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THE ANALYSIS OF NETWORKS IN MANUFACTURING SYSTEMS

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ABSTRACT. Suppose customers (products) arrive at a service facility (machine) at times $t_1, t_2, t_3, \cdots$, where t_n is the arrival time of the n^{th} customer and $w_n + t_n$ is the departure time where w_n = waiting time which includes service time and queueing time. Suppose there is a network of machines with customers (products) experiencing blocking due to machine breakdowns. Let $P(t)$ be the matrix of transition probabilities among states (machines) of the Markov chain $\{X(t), t \geq 0\}$ where $X(t)$ is continuous time stationary and irreducible. Assuming that the Markov chain is strongly mixing, we prove that the times that demand is not met is asymptotically normally distributed.

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1. INTRODUCTION

Consider a manufacturing system with n states (machines) with blocking and customers (products) arriving at the machines. Let the arrival of the n^{th} customer at the first machine be t_n, so that the departure time is $w_n + t_n$, where w_n is the waiting time, and includes both the service time and queueing time. Several authors have considered machine groups each with its own transition matrix, the steady-state queueing times performance analysis criteria and simulation techniques. The performance of a manufacturing system can be judged by the FRACTION of the time it is available to customers. Sometimes as with an Automatic Teller Machine (ATM), it is not a matter of life or death if it fails frequently, as long as the system can RECOVER quickly, in such a manner that the TOTAL DOWNTIME is not large.

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This paper deals with a theoretical, mathematical analysis of a network of machines with blocking. Section II contains a brief literature review, and some definitions. Section III presents the Main Result. Section IV is the Conclusion.

II. Literature Review

In [1], the authors state that, because scheduling is done by human operators (i.e. workers) who operate regardless of the general rules, there is a hint that there are some HIDDEN variables, called SOFT VARIABLES which are extremely difficult to model and for which input data describing them cannot be easily got. According to [1], the MAJOR stochastic inputs are MACHINE breakdowns and VARYING demand patterns.

In [2], estimates of state probabilities and moments of different variables such as the queue length have been found. On the other hand, Lavenberg [3] discussing his Numerical Results, noted that the mean queueing time increases and the coefficient of variation of the queueing time decreases as the mean service time increases. He further noted that the m^{th} moment about the origin of the steady-state queueing time for the M/G/1 finite capacity queue does *not* depend on the service time distribution only through its first $(m+1)$ moments. Finally, he noted that the mean queueing time does *not* necessarily increase as the variance of the service time increases.

The times that demand is not met, is of interest (see [4]) and considering the asymptotic distribution of the times that demand for service is *not* met, is useful. According to [5], an approximation of the service time distribution F by stochastically smaller distributions, say F_n, leads to an approximation of the stationary distribution π of the original M/G/c queue by the stationary distributions π_n of the M/G/c queue with service time distributions F_n. It is stated that all the approximations are in weak convergence. This result is applicable here since convergence in distribution is even weaker than convergence in probability (usually considered as weak convergence).

Moreover, in [6], sufficient conditions are given, under which a time-non-homogeneous countable Markov chain with the transition intensity matrix $\Lambda(t)$ shares the limiting distribution with a time-homogeneous ergodic Markov chain, with the transition intensity matrix Λ.

Definition 1: A stochastic matrix P (equivalently the corresponding Markov chain $\{X_n\}$) is called ERGODIC if

$$\lim_{n\to\infty} P_{ij}^{(n)} = \pi_j$$

$\forall$ states i and j where the limiting vector $\pi = (\pi_0, \pi_1 \ \cdots)$ is called the STATIONARY vector of P and has the properties that $\pi P = \pi, \ \ \pi_j \geq 0 \ \ \forall \ \ j$, and

$$\sum_{j=0}^{\infty} \pi_j = 1.$$

A Markov chain for which $P_{ij}^{(n)}$converges to π_j at a geometric rate introduced in [8], is defined as follows:

Definition 2: An irreducible Markov chain for which $\left|P_{ij}^{(n)} - \pi_j\right| \leq C_{ij}\beta_{ij}^n$ where $\beta_{ij} < 1$ is called a geometrically ergodic Markov chain.

