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ON APPROXIMATE CHARACTER AMENABILITY OF BANACH ALGEBRAS

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ABSTRACT. We introduce the notions of approximate left character amenability, approximate right character amenability and approximate character amenability. General theory is developed for these notions, and studied for Banach algebras defined over locally compact groups and second dual of Banach algebra.

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1. INTRODUCTION

The notion of amenability in Banach algebra was initiated by Johnson in [12]. Since then, amenability has become a major issue in Banach algebra theory and in harmonic analysis. For details on amenability in Banach algebras see [14].

In [8], Ghahramani and Loy introduced generalized notions of amenability with the hope that it would yield Banach algebra without bounded approximate identity which nontheless had a form of amenability. All known approximate amenable Banach algebras have bounded approximate identities until recently when Ghahramani and Read in [10] give examples of Banach algebras which are boundedly approximately amenable but which do not have bounded approximate identities. This answers a question open since the year 2004 when Ghahramani and Loy founded the notion of approximate amenability.

The authors of [8] gave examples to show that for most of these new notions, the corresponding class of Banach algebras is larger than that for the classical amenable Banach algebra introduced by Johnson in [12]. They developed general theory for these notions, and studied them for several concrete classes of Banach algebra.

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Let A be a Banach algebra over $\mathbb{C}$ and $\varphi : A \to \mathbb{C}$ be a character on A, that is, an algebra homomorphism from A into $\mathbb{C}$, and let Φ_A denote the character space of A (that is the set of all characters on A). In [15], Monfared introduced the notion of character amenability in Banach algebras. His definition of this notion requires continuous derivations from A into dual Banach A-bimodules to be inner, but only those modules are concerned where either of the left or right module action is defined by characters on A. As such character amenability is weaker than the classical amenability introduced by Johnson in [12], so all amenable Banach algebras are character amenable.

In this paper, we shall continue in the spirit of [8] and [15] by applying the concept of approximate amenability to that of character amenability and introduce the notions of approximate left character amenability, approximate right character amenability and approximate character amenability. We develop general theory on these notions and study them for Banach algebras defined over locally compact groups and second duals of Banach algebras.

2. PRELIMINARY

First, we recall some standard notions; for further details, see [3] and [5].

Let A be an algebra. The character space of A is denoted by Φ_A. Let X be an A-bimodule. A *derivation* from A to X is a linear map $D : A \to X$ such that

$$D(ab) = D(a) \cdot b + a \cdot D(b) \quad (a, b \in A)\,.$$

For example, for $x \in X$, the map $\delta_x : A \to X$ defined by $\delta_x(a) = a \cdot x - x \cdot a \quad (a \in A)$ is a derivation; derivations of this form are called the *inner derivations*.

Let X be a Banach space, the second dual of X is X'', and the canonical embedding of X in X'' is denoted by i. The image of X in X'' under i is denoted by $\widehat{X}$. The weak* topology on X' is denoted by $\sigma(X', X)$. We shall use the Goldstine's theorem: for each $\Lambda \in X''$, there is a net (x_α) in X such that $\|x_\alpha\| \le \|\Lambda\|$ and $x_\alpha \to \Lambda$ in $(X'', \sigma(X'', X'))$.

Let A be a Banach algebra, and let X be an A-bimodule. Then X is a Banach A-bimodule if X is a Banach space and if there is a constant $k > 0$ such that

$$\|a \cdot x\| \le k\,\|a\|\,\|x\|\,, \quad \|x \cdot a\| \le k\,\|a\|\,\|x\| \quad (a \in A,\, x \in X)\,.$$

