal{H}^1(A, X') = \{0\}$ for each Banach A-bimodule X and *weakly amenable* if $\mathcal{H}^1(A, A') = \{0\}$. Further, as in [4], A is *n-weakly amenable* for $n \in \mathbb{N}$ if $\mathcal{H}^1(A, A^{(n)}) = \{0\}$, and A is *permanently weakly amenable* if it is n-weakly amenable for each $n \in \mathbb{N}$. For instance, each C^*-algebra is permanently weakly amenable [4, Theorem 2.1]. As we stated, each group algebra is n-weakly amenable whenever n is odd.

3. n-WEAK AMENABILITY OF THE SECOND DUAL

Arens in [1] defined two products, $\Box$ and $\Diamond$, on the bidual A'' of Banach algebra A; A'' is a Banach algebra with respect to each of these products, and each algebra contains A as a closed subalgebra. The products are called the *first* and *second Arens products* on A'', respectively. For the general theory of Arens products, see [3, 5, 6]. We recall briefly the definitions. For $\Phi \in A''$, we set

$$\langle a,\, \lambda \cdot \Phi\rangle = \langle \Phi,\, a \cdot \lambda\rangle, \quad \langle a,\, \Phi \cdot \lambda\rangle = \langle \Phi,\, \lambda \cdot a\rangle \quad (a \in A,\, \lambda \in A')\,,$$

so that $\lambda \cdot \Phi, \Phi \cdot \lambda \in A'$. Let $\Phi, \Psi \in A''$. Then

$$\langle \Phi \,\Box\, \Psi,\, \lambda\rangle = \langle \Phi,\, \Psi \cdot \lambda\rangle, \quad \langle \Phi \,\Diamond\, \Psi,\, \lambda\rangle = \langle \Psi,\, \lambda \cdot \Phi\rangle \quad (\lambda \in A')\,.$$

Suppose that $\Phi, \Psi \in A''$ and that $\Phi = \lim_\alpha a_\alpha$ and $\Psi = \lim_\beta b_\beta$ for certain nets (a_α) and (b_β) in A. Then $\Phi\Box\Psi = \lim_\alpha \lim_\beta a_\alpha b_\beta$ and $\Phi\Diamond\Psi = \lim_\beta \lim_\alpha a_\alpha b_\beta$, where all limits are taken in the weak-$*$ topology $\sigma(A'', A')$ on A''. We define the product $\Box$ on $A^{(2n)}$ for $n \in \mathbb{N}$ inductively. Indeed, assume that $\Box$ is defined on $A^{(2n)}$, and

set $B = (A^{(2n)}, \Box)$. Then $(A^{(2n+2)}, \Box) = (B'', \Box)$. Let $\varphi \in \Phi_A$. Then we show below that $\varphi^{(2n)}$ is a character on $(A^{(2n)}, \Box)$.

The following result is well known, see [5, Theorem 2.17]:
Theorem 3.1: Let A be a Banach algebra. Then both $(A'', \Box)$ and $(A'', \Diamond)$ are Banach algebras containing A as a closed subalgebra.
Let A be a Banach algebra, and let $\varphi : A \to \mathbb{C}$ be a character on A. For $n \in \mathbb{N}$, denote by $\varphi^{2n} : A^{(2n)} \to \mathbb{C}$ the $(2n)^{th}$ dual of φ. Then we have the following:
Lemma 3.2: For each $n \in \mathbb{N}$, the functional φ^{2n} is a character on $(A^{(2n)}, \Box)$ with $\varphi^{2n} \mid A = \varphi$.
Proof: We prove this by induction on $n \in \mathbb{N}$.
Let $\Phi, \Psi \in A''$ with $\Phi = \lim_\alpha a_\alpha$ and $\Psi = \lim_\beta b_\beta$, as above. Since φ^2 is weak-* continuous, certainly,

$$\varphi^2(\Phi \Box \Psi) = \lim_\alpha \lim_\beta \varphi(a_\alpha)\varphi(b_\beta) = \varphi^2(\Phi)\varphi^2(\Psi),$$

and so φ^2 is a character on $(A'', \Box)$.
For arbitrary n, the results follows by induction, since $A^{(2n+2)} = A^{(2n)}$. $\Box$