Vere-Jones [7] proved that $\beta < 1$ could replace β_{ij} in Definition 2 above, independent of i and j. Moreover, Nummelin and Tweedie [8] showed that for geometrically ergodic Markov chains, there exist $\beta < 1$ and C_i depending on i s.t. $\left|P_{ij}^{(n)} - \pi_j\right| < C_i\beta^n$. When the coefficient C can be chosen independently of i, Cohen [9] calls the Markov chain uniformly geometrically ergodic.

Note: Strictly stationary or strongly ergodic Markov chains are uniformly geometrically ergodic (See [10, Lemma 1]).

Since the same collection of products are fed into machines at intervals in order to produce parts, the demand function for service $f(t)$, has period τ. Then $f(t + \tau) = f(t)$ where τ depends on the type of products, machine efficiency and the finished goods.

Definition 3: (see [11])

The strictly stationary sequence $\{U_j\}$ is said to be uniformly mixing if, for all $D \in m_{k+n}^{\infty}, \quad |P\{D|m_{-\infty}^{k}\} - P(D)| \leq \phi(n) \downarrow 0$ as $n \to \infty$, where the σ-algebra m_{k+n}^{∞} describes the future of the sequence $\{U_j\}$ and so is generated by $\{U_{k+n}, U_{k+n+1}, \cdots\}$ while the σ-algebra $m_{-\infty}^{k}$ is generated by $\{U_1, U_2, \cdots, U_k\}$, and $\phi(n)$ is said to be the mixing coefficient.

Definition 4: The stationary sequence is said to be strongly mixing if, in addition to the conditions in Definition 3 above,

$$\sum_{n=1}^{\infty} \phi(n) < \infty.$$

III. The Main Results

Let $P(t)$ be the matrix of transition probabilities of the Markov chain $\{X(t), \ \ t \geq 0\}$, where $\{X(t)\}$ is continuous time stationary and irreducible. Note that the restriction on the general service time distribution that it is not degenerate at zero ensures that the imbedded Markov chain is irreducible and aperiodic. Let Q be the matrix of the rate (in time) with which transitions occur.

$$\text{Then} \ \ P(t) = e^{Qt}, \ \ P(0) = I_n \tag{1}$$

where equation (1) is the solution of the Kolmogorov systems

$$\dot{P}(t) = QP(t) \quad \text{... backward equation}$$

$$\text{or} \ \ \dot{P}(t) = P(t)Q \quad \text{... forward equation}$$

with initial condition, $P(0) = I_n$.
From either of these systems, $P(0) = Q$.

$$\text{Let} \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

where $\lambda_i, \quad i = 1, 2, \cdots, n$ are the eigen values of Q and $\vee$ is the matrix of eigenvectors corresponding to λ_i. Then $Q = \vee\Lambda\vee^{-1}$ where $\vee$ is called the matrix of the right eigenvectors of Q and $\vee^{-1}$ is called the left eigenvectors of Q. For ease of notation, let $\vee = B$ and $\vee^{-1} = \tilde{C}$. Then $Q = B\Lambda\tilde{C}$.

Note: Since the eigenvalues of P lie in the interval $|\lambda| \leq 1$ and since the eigenvalues of Q are those that correspond to those of P in the interval $0 \leq \lambda \leq 1$, it implies that the eigenvalues of Q all have negative real parts, with 1 as one of the eigenvalues.

$$\text{Let} \quad e^{\Lambda} = \begin{pmatrix} e^{\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n} \end{pmatrix}$$

$$\text{Then} \quad P(t) = e^{Qt} = Be^{\Lambda t}\tilde{C} \tag{2}$$

where

$$P_{ij}(t) = \sum_{k=1}^{n} b_{i_k} e\lambda_k t \tilde{C}_{k_j}$$

are the elements of equation (2).
Let $I_i(X(t))$ be the indicator function of state i and let $G(X(t), t)$ be defined as

$$G(X(t), t) = \sum_{i=1}^{n} I_i(X(t))g_i(t) \tag{3}$$

$$\text{where} \quad I_i(X(t)) = \begin{cases} 1, & \text{if } X(t) = i \\ \\ 0, & \text{if } X(t) \neq i \end{cases} \tag{4}$$