By renorming X, we can suppose that $k = 1$. For example, A itself is Banach A-bimodule, and X', the dual space of a Banach A-bimodule X, is a Banach A-bimodule with respect to the module operations specified for by

$$\langle x,\, a \cdot \lambda\rangle = \langle x \cdot a,\, \lambda\rangle, \quad \langle x,\, \lambda \cdot a\rangle = \langle a \cdot x,\, \lambda\rangle \quad (x \in X)$$

for $a \in A$ and $\lambda \in X'$; we say that X' is the *dual module* of X.
Let A be a Banach algebra, and let X be a Banach A-bimodule. Then $\mathcal{Z}^1(A, X)$ is the space of all continuous derivations from A into X, $\mathcal{N}^1(A, X)$ is the space of all inner derivations from A into X, and the first cohomology group of A with coefficients in X is the quotient space

$$\mathcal{H}^1(A, X) = \mathcal{Z}^1(A, X)/\mathcal{N}^1(A, X)\,.$$

The Banach algebra A is *amenable* if $\mathcal{H}^1(A, X') = \{0\}$ for each Banach A-bimodule X.
A derivation $D : A \to X$ is *approximately inner* if there is a net (x_v) in X such that

$$D(a) = \lim_v (a \cdot x_v - x_v \cdot a) \quad (a \in A),$$

the limit being taken in $(X, \|.\|)$. That is, $D(a) = \lim_v \delta_{x_v}(a)$, where (δ_{x_v}) is a net of inner derivations. The Banach algebra A is *approximately amenable* if, for each Banach A-bimodule X, every continuous derivation $D : A \to X'$ is approximately inner.
We let $M^A_{\varphi_r}$ denote the class of Banach A- bimodule X for which the right module action of A on X is given by $x \cdot a = \varphi(a)x \quad (a \in A, x \in X, \varphi \in \Phi_A)$, and $M^A_{\varphi_l}$ denote the class of Banach A- bimodule X for which the left module action of A on X is given by $a \cdot x = \varphi(a)x \quad (a \in A, x \in X, \varphi \in \Phi_A)$. If the right module action of A on X is given by $x \cdot a = \varphi(a)x$, then it is easy to see that the left module action of A on the dual module X' is given by $a \cdot f = \varphi(a)f \quad (a \in A, f \in X', \varphi \in \Phi_A)$. Thus, we note that $X \in M^A_{\varphi_r}$ (resp. $X \in M^A_{\varphi_l}$) if and only if $X' \in M^A_{\varphi_l}$ (resp. $X' \in M^A_{\varphi_r}$).
Let A be a Banach algebra and let $\varphi \in \Phi_A$, we recall from [16], see also [15] that
(i) A is left φ-amenable if every continuous derivation $D : A \to X'$ is inner for every $X \in M^A_{\varphi_r}$;
(ii) A is right φ-amenable if every continuous derivation $D : A \to X'$ is inner for every $X \in M^A_{\varphi_l}$;
(iii) A is left character amenable if it is left φ-amenable for every $\varphi \in \Phi_A$;
(iv) A is right character amenable if it is right φ-amenable for every

$\varphi \in \Phi_A$;
(v) A is character amenable if it is both left and right character amenable.

3. DEFINITIONS AND GENERAL THEORY

We introduce the following definitions.

Definition 3.1: Let A be a Banach algebra and let $\varphi \in \Phi_A$.Then we say that
(i) A is approximately left φ-amenable if every continuous derivation $D : A \to X'$ is approximately inner for every $X \in M^A_{\varphi_r}$;
(ii) A is approximately right φ-amenable if every continuous derivation $D : A \to X'$ is approximately inner for every $X \in M^A_{\varphi_l}$;
(iii) A is approximately left character amenable if it is approximately left φ-amenable for every $\varphi \in \Phi_A$;
(iv) A is approximately right character amenable if it is approximately right φ-amenable for every $\varphi \in \Phi_A$;
(v) A is approximately character amenable if it is both approximately left and approximately right character amenable.
Clearly, approximate character amenability is weaker than character amenability, approximate amenability, and amenability, and so, every character amenable, approximately amenable and amenable Banach algebra is approximately character amenable.