For a Banach algebra A, let A^{op} be the Banach algebra with the underlying Banach space as A and with product $\circ$ given by $a \circ b = ba$.
Proposition 3.3: Let A be a Banach algebra. Then:
(i) A is permanently weakly amenable if and only if A^{op} is permanently weakly amenable.
(ii) Let A be commutative. Then $(A'', \Box)$ is permanently weakly amenable if and only if $(A'', \Diamond)$ is permanently weakly amenable.
Proof: (i) This is trivial.
(ii) Since A is commutative, then $\lambda \cdot \Phi = \Phi \cdot \lambda \quad (\lambda \in A', \Phi \in A'')$, and $\Phi \Box \Psi = \Psi \Diamond \Phi \quad (\Phi, \Psi \in A'')$, and so $(A'', \Diamond) = (A'', \Box)^{op}$.
Thus by (i), $(A'', \Box)$ is permanently weakly amenable if and only if $(A'', \Diamond)$ is permanently weakly amenable. $\Box$

Proposition 3.4: Let A be a Banach algebra and suppose A admits a continuous anti-isomorphism. Then $(A'', \Box)$ is permanently weakly amenable if and only if $(A'', \Diamond)$ is permanently weakly amenable.
Proof: Let $\tau : A \to A$ be continuous anti-isomorphism of A. Let $\Phi, \Psi \in (A'', \Box)$ and let (a_α) and (b_β) be nets in A such that $\Phi = \lim_\alpha a_\alpha$ and $\Psi = \lim_\beta b_\beta$. Let $\tau'' : (A'', \Box) \to (A'', \Diamond)$ be the

second dual of τ. Then

$$\begin{aligned}\tau''(\Phi\Box\Psi) &= \lim_\alpha \lim_\beta \tau''(a_\alpha b_\beta)\\ &= \lim_\alpha \lim_\beta \tau''(b_\beta)\tau''(a_\alpha)\\ &= \tau''(\Psi)\Diamond\tau''(\Phi).\end{aligned}$$

Thus τ'' is an isomorphism from $(A'',\Box)$ onto $(A'',\Diamond)^{op}$ and so, by Proposition 3.3 (i), $(A'',\Box)$ is permanently weakly amenable if and only if $(A'',\Diamond)$ is permanently weakly amenable. $\Box$

We recall that a Banach algebra A is a *dual Banach algebra* [5, Definition 2.23] if there is a closed submodule X of A' such that $X' = A$ as a Banach space.

Theorem 3.5: Suppose A is a dual Banach algebra. If $(A'',\Box)$ is $(2n-1)$-weakly amenable, then A is $(2n-1)$-weakly amenable for $n \in \mathbb{N}$.

Proof: Since A is dual Banach algebra, let $A = X'$, for some Banach space X such that $\hat{X}$ is a submodule of the dual module $A' = X''$. Let $i : X \to A'$ be the canonical map and $i' : A'' \to X' = A$ be the dual map of i. We first show that i' is a homomorphism from $(A'',\Box)$ onto A.

Let $\Phi, \Psi \in A''$, such that $\Phi = \lim_\alpha \widehat{a}_\alpha$, $\Psi = \lim_\beta \widehat{b}_\beta$ for nets (a_α), (b_β) in A, where the limits are taken in the weak-$*$ topology $\sigma(A'',A')$ on A''. Then we have

$$\begin{aligned}i'(\Phi\Box\Psi) &= i'(\lim_\alpha \lim_\beta \widehat{a_\alpha}\widehat{b_\beta})\\ &= \lim_\alpha \lim_\beta i'((a_\alpha b_\beta))\\ &= \lim_\alpha \lim_\beta (a_\alpha b_\beta)\\ &= \lim_\alpha a_\alpha \lim_\beta b_\beta\\ &= \lim_\alpha (i'(a_\alpha) \lim_\beta i'(b_\beta))\\ &= i'(\Phi)i'(\Psi)\end{aligned}$$

where the limits are taken in the weak-$*$ topology $\sigma(A'',A')$ on A''. Thus $i' : A'' \to A$ is an algebra homomorphism from A'' onto A.

For $i : X \to A'$ with $A = X'$, clearly $i^{(2n)} : A^{(2n-1)} \to A^{(2n+1)}$ and $i^{(2n-1)} : A^{(2n)} \to A^{(2n-2)}, n \in \mathbb{N}$.