From equation (4), the rhs of equation (3) is the number of occurrences (equivalent to the occupation time) of state i in the first n steps. On the time interval $(0, m\tau)$ let this number of occurrences be given by

$$U_i = \int_0^{m\tau} \sum_{i=1}^{n} I_i(X(t))g_i(t)dt,$$

which is the occupation time in state i. Let

$$\int_0^{m\tau} = m\int_0^{\tau} = \int_0^{h} + \int_h^{2h} + \cdots + \int_{(m-1)h}^{mh}$$

where $\tau = kh - (k-1)h$ or $\tau = h, \quad k = 1, 2, \cdots, m$.
Then,

$$S_m = \sum_{k=1}^{m} \int_0^{\tau} \sum_{i=1}^{n} I_i(X(t + (k-1))\tau)g_i(t + (k-1)\tau)dt$$

where S_m is the amount of time that the demand for service is not being met within $(0, m\tau)$.

$$\text{Since} \quad g_i(t) = \begin{cases} f(t) - C_i, & C_i < f(t) \\ \\ 0, & C_i \geq f(t) \end{cases}$$

$\Rightarrow \quad g_i(t + (k-1)\tau) = f(t + (k-1)\tau) - C_i$, for $\quad C_i < f(t)$ for $k = 1, 2, \cdots, m$ and since f is τ-periodic,
$\Rightarrow \; f(t + (k-1)\tau) - C_i = f(t) - C_i$

$$\therefore \quad g_i(t + (k-1)\tau) = f(t) - C_i = g_i(t) \tag{5}$$

Let

$$S_m = \sum_{k=1}^{m} \int_0^{\tau} G(X(t+(k-1)\tau, t)dt = \sum_{k=1}^{m} Y_k \tag{6}$$

where $Y_k = \int_0^{\tau} G(X(t+(k-1))\tau, t)dt$.

Since the Markov chain $\{X(t)\}$ is stationary, the sequence $\{Y_k\}$ is also stationary. Hence,

$$\begin{aligned} E(Y_k) &= \left[\int_0^{\tau} G(X(t+(k-1))\tau, t)dt\right] \\ &= \int_0^{\tau} E[G(X(t)), t]dt, \end{aligned}$$

by (5),

$$= E\left[\int_0^{\tau} G(X(t), t)dt\right] = E[Y_1].$$

$$\text{If } \ Z_k = Y_k - E[Y_k], \ \text{ then}$$

$$Z_k = Y_k - E[Y_1] \tag{7}$$

$$\therefore \quad \sum_{k=1}^{m} Z_k = \sum_{k=1}^{m} Y_k - mE[Y_1] \quad \text{and hence,}$$

$$\sum_{k=1}^{m} Z_k = S_m - mE[Y_1] \tag{8}$$

We use the Central Limit Theorem for strongly mixing sequences of bounded random variables to establish the asymptotic normality of the sequence in (8).

Theorem: If the random variables Z_k are bounded and if $\{Z_k\}_{k=1}^{m}$ is stationary and strongly mixing with $\sum_{m=1}^{\infty} \phi(m) < \infty$ (see Definition (4) above), then

$$\lim_{m\to\infty} \frac{1}{m} \, var S_m = \sigma^2 = E[Z_1^2] + 2\sum_{l=1}^{\infty} E[Z_i Z_{1+l}] \tag{9}$$

where S_m converges absolutely.
If $\sigma > 0$ then

$$\lim_{m\to\infty} \frac{S_m - mE[Y_1]}{\sigma\sqrt{m}} \sim N(E[Y_1], \ \sigma^2)$$

We need to establish equation (9).

Proof of (9):

$$S_m = mE[Y_1] + \sum_{k=1}^{m} Z_k, \quad \text{by (8)}$$

Since $\{Y_k\}$ is stationary, so is $\{Z_k\}$, by (7).