Definition 3.2: Let A be a Banach algebra and $\varphi \in \Phi_A$. A left [right] approximate φ-mean is a net $(m_v) \subset A''$ such that
(i) $m_v(\varphi) = 1$;
(ii) $\|\varphi(a)m_v - m_v \cdot a\| \to 0$ $[\|a \cdot m_v - \varphi(a)m_v\| \to 0]$ $\quad (a \in A)$.

Theorem 3.3: Let A be a Banach algebra and $\varphi \in \Phi_A$. Then the following statements are equivalent:
(i) A has a left [right] approximate φ-mean;
(ii) A is left [right] approximately φ-amenable;
(iii) Given $(ker\varphi)''$ a dual A-bimodule structure by taking left [right] action to be $a \cdot m = \varphi(a)m$ $[m \cdot a = \varphi(a)m]$ $\quad (a \in A, m \in A'')$ and taking the right [left] action to be the natural one. Then any continuous derivation $D : A \to (ker\varphi)''$ is approximately inner.

Proof: The equivalence of (i) and (ii) can be shown as in the classical case of φ-amenable Banach algebra by simple modifications of the arguments used in the proof of [13, Theorem 1.1]. Trivially, (ii) implies (iii). We only have to proof (iii) $\Rightarrow$ (i). For this, let $b \in B$ be such that $\varphi(b) = 1$, and let $D : A \to (ker\varphi)''$ be defined by $D(a) = \varphi(a)b - ba, \quad (a \in A)$. It is easy to see that D is a derivation, and so it is approximately inner by (iii). Thus there exists a net (n_v) in $(ker\varphi)''$ with

$$D(a) = \lim_v(\varphi(a)n_v - n_v \cdot a) \quad (a \in A)).$$

That is,

$$\varphi(a)b - ba = \lim_v(\varphi(a)n_v - n_v \cdot a) \quad (a \in A)) \qquad (3.1).$$

Set $m_v = b - n_v$. Then clearly $m_v(\varphi) = 1$ and

$$\|\varphi(a)m_v - m_v \cdot a\| = \|\varphi(a)b - ba - (\varphi(a)n_v - n_v \cdot a)\| \to 0 \quad (a \in A)$$

by (3.1). Thus A has a left aproximate φ-mean.
The right version of (iii) $\Rightarrow$ (i) is similar. □

Proposition 3.4: Let A be a Banach algebra.
(i) Let $\varphi \in \Phi_A$ and suppose A is left [right] approximately φ-amenable. Then A has a left [right] approximate identity.
(ii) Suppose that A is approximately left [right] character amenable. Then A has a left [right] approximate identity.
(iii) Suppose that A is approximately character amenable. Then A has left and right approximate identity.

Proof: Let $X = A''$ with the left action as $a \cdot m = \varphi(a)m$, $(a \in A, m \in A'')$ and zero right action. Then the natural injection $i : A \to A''$, defined by $i(a) = \widehat{a} \quad (a \in A)$ is a derivation. Thus, since A is approximately left φ-amenable, there is a net (n_v) in A'' with

$$i(a) = \lim_v(a \cdot n_v - n_v \cdot a) = \lim_v \varphi(a)n_v.$$

That is $\varphi(a)n_v \to \widehat{a}$ for each $a \in A$. Take finite sets $F \subset A, G \subset A'$ and $\epsilon > 0$. Let $H = \{\varphi(a)\lambda : a \in A, \lambda \in G\}$. Then there is $u = u(F, G, \epsilon)$ such that $\|\widehat{a} - \varphi(a)n_v\| < \frac{\epsilon}{2^k}$ for $a \in F$. By Goldstine's theorem, there is $b_v \in A$ such that

$$|\langle \Psi, b_v \rangle - \langle n_v, \Psi \rangle| < \frac{\epsilon}{2} \quad (\Psi \in H).$$