Let $a \in A$, then for $x \in X$, we have

$$\langle i'(\hat{a}), x\rangle = \langle \hat{a}, i(x)\rangle = \langle a, x\rangle.$$

Thus $i'(\hat{a}) = a$. and in general $i^{(2n-1)}(\hat{b}) = b$ for $b \in A^{(2n-2)}$ and $\hat{b} \in A^{(2n)}$. Let $D : A \to A^{(2n-1)}$ be a derivation. Then $\overline{D} \equiv i^{(2n)} \circ D \circ i' : A'' \to A^{(2n+1)}$ is clearly a derivation, since for $\Phi_1, \Phi_2 \in A''$ and $\Psi \in A^{(2n)}$, we have

$$\begin{aligned}
\langle \overline{D}(\Phi_1 \Box \Phi_2), \Psi \rangle &= \langle (i^{(2n)} \circ D \circ i')(\Phi_1 \Box \Phi_2), \Psi \rangle \\
&= \langle D(i'(\Phi_1) i'(\Phi_2)), i^{(2n-1)}(\Psi) \rangle \\
&= \langle D(i'(\Phi_1) i'(\Phi_2) + i'(\Phi_1) D(i'(\Phi_2))), i^{(2n-1)}(\Psi) \rangle \\
&= \langle D(i'(\Phi_1)), i'(\Phi_2) i^{(2n-1)}(\Psi) \rangle + \langle D(i'(\Phi_2)), i^{(2n-1)}(\Psi) i'(\Phi_1) \rangle \\
&= \langle D(i'(\Phi_1)), i^{(2n-1)}(\Phi_2 \Box \Psi) \rangle + \langle D(i'(\Phi_2)), i^{(2n-1)}(\Psi \Box \Phi_1) \rangle \\
&= \langle i^{(2n)}(D(i'(\Phi_1))), \Phi_2 \Box \Psi \rangle + \langle i^{(2n)}(D(i'(\Phi_2))), \Psi \Box \Phi_1 \rangle \\
&= \langle (i^{(2n)} \circ D \circ i')(\Phi_1) \cdot \Phi_2 + \Phi_1 \cdot (i^{(2n)} \circ D \circ i')(\Phi_2), \Psi \rangle \\
&= \langle \overline{D}(\Phi_1) \cdot \Phi_2 + \Phi_1 \cdot \overline{D}(\Phi_2), \Psi \rangle.
\end{aligned}$$

Hence $\overline{D}$ is a derivation. Since A'' is $(2n-1)$- weakly amenable, there exists $\lambda \in A^{(2n+1)}$ such that

$$\overline{D}(\Phi) = \Phi \cdot \lambda - \lambda \cdot \Phi \quad (\Phi \in A'').$$

Now $A^{(2n-2)}$ and $A^{(2n)}$ are A-bimodules and the canonical map $j : A^{(2n-2)} \to A^{(2n)}$ is an A-bimodule morphism, and so is $j' : A^{(2n+1)} \to A^{(2n-1)}$.
Let $\gamma = j'(\lambda), \lambda \in A^{(2n+1)}$. Then for $a \in A, b \in A^{(2n-2)}$, we have

$$\begin{aligned}
\langle D(a), b \rangle &= \langle D(i'(\hat{a})), i^{(2n-1)}(\hat{b}) \rangle \\
&= \langle i^{(2n)} D(i'(\hat{a})), \hat{b} \rangle \\
&= \langle \overline{D}(\hat{a}), j(b) \rangle \\
&= \langle \hat{a} \cdot \lambda - \lambda \cdot \hat{a}), j(b) \rangle \\
&= \langle j'(\hat{a} \cdot \lambda - \lambda \cdot \hat{a}), b \rangle \\
&= \langle a \cdot j'(\lambda) - j'(\lambda \cdot a, b \rangle \\
&= \langle a \cdot \gamma - \gamma \cdot a), b \rangle.
\end{aligned}$$

Thus, $D(a) = a \cdot \gamma - \gamma \cdot a \quad (a \in A, \gamma \in A^{(2n-1)})$, and so $D : A \to A^{(2n-1)}$ is inner. Thus A is $(2n-1)$-weakly amenable. $\square$

4. n-WEAK AMENABILITY OF SEMIGROUP ALGEBRAS

Let S be a non-empty set. Then

$$\ell^1(S) = \left\{ f \in \mathbb{C}^S : \sum_{s \in S} |f(s)| < \infty \right\},$$

with the norm $\|\cdot\|_1$ given by $\|f\|_1 = \sum_{s\in S}|f(s)|$ for $f \in \ell^1(S)$. We write δ_s for the characteristic function of $\{s\}$ when $s \in S$.
Now suppose that S is a semigroup. For $f, g \in \ell^1(S)$, we set

$$(f \star g)(t) = \Big\{\sum f(r)g(s) : r, s \in S,\, rs = t\Big\} \quad (t \in S)$$