Lemma: Given a function f, the standard estimator of $f(Z)$ is $\frac{1}{n}\sum_{i=1}^{n} f(Z_i)$ where Z_i are n random variables from the distribution of Z.

$$\therefore \quad E\left[\sum_{k=1}^{m} Z_k\right] = mE[Z_m] = mE[Z_i]$$

since stationary random variables $\{Z_k\}$ all have the same distribution and therefore the same expected value.

$$\therefore \quad E[S_m] = mE[Y_1] + mE[Z_1]$$

But $E[Z_1] = 0$, by (7).

$$\begin{aligned} \therefore \quad E[S_m] &= mE[Y_1] \\ &\Rightarrow \quad (E[S_m])^2 = m^2(e[Y_1])^2 \end{aligned} \tag{10}$$

Similarly,

$$S_m^2 = m^2(E[Y_1])^2 + 2mE[Y_1]\sum_{k=1}^{m} Z_k + \left(\sum_{k=1}^{m} Z_k\right)^2$$

$$\begin{aligned} \therefore \quad E[S_m^2] &= m^2(E[Y_1])^2 + 2mE[Y_1] \,.\, mE[Z_1] + E\left(\textstyle\sum_{k=1}^{m} Z_k\right)^2 \\ &= m^2(E[Y_1])^2 + E\left(\textstyle\sum_{k=1}^{m} Z_k\right)^2, \quad \text{since } E[Z_1] = 0 \end{aligned}$$

$$\begin{aligned} \therefore \quad var(S_m) &= E[S_m^2] - (E[S_m])^2 \\ &= E\left(\textstyle\sum_{k=1}^{m} Z_k\right)^2, \quad \text{by (10)} \\ &= E\left(\textstyle\sum_{k=1}^{m} Z_k^2\right) + 2E\left[\textstyle\sum_{i,j=1,\, i\neq j}^{m} Z_i Z_j\right] \\ &= mE[Z_1^2] + 2m\textstyle\sum_{j=1}^{m} E[Z_1 Z_{j+1}] \end{aligned}$$

$$\therefore \quad \lim_{m\to\infty} \frac{1}{m} Var(S_m) = E[Z_1^2] + 2\sum_{j=1}^{\infty} E[Z_1 Z_{j+1}] \tag{11}$$

and by (6) above,

$$\lim_{m\to\infty} \frac{1}{m}\, Var(S_m) = \sigma^2.$$

Again, let the sequence $\{Z_k\}$ be strongly mixing. Define events A, B as

$$\begin{aligned} A &= \{X(k\tau)|m_{-\infty}^{k\tau}\} \\ B &= \{(X(k+m-1)\tau)|m_{k\tau}^{\infty}\}. \end{aligned}$$

Then

$$|P(A\cap B) - P(A)\ P(B)| \leq \sum_{i,j=1}^{n} \pi_i\,|P_{ij}((m-1)\tau) - \pi_j| = \gamma_m$$

where

$$\lim_{m\to\infty} \gamma_m = 0,$$

where π_j are the equilibrium or stationary probabilities of state j, defined by

$$\pi_j = \sum_{i=1}^{n} \pi_i P_{ij}(t), \text{ such that}$$

$$\sum_{j=1}^{n} \pi_j = 1.$$

Clearly, $\lim_{m\to\infty} P_{ij}((m-1)\tau) = \pi_j$,
since the Markov chain $\{X(t)\}$ is continuous time stationary.
We now need to calculate the mean and variance of the times that demand for service is not met within $(0, m\tau)$. First,

$$E[S_m] = E\left[\sum_{k=1}^{m} Y_k\right] = mE[Y_1],$$

since $\{Y_k\}$ is stationary.

$$\begin{aligned} E[Y_1] &= E\int_0^\tau \textstyle\sum_{i=1}^n I_i(X(t))g_i(t)dt \\ &= E\int_0^\tau \textstyle\sum_{i=1}^n E[g_i(t)I_i(X(t))]dt. \end{aligned}$$