Thus for each $a \in F, \lambda \in G$, we have

$$|\langle \lambda, b_v a \rangle - \langle \lambda, a \rangle| \leq |\langle \lambda, b_v a \rangle - \langle \varphi(a)n_v, \lambda \rangle| + |\langle \varphi(a)n_v - \widehat{a}, \lambda \rangle|$$
$$\leq |\langle \varphi(a)\lambda, b_n \rangle - \langle n_v, \varphi(a)\lambda \rangle| + \frac{\epsilon}{2} \leq \epsilon.$$

Thus $(b_v)_{(F,G,\epsilon)}$ is a weak left approximate identity for A. Since A has a weak left (or right) approximate identity imply A has a left (or right) approximate identity [8, Lemma 2.1]. Then we conclude that A has a left approximate identity.
An analogous argument works on the right and (ii) and (iii) follows from (i). □

Let A be a Banach algebra, $A\hat{\otimes}A$ denotes the projective tensor product and $\pi : A\hat{\otimes}A \to A$ defined by $\pi(a\otimes b) = ab \quad (a, b \in A)$ be the diagonal operator. Then $A\hat{\otimes}A$ is Banach A-bimodule with the module actions $a\cdot(b\otimes c) = ab\otimes c, (b\otimes c)\cdot a = b\otimes ca \quad (a, b, c \in A)$.

Theorem 3.5: Let A be a Banach algebra and $\varphi \in \Phi_A$. A is left [right] approximately φ-amenable if and only if either of the following equivalent conditions holds:

(i) there is a net $(M_v) \subset (A\hat{\otimes}A)''$ such that for each $a \in A$

$$\varphi(a)M_v - M_v \cdot a \to 0 \ [a \cdot M_v - \varphi(a)M_v \to 0]$$

and $\langle M_v, \varphi\otimes\varphi\rangle = \pi''(M_v)(\varphi) \to 1$;

(ii) there is a net $(M'_v) \subset (A\hat{\otimes}A)''$ such that for each $a \in A$

$$\varphi(a)M'_v - M'_v \cdot a \to 0 \ [a \cdot M'_v - \varphi(a)M'_v \to 0]$$

and $\langle M'_v, \varphi\otimes\varphi\rangle = \pi''(M'_v)(\varphi) = 1$.

Proof: Suppose that A is approximately left φ-amenable. Consider the Banach A-bimodule $A\hat{\otimes}A$ with the module actions given by $a\cdot(b\otimes c) = \varphi(a)b\otimes c, (b\otimes c)\cdot a = b\otimes ca \quad (a, b, c \in A)$. Consider the quotient Banach A-bimodule $X = (A\hat{\otimes}A)'/\mathbb{C}\cdot(\varphi\otimes\varphi)$. Let $u \in (A\hat{\otimes}A)''$ be such that $u(\varphi\otimes\varphi) = 1$, and let $\delta_u : A \to (A\hat{\otimes}A)''$ be the inner derivation by u. Then the image of δ_u is a subset of $X' = \{\varphi\otimes\varphi\}^\circ$, and since A is approximately left φ-amenable, there is a net (u_v) in X' such that $\delta_u(a) = \lim_v \delta_{u_v}(a)$. That is

$$\varphi(a)u - u\cdot a = \lim_v(\varphi(a)u_v - u_v\cdot a) \qquad (3.2).$$

Set $M'_v = u - u_v$. Then for $a \in A$, we have

$$\varphi(a)M'_v - M'_v\cdot a = \varphi(a)u - u\cdot a - (\varphi(a)u_v - u_v\cdot a) \to 0$$

by (3.2). Also

$$\pi''(M'_v)(\varphi) = \pi''(u - u_v)(\varphi) = \langle u, \varphi\otimes\varphi\rangle = 1.$$

Thus we have shown (ii).

