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PICARD ITERATION PROCESS FOR A GENERAL CLASS OF CONTRACTIVE MAPPINGS

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(To the memory of Prof. J.O.C. Ezeilo).

ABSTRACT. Let (E, ρ) be a metric space. Let $T : E \to E$ be a map with $F(T) := \{x \in E : Tx = x\} \neq \emptyset$ such that $\rho(Tx, p) \leq a\rho(x, p)$, $\forall x \in E$, $p \in F(T)$ and some $a \in [0, 1)$. It is shown that the class of mappings satisfying this condition is more general than the class of contraction mappings with fixed points. Several classes of nonlinear operators studied by various authors are shown to belong to this class. Finally, it is shown that the Picard iteration process converges to the unique fixed point of T. Our theorem improves a recent result of Akewe and Olaleru (British Journal of Mathematics and Computer Science, 3(3): 437-447, 2013) and a host of other results.

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1. INTRODUCTION

The following theorem is the main result in Akewe and Olaleru [1].
Theorem(AO) (Theorem 2.2.1, p. 441, of [1], *) Let* $(E, \| \cdot \|)$ *be a Banach space,* $T : E \to E$ *be a selfmap of* E *with a fixed point* p *satisfying the condition*

$$\| Tx - p \| \leq a \| x - p \|, \forall x \in E, \ a \in [0, 1). \qquad (1.0)$$

For $x_0 \in E$, *let* $\{x_n\}_{n=0}^{\infty}$ *be the multistep iterative scheme defined by*

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T y_n^1, \\ y_n^i &= (1 - \beta_n^i)x_n + \beta_n^i T y_n^{i+1}, \ i = 1, \ 2, \ \ldots, \ k-2, \qquad (1.1) \\ y_n^{k-1} &= (1 - \beta_n^{k-1})x_n + \beta_n^{k-1} T x_n, \ k \geq 2 \end{aligned}$$

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converging to p (that is $Tp = p$)(sic), where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n^i\}_{n=0}^{\infty}$ are real sequences in $[0,1)$ for $i = 0,1,2,3,...,k-1$. Then, the multistep iterative scheme converges strongly to p.

2. EXAMPLES

We first note that the class of mappings satisfying contractive condition (1.0) is fairly large. In fact, we give an example to show that the class of contraction mappings with fixed points is a *proper subclass* of the class of mappings satisfying inequality (1.0).

Example 2.1. Let $E = l_\infty$, $B := \{x \in l_\infty :\| x \| \leq 1\}$ and let $T : E \to B \subseteq E$ be defined by

$$Tx = \begin{cases} \frac{11}{12}\Big(0, x_1^2, , x_2^2, x_3^2...\Big), \ if \ ||x||_\infty \leq 1, \\ \\ \frac{11}{12||x||_\infty^2}\Big(0, x_1^2, , x_2^2, x_3^2, ...), \ if \ ||x||_\infty > 1, \end{cases}$$

for $x = (x_1, x_2, x_3, ...) \in l_\infty$. Then, $Tp = p$ if and only if $p = 0$. We compute as follows:

$$||Tx - p||_\infty = \begin{cases} \frac{11}{12}||(0, x_1^2, , x_2^2, x_3^2...)||_\infty, \ if \ ||x||_\infty \leq 1, \\ \\ \frac{11}{12||x||_\infty^2}||(0, x_1^2, , x_2^2, x_3^2...)||_\infty, \ if \ ||x||_\infty > 1, \end{cases}$$

so that

$$||Tx - p||_\infty \leq \begin{cases} \frac{11}{12}||x||_\infty^2 \leq \frac{11}{12}||x||_\infty, \ if \ ||x||_\infty \leq 1, \\ \\ \frac{11}{12}.1, \ if \ ||x||_\infty > 1, \end{cases}$$

Hence, we obtain that

$$||Tx - p||_\infty \leq \frac{11}{12}\Big|\Big|x - p\Big|\Big|_\infty \ \forall \ x \in l_\infty, \ p = 0.$$

Hence, T satisfies contractive condition (1.0). But the map T is not a contraction. To see this, take $x = (\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, ...); y = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, ...)$. Then,

$$||x - y||_\infty = \frac{1}{4}; \ ||Tx - Ty||_\infty = \frac{11}{12}\Big|\Big|\Big(0, \frac{5}{16}, \frac{5}{16}, ...\Big)\Big|\Big|_\infty = \frac{55}{192}.$$

Suppose there exists $a \in [0,1)$ such that $||Tx - Ty||_\infty \leq a||x - y||_\infty \ \forall \ x, y \in E$, then we must have $\frac{55}{192} \leq \frac{a}{4}$ which yields that $a \geq \frac{220}{192} > 1$, a contradiction. So, T is not a contraction map.

It is clear that every contraction map with a fixed point satisfies inequality (1.0). This completes the example.