(since $E|Y_i| < \infty$)

$$\begin{aligned} &= \int_0^\tau \textstyle\sum_{i=1}^n g_i(t)E[I_i(X(t))]dt \\ &= \int_0^\tau \textstyle\sum_{i=1}^n g_i(t)P\{X(t)=i\}dt, \end{aligned}$$

since $E[I_A(\ .\ )] = P(A)$

$$\begin{aligned} &= \int_0^\tau \textstyle\sum_{i=1}^n \pi_i g_i(t)dt \\ &= \textstyle\sum_{i=1}^n \pi_i \int_0^\tau g_i(t)dt, \quad \text{by Fubini's theorem.} \end{aligned}$$

$$\begin{aligned}\therefore \quad E[S_m] &= m\sum_{i=1}^{n} \pi_i \int_0^{\tau} g_i(t)dt,\\ \Rightarrow \lim_{m\to\infty} \frac{1}{m}\, E[S_m] &= E[Y_1]\end{aligned}$$

Since $Z_1 = Y_1 - E[Y_1]$ by (7),

$$E[Z_1^2] = E[(Y_1 - E[Y_1])^2] = E[Y_1^2] - (E[Y_1])^2$$

and using $g_i = g_i(r),\ \ g_j = g_j(s)$, since

$$Y_1 = \int_0^{\tau} \sum_{i=1}^{n} I_i(X(t))g_i(t)dt,$$

it follows that

$$\begin{aligned}Y_1^2 &= \int_0^{\tau} \sum_{i=1}^{n} (I_i g_i)^2 dr + 2\int_0^{\tau} \sum_{i,j=1,\, i\neq j}^{n} g_i g_j I_i I_j \; drds,\\ \Rightarrow\ E[Y_1^2] &= \int_0^{\tau} \sum_{i=1}^{n} \pi_i g_i^2 dr + 2\int_0^{\tau} \sum_{i,j=1,\, i\neq j}^{n} g_i g_j \pi_i P_{ij}(t)\; drds,\end{aligned}$$

since $E[I_A^2] = E[I_A] = P(A)$ since $I^2 = I = 0$ or 1 and $P(I_i) = \pi_i$. Also,

$$\begin{aligned}(E[Y_1])^2 &= \left(\int_0^{\tau} \sum_{i=1}^{n} g_i \pi_i dt\right)^2\\ &= \int_0^{\tau} \sum_{i=1}^{n} (\pi_i g_i)^2 dr + 2\int_0^{\tau}\int_0^{\tau} \sum_{i,j=1,\, i\neq j}^{n} \pi_i\pi_j g_i g_j \; drds,\end{aligned}$$

and since $\pi_i P = \pi_i$ for stationary probabilities π_i,

$$= \int_0^{\tau} \sum_{i=1}^{n} \pi_i g_i^2 dr + 2\int_0^{\tau}\int_0^{\tau} \sum_{i,j=1,\, i\neq j}^{n} \pi_i\pi_j g_i g_j \; drds.$$

Therefore

$$E[Z_1^2] = 2\int_0^{\tau}\int_0^{v} \sum_{i,j=1}^{n} P_{ij}(s-r) g_i g_j dr\; ds - 2\int_0^{\tau}\int_0^{\tau} \pi_i\pi_j g_i g_j \; drds.$$

From (2), when $k = 1,\ \lambda_1 = 0$ and $v = \tau$ with $b_{i1} = 1,\ \ \tilde{C}_{ij} = \pi_j$.

$$\therefore \quad E[Z_1^2] = 2\int_0^{\tau}\int_0^{v} \sum_{i,j=1}^{n} \pi_i \sum_{k=2}^{n} \exp[(s-r)\lambda_k b_{ik}\tilde{C}_k g_i g_j \; drds].$$

Also,

$$\begin{aligned}E[Z_1 Z_{1+l}] &= E\left[Y_1 Y_{1+l} - Y_{1+l}E[Y_1]\right]\\ &= E[Y_1 Y_{1+l}] - E[Y_1]E[Y_{1+l}]\end{aligned}$$

Using a similar argument to the one above,

$$E[Z_1 Z_{1+l}] = \int_0^\tau \int_0^\tau \sum_{i,j=1}^{n} \pi_i \sum_{k=2}^{n} e^{\lambda_k l\tau} b_{i_k} \tilde{C}_{k_j} e^{(s-r)\lambda_k} g_i g_j \; drds$$