Now suppose (i) holds. Let $X \in M^A_{\varphi_r}$ and $D : A \to X'$ be a derivation. We follow the standard argument in [8, Theorem 2.1] with due care given to the module actions defined on X and X'. For each v, set $f_v(x) = M_v(\mu_x)$, where $a, b \in A, x \in X$, $\mu_x(a\otimes b) = (\varphi(a)D(b))(x)$. Then with $(m_v^\alpha) \subset A\hat{\otimes}A$ converging weak* to M_v, and noting that for $m \in A\hat{\otimes}A$,

$$\mu_{\varphi(a)x - x\cdot a}(m) = (\varphi(a)\mu_x - a\cdot\mu_x)(m) + (\varphi(\pi(m))D(a))(x),$$

we have

$$(\varphi(a)f_v - f_v\cdot a)(x) = f_v(\varphi(a)x - a\cdot x)$$

$$= M_v(\mu_{a\cdot x-\varphi(a)x}) = \lim_\alpha m_v^\alpha(\mu_{a\cdot x-\varphi(a)x})$$
$$= M_v(\varphi(a)\mu_x - a\cdot\mu_x) + lim_\alpha(\varphi(\pi(m_v^\alpha))D(a))(x)$$
$$= (\varphi(a)M_v - M_v\cdot a)(\mu_x) + (\pi''(M_v)(\varphi)D(a))(x).$$

Thus

$$\|(\varphi(a)f_v - f_v\cdot a)(x) - D(a)(x)\|$$
$$= \|(\varphi(a)M_v - M_v\cdot a)(\mu_x) + (\pi''(M_v)(\varphi)D(a))(x) - D(a)(x)\|$$
$$\le \|(\varphi(a)M_v - M_v\cdot a)(\mu_x)\| + \|(\pi''(M_v)(\varphi)D(a))(x) - D(a)(x)\|$$
$$\le \|\varphi(a)M_v - M_v\cdot a\|\|\mu_x\| + \|\pi''(M_v)(\varphi) - 1\|\cdot\|x\|\cdot\|D(a)\|$$
$$= \|\varphi(a)M_v - M_v\cdot a\|\|D\|\|x\| + \|\pi''(M_v)(\varphi) - 1\|\cdot\|x\|\cdot\|D(a)\|.$$

Thus $D(a) = \lim_v(\varphi(a)f_v - f_v\cdot a) \quad (a\in A)$. It follows that A is approximately left φ-amenable. Since (ii) clearly implies (i) the equivalences follow.

The right version can be prove similarly. □

Proposition 3.6: Let A and B be Banach algebras and $\varphi\in\Phi_B$. Suppose $\tau : A\to B$ is a continuous epimorphism. Then if A is approximately left [right] $\varphi\circ\tau$-amenable, then B is approximately left [right] φ-amenable.

Proof: Let $X\in M^B_{\varphi_r}$ and $Y = X$ be the A-bimodule with module actions induced via τ as follows; the left module action defined by $a\cdot x = \tau(a)x$ and right module action defined by $x\cdot a = (\varphi\circ\tau)(a)x \quad (a\in A, x\in X)$. If $D : B\to X'$ is a derivation, then $D\circ\tau : A\to Y'$ is a derivation. Thus if A is approximately left $\varphi\circ\tau$- amenable, there is a net (λ_v) in X' with

$$(D\circ\tau)(a) = \lim_v[(\varphi\circ\tau)(a)\lambda_v - \lambda_v\cdot a].$$

Thus D is approximately inner and so B is approximately left φ-amenable.

An analogous argument works on the right. □

Remark: This argument does not extend to the closure of a homomorphic image, since the net (λ_v) in X' need not be bounded.

Corollary 3.7: Let A and B be Banach algebras and $\varphi\in\Phi_B$. Suppose $\tau : A\to B$ is a continuous epimorphism. Then if A is approximately left [right] character amenable, then B is approximately left [right] character-amenable.

Proof: This follows from Proposition 3.6. □

Corollary 3.8: Let A be a Banach algebra. Suppose A is approximately left [right] character amenable, and I is a closed two-sided ideal of A. Then A/I is approximately left [right] character amenable. If I is left [right] character amenable and A/I is approximately left [right] character amenable, then A is left [right] approximately character amenable.