so that $f\star g \in \ell^1(S)$. It is standard that $(\ell^1(S), \star)$ is a Banach algebra, called the *semigroup algebra on* S. For a further discussion of this algebra, see [3, 5], for example. In particular, with $A = \ell^1(S)$, we identify A' with $C(\beta S)$, where βS is the Stone-Čech compactification of S, and $(A'', \Box)$ with $(M(\beta S), \Box)$, where $M(\beta S)$ is the space of regular Borel measures on βS of S; in this way, $(\beta S, \Box)$ is a compact, right topological semigroup that is a subsemigroup of $(M(\beta S), \Box)$ after the identification of $u \in \beta S$ with $\delta_u \in M(\beta S)$.
There is always one character on the Banach algebra $\ell^1(S)$: this is the *augmentation character*

$$\varphi_S : f \mapsto \sum_{s\in S} f(s)\,, \quad l^1(S) \to \mathbb{C}\,.$$

Let S be a semigroup, and let $o \in S$ be such that $so = os = o$ $(s \in S)$. Then o is a *zero* for the semigroup S. Suppose that $o \notin S$; set $S^o = S \cup \{o\}$, and define $so = os = 0$ $(s \in S)$ and $o^2 = o$. Then S^o is a semigroup containing S as a subsemigroup; we say that S is formed by *adjoining a zero to* S.
We recall that S is a *right zero semigroup* if the product in S is such that

$$st = t \quad (s, t \in S)\,.$$

In this case, $f \star g = \varphi_S(f)g$ $(f, g \in \ell^1(S))$.

Theorem 4.1: Let A be a Banach algebra such that
$ab = \varphi(a)b$ $(a, b \in A)$, where $\varphi \in \Phi_A$. Then:

(i) $\lambda \cdot a = \varphi(a)\lambda$ and $a \cdot \lambda = \langle a,\, \lambda\rangle\varphi$ for each $a \in A$ and $\lambda \in A'$;

(ii) $a \cdot \Phi = \varphi(a)\Phi$ and $\Phi \cdot a = \langle \Phi, \varphi\rangle a$ for each $a \in A$ and $\Phi \in A''$;

(iii) a continuous linear map $D : A \to A''$ is a derivation if and only if $D(A) \subset \ker\varphi$.

Proof: (i) Let $a, c \in A$ and $\lambda \in A'$. Then

$$\langle c,\, \lambda \cdot a\rangle = \langle ac,\, \lambda\rangle = \langle \varphi(a)c,\, \lambda\rangle = \langle c,\, \varphi(a)\lambda\rangle\,,$$

and so $\lambda \cdot a = \varphi(a)\lambda$. Similarly, $a \cdot \lambda = \langle a,\, \lambda\rangle\varphi$.
(ii) Let $a \in A$, $\lambda \in A'$, and $\Phi \in A''$. Then

$$\langle \lambda,\, a \cdot \Phi\rangle = \langle \lambda \cdot a,\, \Phi\rangle = \varphi(a)\langle \lambda,\, \Phi\rangle\,,$$

and so $a \cdot \Phi = \varphi(a)\Phi$. Similarly, $\Phi \cdot a = \langle \Phi, \varphi \rangle a$.

(iii) Let $D : A \to A''$ be a continuous linear map. Then D is a derivation if and only if

$$\varphi(a)\langle \lambda,\, Db\rangle = \varphi(a)\langle \lambda,\, Db\rangle + \langle b,\, \lambda\rangle\langle \varphi,\, Da\rangle \quad (a, b \in A,\, \lambda \in A')\,,$$

and so D is a derivation if and only if $\langle \varphi,\, Da\rangle = 0$ $(a \in A)$. $\square$

The following result is [5, Proposition 2.13].

Theorem 4.2: Let A be a Banach algebra such that $ab = \varphi(a)b$ $(a, b \in A)$, where $\varphi \in \Phi_A$. Then A is weakly amenable.

We shall extend the result by showing that, in fact, A is n-weakly amenable for each odd n.

Theorem 4.3: Let A be a Banach algebra such that $ab = \varphi(a)b$ $(a, b \in A)$, where $\varphi \in \Phi_A$. Then A is $(2k-1)$-weakly amenable for each $k \in \mathbb{N}$.