Remark 1. If an operator satisfies inequality $\rho(Tx,p) \leq a\rho(x,p)\ \forall x \in E,\ p \in F(T), a \in [0,1)$, then p is necessarily unique. For assume that there exists $q \neq p$ such that $Tq = q$. Then, $\rho(p,q) = \rho(Tp,Tq) \leq a\rho(p,q)$ so that $(1-a)\rho(p,q) \leq 0$ which yields $p = q$.

Example 2.2. Zamfirescu [11] studied mappings T satisfying the following contractive condition: for $\delta \in [0,1)$

$$\| Tx - Ty \| \leq \delta \| x - y \| + 2\delta \| x - Tx \| . \qquad (1.2)$$

We observe that if p is a fixed point of T, we obtain from inequality (1.2) that if $\delta < \frac{1}{5}$, then

$$\| Tx - p \| \leq a \| x - p \| \text{ where } a = \frac{3\delta}{1-2\delta}.$$

Example 2.3. Another general class of mappings generalizing the

contraction mappings was introduced by Hardy and Rogers [7] as follows: Let (M,d) be a complete metric space and $T : M \to M$ satisfy the following contractive condition: $\forall\ x,y \in M$,

$$d(Tx,Ty) \leq a_1 d(x,y) + a_2 d(x,Tx) + a_3 d(y,Ty) + a_4 d(x,Ty) + a_5 d(y,Tx),$$

where $a_i \geq 0\ \forall i = 1,2,3,4,5,\ \sum_{i=1}^{5} a_i < 1$. Hardy and Rogers proved that T has a unique fixed point. Several authors proved fixed point theorems for mappings satisfying special cases of the contractive condition of Hardy and Rogers. We observe that if p denotes the unique fixed point of T in the theorem of Hardy and Rogers, then the following inequality holds:

$$d(Tx,p) \leq ad(x,p)\ \forall\ x \in M,$$

where

$$a := \frac{a_1 + a_2 + a_3}{1 - a_2 - a_5} \in (0,1)$$

.

Remark 2. It is known that whenever a recursive formula is used successfully to approximate a fixed point of an operator, any discussion of a more cumbersome recursion formula for the same problem is absolutely unnecessary.

In particular, whenever the classical Picard formula, $x_n = Tx_{n-1}$, $n = 1, 2, ...,$ $x_0 \in D(T)$, converges to fixed points of mappings T which are not assumed to be differentiable, any discussion of a more cumbersome iterative formula for such a problem is totally undesirable (see e.g, [5] for more comments). The Picard iteration process requires much less computation time than any multi-step method, thereby making it less expensive to implement and more appealing to possible users.

Noor [8] introduced the *three-step* method for approximating solutions of *general variational inequality problems*. In the problem under consideration in this paper, *Theorem(AO),* the use of this *three-step* method of Noor is unnecessary since the simpler Picard iterative scheme is appplicable (see the theorem below).

3. A STRONG CONVERGENCE THEOREM

Theorem. 1 Let (E, ρ) *be a metric space. Let* $T : E \to E$ *be a map with* $F(T) := \{x \in E : Tx = x\} \neq \emptyset$ *be such that*

$$\rho(Tx, p) \leq a\rho(x, p) \ \forall x \in E, \ p \in F(T), a \in [0, 1). \qquad (*)$$

Then, the Picard sequence $\{x_n\}_{n=0}^{\infty}$ *defined by*

$$x_{n+1} = Tx_n, \ x_0 \in E,$$

$n = 0, 1, 2, ...$ *converges to* p.

Proof. Let $p \in F(T)$ and T satisfy inequality $(*)$. Then, the following inequality holds:

$$\rho(x_{n+1}, p) \leq a^{n+1}\rho(x_0, p) \ \forall n \in \mathbb{N}. \qquad (1.3)$$

The proof of inequality (1.3) is by induction. Observe that if $n = 0$, we have

$$\rho(x_1, p) = \rho(Tx_0, p) \ \leq a\rho(x_0, p),$$

and so, for $n = 1$,

$$\rho(x_2, p) = \rho(Tx_1, p) \leq a\rho(x_1, p) \leq a^2\rho(x_0, p),$$

and inequality (1.3) holds for n=1. Assume it holds for $n = k \in \mathbb{N}$. Then,

$$\rho(x_{(k+1)+1}, p) = \rho(Tx_{k+1}, p) \leq a\rho(x_{k+1}, p) \leq a^{k+2}\rho(x_0, p),$$

and so inequality (1.3) holds for $k + 1$. By induction, inequality (1.3) holds for all $n \in \mathbb{N}$. Since $a \in [0, 1)$, it follows that $\{x_n\}_{n=0}^{\infty}$

converges to p. ■

Remark 3. The stability of the Picard iteration process, whenever it converges, is well known (see e.g., Berinde [2], [3], Bosede and Rhoades [4], Harder and Hicks [6], Rhoades [9], Ostrowski [10]).

In the light of *Remark 3*, our theorem is a significant improvement on the result of Akewe and Olaleru [1], and the results of a host of other authors.

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