Proof: Immediate from Proposition 3.6. In the reverse direction, the standard argument [15, Theorem 2.6 (iii)] applies. $\square$

Proposition 3.9: Let A be a Banach algebra without identity and A_e denote the Banach algebra obtained by adjoining an identity e. Let $\varphi \in \Phi_A$ and let φ_e be the unique extension of φ to an element of Φ_{A_e}. Then A is approximately left [right] φ-amenable if and only if A_e is approximately left [right] φ_e-amenable. In particular, the unitization algebra A_e is approximately left [right] character amenable if and only if A is approximately left [right] character amenable.

Proof: This can be shown by simple modifications of the argument used in the proof of [8, Proposition]. $\square$

Given two Banach algebras A and B, for $f \in A'$ and $g \in B'$, let $f \otimes g$ denote the element of $(A\hat{\otimes}B)'$ satisfying $(f \otimes g)(a \otimes b) = f(a)g(b)$. With this, we have that $\Phi_{A\hat{\otimes}B} = \{\varphi \otimes \psi : \varphi \in \Phi_A, \psi \in \Phi_B\}$.

The following result is from [13]:

Theorem 3.10: Let A and B be Banach algebras and let $\varphi \in \Phi_A$ and $\psi \in \Phi_B$. Then $A\hat{\otimes}B$ is $(\varphi \otimes \psi)$-amenable if and only if A is φ-amenable and B is ψ-amenable.

Remark: (i) We note that the concept of φ-amenable Banach algebra in [13] is the same as right φ-amenable in [15], see also [13, Theorem 1.1].

(ii) Any statement about right φ-amenability turns into an analogous statement about left φ- amenability by simply replacing A by its opposite algebra.

With the above remark and Theorem 3.10, we have the following result:

Corollary 3.11: Let A and B be Banach algebras. Then $A\hat{\otimes}B$ is character amenable if and only if A and B are character amenable.

Proposition 3.12: Let A and B be Banach algebras. Suppose that A is approximately character amenable and has a bounded approximate identity, and that B is character amenable. Then $A\hat{\otimes}B$ is approximately character amenable.

Proof: The classical argument [12, Theorem 5.4] suffices to show this with due care given to the way the left and right module actions are defined. $\square$

The next result is an analogue of a well known splitting property [8, Theorem 2.2] whose proof follows the argument of [2, Theorem 2.3]. The proof is similar.

Theorem 3.13: Let A be an approximately left character amenable Banach algebra and let X be a right Banach A-bimodule such that $x \cdot a = \varphi(a)x$ for some $\varphi \in \Phi_A \quad (x \in X, a \in A)$. Let

$$\sum : 0 \to X' \overrightarrow{f} \ Y \ \overrightarrow{g} \ Z \to 0$$

be an admissible short exact sequence of Banach left A-modules. Then $\sum$ approximately splits.

Let A be a Banach algebra, we recall that a subspace X of A is called weakly complemented in A, if $X^\circ = \{\lambda \in A' : \langle \lambda, x \rangle = 0, x \in X\}$ is complemented in A'. Also, we let $A^2 = span\{ab : a, b \in A\}$ and $\overline{A^2}$ be the closure of A^2 in A.

Corollary 3.14: Let A be an approximately left character amenable Banach algebra and let I be a weakly complemented left ideal of A. Then I has a right approximate identity. In particular, $\overline{I^2} = I$.

Proof: We use the argument of [8, Corollary 2.4, Lemma 2.2]. By Proposition 3.9, we suppose that A is unital. We consider the sequence of left A-module

$$\sum : 0 \to I \ \overrightarrow{i} \ A \to A/I \to 0,$$

and its second dual;

$$\sum^{''} : 0 \to I'' \ \overrightarrow{i''} \ A'' \to (I^\circ)' \to 0.$$

$\sum''$ is admissible since I is weakly complemented, and so by Theorem 3.13, there is a net of maps (I_v), each a left inverse to i'' such that $a \cdot I_v - I_v \cdot a \to 0$ for each $a \in A$. Now A'' has a right identity m, so that for $a \in I_v$,

$$(I_v \cdot a)(m) = I_v(i''\hat{a} \cdot m) = I_v(i''\hat{a}).$$

Thus

$$(a \cdot I_v)(m) = (a \cdot I_v - I_v \cdot a)(m) + \hat{a},$$

and so $a \cdot I_v \to \hat{a}, \quad a \in I$. The argument of [8, Lemma 2.2] gives the result. □