Proof: Let $k \in \mathbb{N}$, and let $D : A \to A^{(2k-1)}$ be a continuous derivation. The $(2k-2)$-dual of D is the map $D^{(2k-2)} : A^{(2k-2)} \to A^{(4k-3)}$; of course this map is continuous with respect to the respective weak-$*$ topologies on $A^{(2k-2)}$ and $A^{(4k-3)}$. Let $P : A^{(4k-3)} \to A^{(2k-1)}$ be the canonical projection that is the dual of the natural injection of $A^{(2k-2)}$ into $A^{(4k-4)}$, and define

$$\overline{D} = P \circ D^{(2k-2)} : A^{(2k-2)} \to A^{(2k-1)}\,.$$

The map $\overline{D}$ is a continuous linear operator with $\|\overline{D}\| = \|D\|$, $\overline{D}$ is continuous with respect to the respective weak-$*$ topologies on $A^{(2k-2)}$ and $A^{(2k-1)}$, and $\overline{D} \mid A = D$.

Since D is a derivation and $ba = \varphi(b)a$ $(a, b \in A)$, we have

$$\varphi(b)\langle \Phi_0,\, Da\rangle = \varphi^{(2k-2)}(\Phi_0)\langle b,\, Da\rangle + \varphi(a)\langle \Phi_0,\, Db\rangle \quad (a, b \in A) \tag{1}$$

for each $\Phi_0 \in A^{(2k-2)}$. Now take $\Psi \in A^{(2k-2)}$. Since A is weak-$*$ dense in $A^{(2k-2)}$, it follows from (1) that

$$\varphi^{(2k-2)}(\Psi)\langle \Phi_0,\, Da\rangle = \varphi^{(2k-2)}(\Phi_0)\langle \Psi,\, Da\rangle + \varphi(a)\langle \Phi_0,\, \overline{D}\Psi\rangle \tag{2}$$

for each $a \in A$ and $\Psi \in A^{(2k-2)}$.

Let $\Lambda \in A^{(2k-1)}$, and let the inner derivation from A to $A^{(2k-1)}$ specified by Λ be D_Λ. Then

$$\langle \Psi,\, D_\Lambda a\rangle = \varphi^{(2k-2)}(\Psi)\langle a,\, \Lambda\rangle - \varphi(a)\langle \Psi,\, \Lambda\rangle \quad (a \in A,\, \Psi \in A^{(2k-2)})\,. \tag{3}$$

Choose $\Phi_0 = a_0 \in A$ with $\varphi(a_0) = 1$, and then define

$$\langle \Psi,\, \Lambda\rangle = \langle a_0,\, \overline{D}\Psi\rangle \quad (\Psi \in A^{(2k-2)})\,.$$

Clearly Λ is a continuous linear function on $A^{(2k-2)}$ with $\|\Lambda\| \leq \|D\|\,\|a_0\|$, and so $\Lambda \in A^{(2k-1)}$. By (3) we have

$$\langle \Psi,\, D_\Lambda a\rangle = \varphi^{(2k-2)}(\Psi)\langle a_0,\, Da\rangle - \varphi(a)\langle a_0,\, \overline{D}\Psi\rangle \quad (a \in A,\, \Psi \in A^{(2k-2)})\,.$$

It follows from (2) that $\langle \Psi,\, D_\Lambda a\rangle = \langle \Psi,\, Da\rangle$ $(a \in A,\, \Psi \in A^{(2k-2)})$, and so $D = D_\Lambda$. Thus D is inner. □

Corollary 4.4: Let S be a right zero semigroup. Then $\ell^1(S)$ is $(2k-1)$-weakly amenable for each $k \in \mathbb{N}$.

Proof: The Banach algebra $\ell^1(S)$ clearly satisfies the condition in the above theorem, taking $\varphi = \varphi_S$. □

Theorem 4.5: Let S be an infinite right zero semigroup. Then $\ell^1(S)$ is not 2-weakly amenable.

Proof: Set $A = \ell^1(S)$, so that $A'' = M(\beta S)$. Clearly $\varphi_S(\delta_u) = 1$ $(u \in \beta S)$.

Let $u, v \in \beta S \setminus S$ with $u \neq v$, let $\Lambda \in C(\beta S)$, and then define

$$D(f) = \langle f,\, \Lambda\rangle(\delta_u - \delta_v) \quad (f \in A)\,.$$

Then $D : A \to A''$ is a continuous linear map; by Lemma 4.1(iii), D is a derivation because $\langle \varphi_S,\, D(f)\rangle = 0$ $(f \in A)$.

Since βS is a Hausdorff and totally disconnected space, there exists a clopen set $F \subset \beta S$ with $u \in F$ and $v \notin F$. We choose $s_0, t_0 \in F \cap S$ with $s_0 \neq t_0$, and we define $\Lambda \in C(\beta S)$ by setting $\langle f,\, \Lambda\rangle = f(s_0)$ $(f \in A)$.