Since

$$\begin{aligned} \textstyle\sum_{i=1}^{\infty} e^{\lambda_k l\tau} &= e^{\lambda_k \tau} + e^{2\lambda_k \tau} + \cdots \\ &= \tfrac{e^{\lambda_k \tau}}{1-e^{\lambda_k \tau}} = \tfrac{-1}{1-e^{-\lambda_k \tau}} \end{aligned}$$

we have by equation (11) above,

$$\sigma^2 = 2 \sum_{i,j=1,\, i\neq j}^{n} \pi_i \sum_{k=2}^{n} b_{ik} \tilde{C}_{kj} \left[\int_0^\tau \int_0^v g_i g_j e^{\lambda_k (s-r)} \; drds \right.$$

$$\left. \times \frac{-1}{1-e^{-\lambda_k \tau}} \int_0^\tau g_i e^{-\lambda_k \tau} dr \; . \; \int_0^\tau g_j e^{-\lambda_k s} ds \right]$$

By equation (6) above we have

$$Var(S_m) = m\sigma^2 \;\; \text{or} \;\; \frac{1}{m} Var(S_m) = \sigma^2$$

$\Rightarrow \lim_{m\to\infty} \frac{1}{m} Var\ S_m = \sigma^2$, where $\sigma^2 = Var(Y_1) > 0$.

IV. Conclusion

We have found that the variance of the random variables which are asymptotically normally distributed, is $Var(Y_1)$ and the mean is $E(Y_1)$.

$\therefore$ the average time the demand is not met in $(0, m\tau)$ is

$$\lim_{m\to\infty} \frac{1}{m} E(S_m) = \tau,$$

i.e.

$$\sum_{i=1}^{n} \pi \int_0^\tau g_i(t) dt = \tau.$$

The analysis of manufacturing systems is a broad topic which should generate more results especially in the non-simulation area of study. However, as has been noted by several authors, it requires a lot of time.

REFERENCES

[1] V. Doerrsam, K. Furmans (1995): "Application Case Study of a Queueing Network Simulation Tool for Analyzing and Optimizing a Manufacturing System". Proceedings of the 1995 European Simulation Symposium, Erlangen.

[2] Slawomir Pilarski and Tiko Kameda (1993): "Simple Bounds on the Convergence rate of an Ergodic Markov Chain". *Information Processing Letters* 45 (1993), 81-87 (Elsevier).

[3] Stephen S. Lavenberg (1975): "The Steady-State Queueing Time Distribution for the M/G/1 Finite Capacity Queue". *Management Science* Vol. 21, No. 5, January 1975, pp. 501-506.

[4] Ole Platz (1983): "A Central Limit Theorem for a Security of Supply Model". JAP 20, 220-225.

[5] Lothar Breuer (2008): "Continuity of the M/G/c Queue". *Queueing Systems* (2008) 58: 321-331.

[6] V. Abramov, R. Liptser (2004): "On Existence of Limiting Distribution for Time-Nonhomogeneous Countable Markov Process". *Queueing Systems* 46, 353-361.

[7] D. Vere-Jones(1962): "Geometric Ergodicity in Denumerable Markov Chains". *Quart. J. Math.* Oxford Ser. (1962) 7-28.

[8] E. Nummelin and R. Tweedie: "Geometric Ergodicity and R-Positivity for General Markov Chains", Preprint.

[9] J. E. Cohen:"Ergodicity of Age Structure in Populations with Markovian Vital Rates, I: Countable States", *J. Am. Statist. Assoc.* (1976) 335-339.

[10] C.C. Huang, D. Isaacson and B. Vinograde (1976): "The Rate of Convergence of Certain Nonhomogeneous Markov Chains", Z. Wahrschein lichkeits theorie und Verw. Gebiete 35 (1976), 141-146.

[11] Ugbebor, Olabisi O. (1976): Sample Path Properties of Brownian Motion. A Ph.D. Thesis Submitted to the University of London, London, England.

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