4. SECOND DUAL OF BANACH ALGEBRA

Let A be a Banach algebra. Here we study the relation between approximate character amenability of A and that of its second A''. Arens in [1] defined two products, $\Box$ and $\Diamond$, on the bidual A'' of Banach algebra A; A'' is a Banach algebra with respect to each of these products, and each algebra contains A as a closed subalgebra. The products are called the *first* and *second Arens products* on A'', respectively. For the general theory of Arens products, see [3, 5, 6]. We recall briefly the definitions. For $\Phi \in A''$, we set

$$\langle a,\, \lambda \cdot \Phi\rangle = \langle \Phi,\, a\,\cdot\,\lambda\rangle, \quad \langle a,\, \Phi \cdot \lambda\rangle = \langle \Phi,\, \lambda\,\cdot\, a\rangle \quad (a \in A,\, \lambda \in A')\,,$$

so that $\lambda \cdot \Phi, \Phi \cdot \lambda \in A'$. Let $\Phi, \Psi \in A''$. Then

$$\langle \Phi \,\Box\, \Psi,\, \lambda\rangle = \langle \Phi,\, \Psi\,\cdot\,\lambda\rangle, \quad \langle \Phi \,\Diamond\, \Psi,\, \lambda\rangle = \langle \Psi,\, \lambda\,\cdot\,\Phi\rangle \quad (\lambda \in A')\,.$$

Suppose that $\Phi, \Psi \in A''$ and that $\Phi = \lim_\alpha a_\alpha$ and $\Psi = \lim_\beta b_\beta$ for certain nets (a_α) and (b_β) in A. Then $\Phi\Box\Psi = \lim_\alpha \lim_\beta a_\alpha b_\beta$ and $\Phi\Diamond\Psi = \lim_\beta \lim_\alpha a_\alpha b_\beta$, where all limits are taken in the weak-$*$ topology $\sigma(A'', A')$ on A''.

The following result is well known, see [5]:

Proposition 4.1: Let A be a Banach algebra. Then both $(A'', \Box)$ and $(A'', \Diamond)$ are Banach algebras containing A as a closed subalgebra.

For a Banach algebra A, let A^{op} be the Banach algebra with the underlying Banach space as A and with product $\circ$ given by
$a \circ b = ba$. A^{op} is called the opposite algebra of A.

Proposition 4.2: Let A be a Banach algebra and $\varphi \in \Phi_A$. Then A is approximately left [right] φ-amenable if and only if A^{op} is approximately right [left] φ-amenable. In particular, A is approximately left [right] character amenable if and only if A^{op} is approximately right [left] character amenable.

Proof This trivial. $\Box$

Proposition 4.3: Let A be a Banach algebra. Suppose A is commutative. Then $(A'', \Box)$ is approximately left [right] character amenable if and only if $(A'', \Diamond)$ is approximately right [left] character amenable.

Proof: Since A is commutative, then $\lambda \cdot \Phi = \Phi \cdot \lambda \quad (\lambda \in A', \Phi \in A'')$, and $\Phi\Box\Psi = \Psi\Diamond\Phi \quad (\Phi, \Psi \in A'')$, and so $(A'', \Diamond) = (A'', \Box)^{op}$. Thus the result follows by Proposition 4.2. $\Box$

Proposition 4.4: Let A be a Banach algebra and suppose A admits a continuous anti-isomorphism. Then $(A'', \Box)$ is approximately left [right] character amenable if and only if $(A'', \Diamond)$ is approximately left [right] character amenable.