Let D be as above, with our chosen value of Λ, and assume towards a contradiction that $D = D_\mu$ for some $\mu \in M(\beta S)$. Then, using Lemma 4.1(i), we see that

$$f(s_0)(\lambda(u) - \lambda(v)) = \langle f,\, \lambda\rangle\langle \varphi_S,\, \mu\rangle - \varphi_S(f)\langle \lambda,\, \mu\rangle \quad (f \in A, \lambda \in A')\,. \tag{4}$$

Choosing $\lambda = \chi_F$, the characteristic function of F, and $f_0 = \delta_{s_0} - \delta_{t_0} \in A$, we see that $f_0(s_0) = \chi_F(u) = 1$, $\chi_F(v) = 0$, and $\langle f_0,\, \chi_F\rangle = \varphi_S(f_0) = 0$. This is a contradiction of (4), and so D is not inner.

This shows that A is not 2-weakly amenable. □

Theorem 4.6: Let S be an infinite right zero semigroup. Then $\ell^1(S)$ is n-weakly amenable if and only if n is odd.

Proof: By Theorem 4.3, $\ell^1(S)$ is n-weakly amenable whenever n is odd. By Theorem 4.5, $\ell^1(S)$ is not 2-weakly amenable, and hence not n-weakly amenable for any even n. □

We now give a small extension of the above result.

Let $(B, \|\cdot\|_B)$ be a Banach algebra, and let $\varphi \in \Phi_B$. Define $A = B \oplus \mathbb{C}$ (that is the ℓ^1-direct sum of B and $\mathbb{C}$) as a Banach space with

the ℓ^1-norm (that is $\|(b,z)\| = \|b\|_B + |z| \quad (b,z) \in A, b \in B, z \in \mathbb{C}$), and take the product in A to be specified by

$$(b_1, z_1)(b_2, z_2) = (b_1 b_2, \varphi(b_1) z_2 + \varphi(b_2) z_1 + z_1 z_2) \quad (b_1, b_2 \in B,\ z_1, z_2 \in \mathbb{C})\,.$$

Then A is a Banach algebra. Set $p = (0,1)$, an idempotent in A. Then $\mathbb{C}p$ is a one-dimensional ideal in A, and $\mathbb{C}p$ has the trace extension property. We regard φ as an element of Φ_A by setting $\varphi(p) = 1$.

Define $q \in A'$ by setting $\langle p, q\rangle = 1$ and $q \mid B = 0$. Let $n \in \mathbb{N}$. Then we have $A^{(n)} = B^{(n)} \oplus \mathbb{C}p$ as a Banach space with the ℓ^1-norm for n even, and $A^{(n)} = B^{(n)} \oplus \mathbb{C}q$ as a Banach space with the ℓ^∞-norm for n odd. Clearly $b \cdot q = q \cdot b = \varphi(b) q \ \ (b \in B)$, and so these decompositions are B-module decompositions.

Let $n \in \mathbb{N}$, and let $\Lambda \in A^{(n)}$. Then

$$p \cdot \Lambda = \Lambda \cdot p = \begin{cases} \langle p,\, \Lambda\rangle \varphi & (n \ \text{ odd})\,, \\ \langle \Lambda,\, \varphi\rangle p & (n \ \text{ even})\,. \end{cases} \tag{5}$$

Let $D : A \to A^{(n)}$ be a continuous derivation. Then $p \cdot Dp = Dp \cdot p$, and so $Dp = 0$ by [3, Theorem 1.8.2]. Thus we may regard D as a continuous derivation $D : B \to A^{(n)}$. Suppose that n is even. Then we may write

$$Db = D_1(b) + d(b)p \quad (b \in B)\,; \tag{6}$$

here $D_1 : B \to B^{(n)}$ is a continuous derivation and $d \in B'$ is a continuous point derivation at the character φ. A similar conclusion holds in the case where n is odd, with q replacing p in (6).

Now suppose that the product in B is given by $b_1 b_2 = \varphi(b_1) b_2$ $(b_1, b_2 \in B)$. Then

$$\varphi(b_1) d(b_2) + \varphi(b_2) d(b_1) = d(b_1 b_2) = \varphi(b_1) d(b_2) \quad (b_1, b_2 \in B)\,,$$

and so $d = 0$. Thus we may regard D as a continuous derivation $D : B \to B^{(n)}$. It follows that, for each $n \in \mathbb{N}$, the Banach algebra A is n-weakly amenable if and only if B is n-weakly amenable.

Let S be an infinite right zero semigroup, and let $A = \ell^1(S^\circ)$, $B = \ell^1(S)$, and $p = \delta_o$. Then p is an idempotent in A such that $\mathbb{C}p$ is an ideal in A, and so we are in the above situation, with $\varphi = \varphi_S$. Thus we have the following conclusion.

Theorem 4.7: Let S be an infinite right zero semigroup. Then $\ell^1(S^\circ)$ is n-weakly amenable if and only if n is odd. □

It is shown in [4] that the commutative Banach algebra $C^{(1)}(\mathbb{I})$ of all continuously differentiable functions on the unit interval is not

n-weakly amenable for any $n \in \mathbb{N}$. We now give an example of the form $A \oplus X$ with this property.

We recall that a Banach algebra A is a *dual Banach algebra* [5, Definition 2.23] if there is a closed submodule X of A' such that $X' = A$ as a Banach space. For example, let $A = \ell^1(S^\circ)$ and $B = \ell^1(S)$, where $S = \mathbb{N}$ and S is a right zero semigroup. Then A is a dual Banach algebra. Indeed, take set $X = c$, the space of convergent sequences, regarded as a subspace of $\ell^\infty \subset \ell^\infty + \mathbb{C}q = A'$. Clearly $X' = A$ as a Banach space. We *claim* that X is a closed A-submodule of A'. Indeed, take $\lambda \in c$ and $k \in \mathbb{N}$. Then $(\delta_k \cdot \lambda)(n) = \lambda(k)$ and $(\lambda \cdot \delta_k)(n) = \lambda(n)$ for $n \in \mathbb{N}$, and so $\delta_k \cdot \lambda, \lambda \cdot \delta_k \in c$. Also $\delta_o \cdot \lambda = \lambda \cdot \delta_o$ is a constant sequence, and so belongs to X. It follows that X is an A-submodule.

Let A be a Banach algebra, and let X be a Banach A-bimodule. Then we set $\mathfrak{A} = A \oplus X$ as a Banach space with the ℓ^∞-norm, with the product given by

$$(a, x)(b, y) = (ab, a \cdot y + x \cdot b) \quad (a, b \in A,\, x, y \in X)\,.$$

Then $\mathfrak{A}$ is a Banach algebra [3, p. 239]. The n-weak amenability of these algebras is studied in [14], and four conditions which, taken together, are necessary and sufficient for $\mathfrak{A}$ to be n-weakly amenable are given in [14, Theorems 2.1, 2.2].

Theorem 4.8: Let $S = \mathbb{N}$, regarded as an infinite right zero semigroup, let $A = \ell^1(S^\circ)$, and let $X = c$, regarded as a Banach A-module. Set $\mathfrak{A} = A \oplus X$. Then, for each $n \in \mathbb{N}$, $\mathfrak{A}$ is not n-weakly amenable.

Proof: We have $A^{(k)} = X^{(k+1)}$ as Banach A-bimodules for each $k \in \mathbb{Z}^+$. Let $k \in \mathbb{N}$.

By Theorem 4.7, A is not $2k$-weakly amenable, and so there is a continuous derivation $D : A \to X^{(2k+1)}$ which is not inner. Thus $\mathcal{H}^1(A, X^{(2k+1)}) \neq \{0\}$. This shows that clause 2 of [14, Theorem 2.1] fails, and so, by that theorem, $\mathfrak{A}$ is not $(2k+1)$-weakly amenable.

To show that $\mathfrak{A}$ is not $2k$-weakly amenable, it suffices to show that $\mathfrak{A}$ is not 2-weakly amenable; for this, we shall follow the argument in clause 4 of [14, Theorem 2.2].

We have $\mathfrak{A}'' = A'' \oplus X'' = A'' \oplus A'$. Define

$$D : (a, x) \mapsto (0, x)\,, \quad A \oplus X \to A'' \oplus X''\,.$$

Then D is a continuous linear operator, and it is immediately checked that D is a derivation. Assume towards a contradiction that there exists $(\Phi_0, \lambda_0) \in A'' \oplus A'$ such that $D = D_{(\Phi_0, \lambda_0)}$. Then

$$(0,x) = (a\cdot\Phi_0,\ a\cdot\lambda_0 + x\cdot\Phi_0) - (\Phi_0\cdot a, \lambda\cdot a + \Phi_0\cdot x) \quad (a\in A,\ x\in X)\,. \tag{7}$$