Proof: Let $\tau : A \to A$ be continuous anti-isomorphism of A.
Let $\Phi, \Psi \in (A'', \Box)$ and let (a_α) and (b_β) be nets in A such that $\Phi = \lim_\alpha a_\alpha$ and $\Psi = \lim_\beta b_\beta$. Let $\tau'' : (A'', \Box) \to (A'', \Diamond)$ be the second dual of τ. Then

$$\tau''(\Phi\Box\Psi) = \lim_\alpha \lim_\beta \tau''(a_\alpha b_\beta)$$

$$= \lim_{\alpha}\lim_{\beta}\tau''(b_\beta)\tau''(a_\alpha)$$

$$= \tau''(\Psi)\Diamond\tau''(\Phi).$$

Thus τ'' is an isomorphism from $(A'', \Box)$ onto $(A'', \Diamond)^{op}$ and so, the result follows from Proposition 4.2. $\Box$

The amenability, approximate amenability and character amenability versions of the result below on Banach algebra have been proved in [7] and [9], [8] and [16] respectively. The proof of character amenability and approximate amenability cases carries over. The proof is similar to the argument used in [16, Theorem 3.8] and [8, Theorem 2.3].

Theorem 4.5: Let A be a Banach algebra. Suppose $(A'', \Box)$ is approximately left [right] character amenable, then A is approximately left [right] character amenable.

5. ALGEBRAS OVER LOCALLY COMPACT GROUPS

Let G be a locally comapct group. Then we define the group algebra $L^1(G)$ (using the left Haar measure) and the measure algebra $M(G)$ as in [3], see also [4] and [11] for details. We recall that the product $\mu, v \in M(G)$ is specified by the formula

$$\langle \mu * v, \lambda\rangle = \int_G\int_G \lambda(st)d\mu(sdv(t) \quad (\lambda \in C_0(G)),$$

so that $\delta_s * \delta_t = \delta_{st} \quad (s, t \in G)$. It is standard that $M(G)$ is a unital Banach algebra with the convolution product and it is identified with the dual space of all continuous linear functional on the Banach space $C_0(G)$ and that $L^1(G)$ is a closed ideal in $M(G)$. The subspace of $M(G)$ consisting of the continuous measures is denoted by $M_C(G)$.

The map $\varphi : \mu \to \mu(G), \quad M(G) \to \mathbb{C}$ is a character on $M(G)$, called the augumented character.

Let $C_b(G)$ be the space of bounded continuous functions on G. We recall that a function $f \in C_b(G)$ is called left uniformly continuous if the map $g \to \delta_g * f, \quad G \to C_b(G)$ is continuous (wrt the norm topoloy on $C_b(G)$) and $LUC(G)$ denotes the space of bounded left uniformly continuous functions on G.

The following results are from [4, Theorem 1.1] and [8, Theorem 3.1]:

Theorem 5.1: Let G be a locally compact group. Then

(i) $M(G)$ is amenable if and only if G is discrete and amenable

(ii) $M(G)$ is approximately amenable if and only if G is discrete and amenable.

It was shown in [4] that if G is non-discrete, then the complemented ideal $M_C(G)$ satisfies $\overline{M_C(G)^2} \neq M_C(G)$. With this and Corollary 3.14, we have the next result.

Theorem 5.2: Let G be a locally compact group. Then $M(G)$ is approximately left character amenable if and only if G is discrete and amenable.

Proof: Suppose G is non-discrete, then $\overline{M_C(G)^2} \neq M_C(G)$ which is impossible by Corollary 3.14. □

Theorem 5.3: Let G be a locally compact group. Then $(L^1(G)'', \Box)$ is approximately character amenable if and only if G is finite.

Proof: This can be shown be following similar argument in the proof of [8, Theorem 3.3] and using Corollary 3.8, Theorem 4.2 and Corollary 3.14. □

Corollary 5.4: Let G be a locally compact group. Then $LUC(G)'$ is approximately character amenable if and only if G is finite.

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