In particular $a\cdot\Phi_0 = \Phi_0\cdot a$ $(a\in B)$, where $B=\ell^1(S)$. By Lemma 4.4(ii), it follows that $\varphi_S(a)\Phi_0 = \langle\Phi_0,\varphi_S\rangle a$ $(a\in B)$. Take $a\in B$ with $\varphi_S(a)=1$ and $a\notin\mathbb{C}\Phi_0$ to see that necessarily $\Phi_0=0$. Now take $a=0$ in (7) to see that $x=0$ $(x\in X)$, a contradiction. Thus the derivation D is not inner, and so $\mathfrak{A}$ is not 2-weakly amenable. □

Let S be a semigroup. For $s\in S$, we define $L_s(t)=st$, $R_s(t)=ts$ $(t\in S)$. Let F be a non-empty subset of S. Then $s^{-1}F = L_s^{-1}(F) = \{t\in S : st\in F\}$ and $Fs^{-1} = R_s^{-1}(F) = \{t\in S : ts\in F\}$. We recall that S is weakly left (respectively, right) cancellative if $s^{-1}F$ (respective, Fs^{-1}) is finite for each $s\in S$ and each finite subset F of S, and S is weakly cancellative if it is both weakly left cancellative and weakly right cancellative. With this definition, we have the following result:

Theorem 4.9: Let S be an infinite weakly cancellative semigroup. Then $l^1(S)$ is $(2n-1)$-weakly amenable if $l^1(S)''$ is $(2n-1)$-weakly amenable.

Proof: Since S is weakly cancellative, then $l^1(S)$ is a dual Banach algebra [5, Theorem 4.6], and so, the result follows from Theorem 3.5. □

ACKNOWLEDGEMENTS

The authors would like to thank the anonymous referee(s) whose comments improved the original version of this manuscript.

REFERENCES

(1) R. Arens, *The adjoint of a bilinear operation*, Proc. Amer. Math. Soc. **2** 2, 839 - 848, 1951.
(2) T. D. Blackmore, *Weak amenability of discrete semigroup algebras*, Semigroup Forum **55**, 196 - 205, 1997.
(3) H. G. Dales, *Banach algebras and automatic continuity*, London Mathematical Society Monographs, New Series, **24**, The Clarendon Press, Oxford, 2000.
(4) H. G. Dales, F. Ghahramani, and N. Gronbaek, *Derivations into iterated duals of Banach algebras*, Studia Math. **128**, 19 - 54, 1998.
(5) H. G. Dales, A. T.-M. Lau, and D. Strauss, *Banach algebras on semigroups and their compactifications*, Memoirs American Math. Soc., **205**, 1 - 165, 2010.
(6) J. Duncan and S. A. R. Hosseinium, *The second dual of a Banach algebra*, Proc. Royal Soc. Edinburgh, **84A**, 309 - 325, 1978.
(7) G. H. Esslamzadeh, *Banach algebra structure and amenability of a class of matrix algebras with applications*, J. Funct. Anal., **161**, 364 - 383, 1999.

(8) J. M. Howie, *Fundamentals of semigroup theory*, London Math. Soc. Monographs, **12**, Clarendon Press, Oxford, 1995.
(9) B. E. Johnson, *Cohomology in Banach algebras*, Memoirs American Math. Soc., **127** 1972.
(10) B. E. Johnson, *Weak amenability of group algebras*, Bull. London Math. Soc. **23**, 281 - 284, 1991.
(11) B. E. Johnson, *Permanent weak amenability of group algebras of free groups*, Bull. London Math. Soc., **31**, 569 - 573, 1999.
(12) O.T. Mewomo, *On n-weak amenability of Rees semigroup algebras*, Proc. Indian Acad. Sci. (Math. Sci.), **118** (4), 1 - 9, 2008.
(13) Y. Zhang, *Weak amenability of a class of Banach algebras*, Canadian Math. Bull. **44**, 504 - 508, 2001.
(14) Y. Zhang, *Weak amenability of module extensions of Banach algebras*, Trans. Amer. Math. Soc., 4131 - 4151, 2002.

DEPARTMENT OF MATHEMATICS, FEDERAL UNIVERSITY OF AGRICULTURE, ABEOKUTA, NIGERIA.
E-mail addresses: mewomoot@unaab.edu.ng, tosinmewomo@yahoo.com

DEPARTMENT OF MATHEMATICS, FEDERAL UNIVERSITY OF AGRICULTURE, ABEOKUTA, NIGERIA.
E-mail address: jibfem@yahoo.com